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FUNDAMENTAL RELATION OPERATIONS IN THE MATHEMATICAL MODELS OF PROGRAMMING

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Abstract. It is a well known aspiration that aims to use those methods exclusively which guarantee the correctness of the developed program with respect to the posed problem. That is what made it essential to find abstract mathematical definition of program, problem, and solution. We investigate the properties of some relational operations, which are generally applicable in these mathematical models. One of these studied operations is that which can be called the strict composition of relations. It can be originated from the normal relational composition by restricting its domain. The program function of sequential composition of nondeterministic programs is defined by the help of strict composition. This operation is closely connected with the constituting operation of the pre-image of a set, with respect to a given relation. The latter operation comes from a concept of converse-image, to define it in a more rigorous way. We describe a geometrical and matrixaritmethical representation of strong composition and pre-image. These representations make possible a paralell computing of program function of the sequential composition following Bendarek's and Ulam's

ideas [2]. Finally, we investigate the properties of closure and bounded closure of a relation. These operations are applicated to determine the program functions of loops. We analyse the occurrent similar concepts in the literature, investigate their connections to the closure and to the bounded closure, and we try to generalize the theorems which deal with the relationship between the closure and the bounded closure.

I. Introduction

The functions and relations play a central role in the mathematical models of programming. The relations are applicable to the description of nondeterministic programs [17, 4, 19, 7, 14, 15]. Relation means binary relation in the following.

We can imagine a program as a relation, which associates a sequence of the points of the state space to the points of the state space .

DEF.1. [7]. The relation S is called a **program**, if

i) $S \subseteq A \times A^{**}$,

ii) $\mathcal{D}_S = A$,

iii) $(a \in A \wedge \alpha \in S(a)) \Rightarrow \alpha_1 = a$,

iv) $(\alpha \in \mathcal{R}_S \wedge \alpha \in A^*) \Rightarrow (\forall i \; (1 \leq i < |\alpha|) : \alpha_i \neq \alpha_{i+1})$,

v) $(\alpha \in \mathcal{R}_S \wedge \alpha \in A^\infty) \Rightarrow$

$$(\forall i \in N \; (\alpha_i = \alpha_{i+1} \rightarrow (\forall k (k > 0) : \alpha_i = \alpha_{i+k}))).$$

where A^* is the set of the finite sequences of the points of the state space, and A^∞ the set of the infinite ones. Let $A^{**} = A^* \cup A^\infty$. An important element of the above definition is that the domain of the program is identical with the whole space of states. Note, that the program is not necessarily a deterministic relation ! The nondeterminancy is used in its widest meaning, as it occurs in the literature. If the program associates both finite and infinite sequences to a point of the state space , then neither the infinite sequences are more preferable to the finite ones, nor the finite sequences are to the infinite ones [6, 10, 4].

The problems are given in the form of a relation : $F \subseteq A \times A$. To determine, if a program is correct with respect to a problem (specification), we introduce the concept of the program function:

DEF.2. [7]. The **program function** of the program S is the relation $p(S) \subseteq A \times A$, if

$i)\quad \mathcal{D}_{p(S)} = \{a \in A \mid S(a) \subseteq A^*\}$

$ii)\quad \forall a \in \mathcal{D}_{p(S)} : p(S)(a) = \{b \in A \mid \exists \alpha \in S(a) : \tau(\alpha) = b\},$

where $\tau : A^* \to A$ is a function, which associates its last element to the sequence $\alpha = (\alpha_1, ..., \alpha_n)$, i.e. $\tau(\alpha) = \alpha_n$.

A similar concept, called indeterministic programrelation occurs in Mills's work [16].

DEF.3. [7]. The program S is **correct with respect to** problem F, (or with other words : the program S is a **solution of** problem F) , if

$i)\qquad \mathcal{D}_F \subseteq \mathcal{D}_{p(S)},$

$ii)\qquad \forall a \in \mathcal{D}_F : p(S)(a) \subseteq F(a).$

The finite sequences of the points of the state space are more preferable to the infinite ones, if we have a closer look at the above definitions of the program function and of the correctness (or solution). Therefore only those points are the elements of the the domain of program function of the program, which only finite sequences are associated to. It means that the Dijkstra's demonic model of nondeterminacy is applicable to determining program's correctness with respect to a problem [4].

It is necessary to be familiar with the program functions of the program constructs, to prove the correctness of a complicated program with respect to a difficult problem . First we should investigate the program function of sequential composition of programs.

DEF.4. [7]. Let $S, S1, S2 \subseteq A \times A^{**}$ programs. S called the sequential composition of $S1$ and $S2$, $S = (S1; S2)$, if

$$\forall a \in A : S(a) = \{\alpha \in A^{\infty} \mid \alpha \in S1(a)\} \cup \{\chi(\alpha, \beta) \mid \alpha \in S1(a) \cap A^{*} \ \wedge \ \beta \in S2(\tau(\alpha))\}$$

where the sequence $\chi(\alpha, \beta)$ comes from the concatenation of α and β, substituting all those finite subsequences which consist of the same elements, with one of it's elements. (Since the last element of the first sequence is in this case identical with the first element of the second one, they may constitute a length 2 subsequence of identical elements. This subsequence has to be replaced by one of it's elements.)

The simple composition of program functions does not give the program function of the sequential composition. (The simple composition is only applicable to deterministic programs [17, 15].) Let us examine the following example : the program function of the first program associates two points to an x point of the state space. One and only one of these two points is an element of the domain of program function of the second program. That means x is not an element of the domain of the program function of the sequential composition, but x is an element of the domain of the simple composition of the program functions. A solution for this problem can be that if we define a stronger version of the composition of relations. Let us consider those points a of the domain of the first relation to which the first relation associates (also) some point b not in the domain of the second relation. We just have to exlude such a's from the domain of the stronger version of the composition:

DEF.5. : Strict composition of relations : $P \subseteq A \times B$, $Q \subseteq B \times C$

$$Q \odot P ::= \{(a,c) \mid P(a) \subseteq \mathcal{D}_Q \ \wedge \exists b \in B : (a,b) \in P \ \wedge \\ (b,c) \in Q\}.$$

Indeed, it is easy to prove, that $p(S1;S2) = p(S2) \odot p(S1)$.

Mili restricts the domain in his definition of the program function of the sequential constructs in the case of deterministic programs [14], Berghammer and Zierer does so in the case of nondeterministic ones similarly [4]. Mili, Desharnais and Gagné make a more rigorous restriction in an other paper in the sequence statetment rule [15 pp. 244.]. Their requirement is the following: the set of the points associated to an arbitrary point of the domain of first relation has to be a part of the domain of the second one.

Let us investigate the properties of the new relational operation !

II. The properties of the strict composition :

Notations : A, B, C, D are arbitrary sets, H, G, F are relations over these sets :

$$H \subseteq A \times B\ ,\ G \subseteq B \times C\ ,\ F \subseteq C \times D$$

A) Statement : If a is an element of $\mathcal{D}_{GoH} \cap \mathcal{D}_{G \odot H}$ then $GoH(a) = G \odot H(a)$.

B) Statement : $\mathcal{D}_{G \odot H} \subseteq \mathcal{D}_{GoH}$

C) Statement :($\forall a \in \mathcal{D}_H : |H(a)| = 1) \Rightarrow G \odot H = GoH$
$\mathcal{D}_G = B \Rightarrow G \odot H = GoH$.

(All of the three statements can be concluded directly from the definition of the strict composition.)

Remark 1. : If at least one of the two relations is a function itself, then the strict composition gives the same result as the simple composition. (We require that a function should be defined in all points of the base set.) □

D) Statement : The strict composition is an associative operation :

$$(F \odot G) \odot H = F \odot (G \odot H)$$

It will be sufficient to show that the domains are the same, because A), B) and $(FoG)oH = Fo(GoH)$ holds.

On the one hand a is an element of the domain of the relation on the left side if and only if

1) $a)$ $a \in \mathcal{D}_H$
$b)$ $H(a) \subseteq \mathcal{D}_{F \odot G}$, i.e. :
$H(a) \subseteq \mathcal{D}_G \;\wedge\; \forall b \in H(a) : G(b) \subseteq \mathcal{D}_F$,

on the other hand a is an element of the domain of the relation on the right side if and only if

2) $a)$ $a \in \mathcal{D}_{G \odot H}$, i.e. : $a \in \mathcal{D}_H \wedge H(a) \subseteq \mathcal{D}_G$
$b)$ $(G \odot H)(a) \subseteq \mathcal{D}_F$, i.e. : $\forall b \in H(a) : G(b) \subseteq \mathcal{D}_F$

Comparing 1) to 2) follows that a is an element of the domain of the relation on the right side if and only if it is an element of the one on the left side. □

Remark 2.: In consequence of the above statetment the relations over $A \times A$ construct a semigroup in respect of strict composition operation, where A is an arbitrarily given base set.

The following statement expresses, that this semigroup has an identity element.□

E) Statement:

$$\exists !\, 1' \subseteq A \times A \;:\; (\forall R \subseteq A \times A \;:\; (R \odot 1' = R \;\wedge\; 1' \odot R = R)).$$

The uniqueness follows from the fact that the relation $1'$ [21] is left sided and right sided identity element at the same time [9].

$$1' = \{\ (a, b) \mid a = b \;\wedge\; a \in A\ \}$$

Since $1'$ is a function, it is sufficient for us to consider the case when all occurences of the strict composition in statement E) are substituted by the simple composition. (See remark 1.) It is known, that the relations over $A \times A$ construct a semigroup with respect to simple composition, and the identity element is exactly the relation $1'$ [21]. □

After the identity element has been found, the question is raised immediately : which relations have got an inverse , i.e. to which relation R can be found such a relation RR, as :

$$\alpha)\ R \odot RR = 1' \quad \text{and} \quad \beta)\ RR \odot R = 1'$$

If a relation has an inverse, then the uniqueness of this inverse is known [9].

If the α) equation is true, then RR is defined in all points of A, since the domain of the strict composition is a part of the domain of the relation applied at first. It similarly follows from β), that R is defined in all points of A.

From the above line reasoning, it follows that we can apply the statement C) again, i.e. it is sufficient for us to consider the case when all occurences of the strict composition in α and β are substituted by the simple composition. It is known, that only and all the permutations have an inverse with respect to the simple composition from the relations over $A \times A$ [9]. Hence:

F) Statement: Only and all the permutations have an inverse with respect to the strict composition from the relations over $A \times A$, and this inverse is unique. The permutations construct a group with respect to the strict composition.

The statements D),E) and F) are analogue with the statements concerning the simple composition [13, 21, 20].

Let us investigate the connection between the strict composition and the other relational operation !

Def.6.: [21]

Let $R, S \subseteq A \times B$, $R^{(-1)} \subseteq B \times A$.

The complement of S is $\tilde{S} ::= \{(a,b) \mid \neg((a,b) \in S)\}$.

The converse of R is $R^{(-1)} ::= \{(b,a) \mid (a,b) \in R\}$.

The Schröder-rule does not hold in the case of the strict composition (although this rule can play an important role in the construct of the axiomatic relation calculus), nor does the Tarski-rule [20 , 13].

$$QoR \subseteq S \;\Leftrightarrow\; Q^{(-1)}o\tilde{S} \subseteq \tilde{R} \;\Leftrightarrow\; \tilde{S}oR^{(-1)} \subseteq \tilde{Q}$$

(Schröder – rule)

$$R \neq \emptyset \;\Rightarrow\; LoRoL = L \text{ , where } L = A \times A$$

(Tarski – rule)

The example 1 is a counter-example for both rules (for the case of the strict composition).

Example 1.:

$A = \{1,2\},\ R,S,Q \subseteq A \times A,$
$Q = R = \{(1,1),(1,2)\}, S = \emptyset.$
Hence
$Q \odot R \subseteq S$, but $Q^{(-1)} \odot \tilde{S} \not\subseteq \tilde{R}$ and $\tilde{S} \odot R^{(-1)} \not\subseteq \tilde{Q}$
as well as $R \neq \emptyset$ and $L \odot R \odot L = \emptyset$.

G) Statement: (monotonicity) :

a) $\quad R \subseteq S \ \Rightarrow\ R \odot Q \subseteq S \odot Q$
b) but $\quad R \subseteq S \ \Rightarrow\ Q \odot R \subseteq Q \odot S$ does not hold.

Proof :

a) $\mathcal{D}_{R \odot Q} \subseteq \mathcal{D}_{S \odot Q}$, because if $a \in \mathcal{D}_{R \odot Q}$, then $a \in \mathcal{D}_Q \wedge Q(a) \subseteq \mathcal{D}_R$, since $\mathcal{D}_R \subseteq \mathcal{D}_S$ therefore $Q(a) \subseteq \mathcal{D}_S$ holds, hence $a \in \mathcal{D}_{S \odot Q}$.
From the statement A) and the monotonicity of the simple composition [20] it follows : if the domain of $R \odot Q$ is a part of the domain of $S \odot Q$, then $R \odot Q \subseteq S \odot Q$ is valid, too. □

To prove the statement b) we show a counter-example :

Example 2. :

$A = B = C = \{1,2\},\ R,S \subseteq A \times B,\ Q \subseteq B \times C,$
$S = \{(1,1),(1,2)\},\ R = \{(1,1)\},\ Q = \{(1,1)\}.$
Hence : $Q \odot R = \{1,1\},\ Q \odot S = \emptyset.$

It is not surprising, that the often used connection between the simple composition and the constitution of the converse relation does not hold in the case of the strict composition.

H) Statement: If $H \subseteq A \times B$, $G \subseteq B \times C$, then

$$(GoH)^{(-1)} = H^{(-1)} o G^{(-1)} \quad [\,21\,], \text{ but}$$
$$(G \odot H)^{(-1)} \neq H^{(-1)} \odot G^{(-1)}$$

The first part of this statement is well known from the literature. To prove the second part we show a counter-example:

Example 3. : Let $a \in A$, $b_1, b_2 \in B$, $c \in C$.

$$H \subseteq A \times B \ , \ H = \{(a, b_1), (a, b_2)\}$$
$$G \subseteq B \times C \ , \ G = \{(b_1, c)\}$$
$$\text{Hence } G \odot H = \emptyset \ , \ (G \odot H)^{(-1)} = \emptyset$$
$$G^{(-1)} = \{(c, b_1)\} \ , \ H^{(-1)} \odot G^{(-1)} = \{(c, a)\}.$$

If substitute all occurences of the converse-image by a stronger version of this operation, then we get a statement, wich holds. We introduce for this statement the concept of the pre-image.

Def. 7. : [8] The set $H^{-1}(Y)$ called the **pre-image** of Y with respect to relation H, if

$$H \subseteq A \times B \ , \ Y \subseteq B \ , \ \Gamma ::= 2^A \ , \ \Sigma ::= 2^B$$
$$H^{-1}(Y) = \{a \mid a \in D_H \wedge H(a) \subseteq Y\}$$

Remark 3. : $H^{-1} \subseteq \Sigma \times \Gamma$ is a function (the pre-image function of H). □

Remark 4. : Let us suppose a problem is described by its pre- and postcondition (Q_b , R_b) [7] . In this case the truth set of the weakest precondition with respect to R_b postcondition is computable by the help of the pre-image, if the program function is known. (The truth set of the weakest precondition is the largest set, whose elements only finite seqeuences are associeted to, and the postcondition is true in respect of the last elements of these sequences.) :

$$[wp(S, R_b)] = p(S)^{-1}[R_b]. \ [\ 8\ ]. \ \square$$

J) Statement (the connection between the strict composition and the pre-image) :

If $H \subseteq A \times B$, $G \subseteq B \times C$, then
$\forall Y \subseteq C : ((G \odot H)^{-1}(Y) = H^{-1} \odot G^{-1}(Y)),$

Proof:

I. First let us execute equivalent transformations on the right side of the equation :

$H^{-1} \odot G^{-1}(Y) = H^{-1} o G^{-1}(Y)$, since H^{-1}, G^{-1} are functions.

$$G^{-1}(Y) = \{b \mid b \in B \wedge G(b) \subseteq Y \wedge b \in \mathcal{D}_G\}$$

$$H^{-1}(G^{-1}(Y)) = \{a \mid a \in \mathcal{D}_H \ \wedge \ H(a) \ \subseteq \ G^{-1}(Y)\}$$

$(H(a) \subseteq G^{-1}(Y)) \equiv \forall b \in H(a) : (G(b) \subseteq Y \wedge b \in \mathcal{D}_G)$ i.e. :

$$H^{-1}(G^{-1}(Y)) = \{a | \, a \in \mathcal{D}_H \wedge \forall b \in H(a) : (b \in \mathcal{D}_G \wedge G(b) \subseteq Y)\}$$

II. Let us execute equivalent transformations on the left side of the equation, too :

$$(G \odot H)^{-1}(Y) = \{a \mid a \in A \wedge G \odot H(a) \subseteq Y \wedge G \odot H(a) \neq \emptyset\}$$

$$G \odot H(a) \neq \emptyset \ \equiv \ a \in \mathcal{D}_H \wedge H(a) \subseteq \mathcal{D}_G$$

$$(G \odot H(a) \subseteq Y \wedge H(a) \subseteq \mathcal{D}_G) \ \equiv$$

$\equiv \ \forall b \in H(a) : (b \in \mathcal{D}_G \wedge G(b) \subseteq Y)$ i.e. :

$$(G \odot H)^{-1}(Y) = \{a | \, a \in \mathcal{D}_H \wedge \forall b \in H(a) : (b \in \mathcal{D}_G \wedge G(b) \subseteq Y)\}$$

Hence the sets on the left side and on the right side of the equation are the same, for an arbitrary $Y \subseteq C$. $\square$

Remark 5. : $(GoH)^{-1} \neq H^{-1}oG^{-1}$. □

The example 3. is a suitable counter-example again. Let the set $Y = \{c\}$! Hence :

$$\begin{aligned} (GoH)^{-1}(Y) &= \{a\}, \\ G^{-1}(Y) &= \{b_1\}, \\ H^{-1}(G^{-1}(Y)) &= \emptyset \ . \end{aligned}$$

Remark 5./a. :There is not possible to correct the Schröder-rule substituting converse-image by pre-image. □

Remark 6. :The semantics of the sequential composition is definiable by the following equation, (the properties of the weakest precondition are choosen as axioms) [6, 10]:

$$wp(S1; S2, R) = wp(S1, wp(S2, R)).$$

This equation is equivalent to the statement J). □

III. The representation of strict composition by matrices and on geometrical way:

Def.8. : [5, 1] Let $A = \{a_1, a_2, ..., a_n\}$ a set with n-element, $B = \{b_1, b_2, ..., b_m\}$ a set with m-element, $R \subseteq A \times B$ a relation. The $M_R \in \mathcal{L}_{n\times m}$ - matrix is called the **incidence matrix** of R (where $\mathcal{L} = \{$ true , false $\}$) , if

$$M_R[k, j] := \begin{cases} \text{true}, & \text{if} \quad (a_k, b_j) \in R \\ \text{false}, & \text{otherwise} . \end{cases}$$

According to Copilowish's paper the operations of Boolean algebra are choosen for the basic operations of the matrixoperations.(In this case the elements of the matrices are logical values, therefore the operations '+' and '*' over them means logical 'OR' and 'AND'.) Comparing with the choise of integeroperations it simplifies the description of notions.

The incidence matrix of the simple composition of two relations can be computed by the matrixmultiplication of the incidence matrices [5] :

$$\begin{aligned} \text{Let}\quad & A = \{a_1, a_2, ..., a_n\}\ , \quad B = \{b_1, b_2, ..., b_m\}, \\ & C = \{c_1, c_2, ..., c_q\}\ ,\ R \subseteq A \times B\ ,\ G \subseteq B \times C, \\ \text{then}\quad & M_{G o R} \;=\; (M_R * M_G)\ . \end{aligned}$$

Let us constitute the column vectors $D_G \in \mathcal{L}_m$ and $D_{G \odot R} \in \mathcal{L}_n$ to compute the incidence matrix of strict composition :

$$D_G[j] \;:= \sum_{s=1}^{q} M_G[j, s]$$

$$D_G[j] \;=\; \text{true, if and only if } b_j \in \mathcal{D}_G$$

Let the column vector $D_{G \odot R} \in \mathcal{L}_n$ be the following :

$$D_{G \odot R}[k] := \prod_{j=1}^{m} (M_R[k, j] \rightarrow D_G[j]).$$

$$D_{G \odot R}[k] = \;\text{true , if and only if } (a_k \in \mathcal{D}_{G \odot R} \vee a_k \notin \mathcal{D}_R).$$

The incidence matrix of the strict composition can be obtained in two steps, first by constituting the incidence matrix of simple composition, second multiplicating the elements of its rows by that element of column vector $D_{G \odot R}$ which has the same index as the columnindex of the matrixelement :

$$M_{G \odot R}[k, j] = M_{G o R}[k, j] * D_{G \odot R}[k].\ \text{ i.e. :}$$

$$M_{G\odot R}[k,j] = (M_R * M_G)[k,j] * \prod_{l=1}^{m}(M_R[k,l] \rightarrow \sum_{s=1}^{q} M_G[l,s]).$$

Remark 8. : Let $Y \subseteq B$, let us define the $H_y \in \mathcal{L}_m$ and the $H_{R^{-1}(Y)}$ column vectors as it follows :

$$H_y[j] := \begin{cases} \text{true}, & \text{if } b_j \in Y \\ \text{false}, & \text{otherwise .} \end{cases}$$

$$H_{R^{-1}(Y)}[k] := \prod_{j=1}^{m}(M_R[k,j] \rightarrow H_y[j])$$

$$\text{Thus : } a_k \in R^{-1}(Y) \Leftrightarrow H_{R^{-1}(Y)}[k].$$

We can conclude from the similarity of the computing method of the two operations that there is a strong connection beetwen the strict composition and the pre-image. This strong connection is also expressed by the statement J). □

Remark 9. : It is theoretically possible to compute the image of every single point with respect to the strict composition and the pre-image, in the case of finite sets, if a relation is represented by a matrix and a set is given by a vector. Thus the program function of the sequential composition and the truth set of the weakest precondition are computable by the help of matrix- and vectoroperations. Since the strict composition is an associative operation, the program function of a large sequential program can be computed parallel by the help of the recursive doubling technic [18]. □

A possible geometrical representation of the simple compositions of relations and other relational operations is made known by Alfred Tarski [21]: Let $F, G \subseteq R \times R$, where R is the set of real numbers. Let us get a co-ordinate system on the plain S, with real axes, let the axis x perpendicular to the axis y. Associate the set of points P_F and P_G to the relations F, G :

$$P_F := \{w \in S \mid w = (x,y) \wedge (x,y) \in F\}$$
$$P_G := \{w \in S \mid w = (x,y) \wedge (x,y) \in G\}$$

We search the set of points P_{GoF}, according to the definition :

$$p_{GoF} := \{w \in S \mid w = (x,y) \wedge (x,y) \in GoF\}.$$

Let us rotate the set of points P_F into P_F* and the axis y into $y*$ around the axis x with 90 grades. Let x, y and the rotated y $(y*)$ constitute a right-handed co-ordinate system. Let us rotate the set P_G and the axis x around y similarly into P_G* and $x*$. At this time $x* = y* = z$. Let us draw perpendicular lines to all points of P_F* and P_G*, let us project their cross points onto the plain S. The constructed set is P_{GoF}.

To get the set of points $P_{G \odot F} := \{w \in S \mid w = (x,y) \wedge (x,y) \in G \odot F\}$ the above algorithm must be modified. The construction has to be started from a subset of the P_F* instead of P_F*. Let us consider only those lines which are paralell with z and the perpendicular projection of its intersection with P_F* to z is not a part of the perpendicular projection of the set of the points P_G* to the z. Let us exclude those points of P_F* from P_F* which are incidental to a line described above.

The geometrical representation of the pre-image of a set with respect to a given relation can be determined similarly.

Remark 10. : The above algorithm of Tarski and the axioms of McKinsey's complete atomic relation algebra attracted the attention to the connection between the calculus of relation and the projective geometry which is recorded as the algebrian version of mathematical logic [3, 20]. Bendarek and Ulam think the paralell computing of the composition of relations possible on the basis of the above projective algorithm of Tarski [2].□

IV. The closure and bounded closure of relations :

Not only can the program function of sequential composition be computed by the help of the program function of its components, but the program function of a loop is computable on the basis of the loop condition and the program function of the body of the loop by the help of introduction of more complicated relational operations.

The program function of the body of a loop is modified according to the loop condition firstly by restricting and suitably extending its domain to the truth set of the loop condition. After that the closure of the obtained relation has to be determined [7].

Def. 9. : Let $R \subseteq A \times A$. The infinite sequence $(a_0, a_1, ...)$ can be called the **infinite point-chain** of relation R and denoted by $L_R(a_0, a_1, ...)$, if

$$\forall k \in N : a_k \in R(a_{k-1}).$$

a_0 is called the **start point** of the point-chain.

Def. 10. Let $R \subseteq A \times A$. The sequence $(a_0, a_1, .., a_q)$ can be called **length q point-chain** of the relation R, and denoted $L_R(a_0, a_1, ..a_q)$, if

$$q \in N_0 \land \forall k \in [1, q] : a_k \in R(a_{k-1})$$

a_0 is called the **start point** of the point-chain. If $a_q \notin \mathcal{D}_R$, then a_q is called the **end point** of the chain. In the case of an arbitrary relation $R \subseteq A \times A$ and an arbitrary point $a_0 \in A$ the sequence (a_0) is a $L_R(a_0)$ length 0 point-chain at the same time.

Def. 11.[7] Let $R \subseteq A \times A$. The $\overline{R} \subseteq A \times A$ relation is called the **closure** of relation R, if

$$\mathcal{D}_{\overline{R}} = \{a \mid a \in A \land \neg(\exists L_R(a_0, a_1, ...) : a = a_0)\}$$

$$\overline{R}(a) = \{b \mid b \in A \land \neg(b \in \mathcal{D}_R) \land$$
$$\exists L_R(a_0, a_1, ..a_q) : (b = a_q \land a = a_0)\}.$$

We can formulate a more rigorous restriction of the domain :

Def. 12. [7] Let $R \subseteq A \times A$. The relation $\overline{\overline{R}} \subseteq A \times A$ is called the **bounded closure** of the relation R, if

$$D_{\overline{\overline{R}}} = \{a \mid a \in A \land \exists n(a) \in N_0 : \forall q > n(a) : \\ (\forall L_R(a_0, a_1, \ldots, a_q) : a \neq a_0)\},$$

$$\overline{\overline{R}}(a) = \{b \mid b \in A \land \neg(b \in D_R) \\ \land \exists L_R(a_0, a_1, ..a_q) : (b = a_q \land a = a_0)\}.$$

Hence the elements of the domain of the closure of R are those elements of the base set A which are not start points of any infinite point-chains of R. The domain of the bounded closure of R is a smaller set. All the points are excluded from this which are not start points of any infinite point-chains, but the length of the sequences starting from it have not got an upper bound. Both the closure and the bounded closure associate those points to a point of their domains which are end points of the point-chains starting from the given point. (The length of the point-chain may be 0!)

Remark 11. : There are several similar definition in the literature. Mills defines the closure of functions similarly to the definition above of bounded closure [16]. Schmidt associates an oriented graph to every relation (the point-chains are the paths of graphs), and defines the progressively finite and progressively bounded graphs. If a graph is progressively finite, then the domain of the closure of the corresponding relation to the graph is identical with the domain of the relation. (The so called initial part of a graph, the $I(G)$ contains the vertices which correspond to the elements of the domain of the closure.) There is a similar connection between the bounded closure and the progressively bounded graph. Schmidt gives an example that the two concepts are not the same generally. He proves that if the relation is deterministic then the two concepts are identical [19]. Schmidt's these concepts are corresponding to Mili's two ones. The noninfinitely decreasing relation corresponds to the one, whose domain of closure is identical with the domain of the relation. Mili defines also the concept of the relation whose domain of the bounded closure is identical with the domain of the relation. Mili concludes a false statement saying the two concepts are identical in every case [15]. The concepts of the transitive closure and reflexive transitive closure of the relations occur several times. These concepts are only related to the closure and the bounded closure but are not identical with them. □

Remark 12. : In practice we construct our programs only from those loop bodies and loop conditions which hold the statement that when we restrict and similarly expand the program function of the loop body to the truth set of the loop condition, then its bounded closure and closure (i.e. the program function of the loop) are indentical. This follows from the inference rule of the loop [6, 7]. That is why the relationship of these two concepts is essential for us. □

V. The connection between the closure and bounded closure of relations:

Let $R \subseteq A \times A$. Hence

A) Statement: If a is an element of $\mathcal{D}_{\overline{R}} \cap \mathcal{D}_{\overline{\overline{R}}}$ then
$\overline{R}(a) = \overline{\overline{R}}(a)$.

B) Statement: $\mathcal{D}_{\overline{\overline{R}}} \subseteq \mathcal{D}_{\overline{R}}$.

Both statements follow directly from the definitions.

C) Statement: If the set A is finite, then $\overline{R} = \overline{\overline{R}}$.

The length of every finite point-chain is not larger than the number of the elements of the set A. Hence the number of the elements of A is an upper bound for the length of every finite point-chain. □

D) Statement: If A is a countable infinite set, then

$$\exists R \subseteq A \times A : \overline{R} \neq \overline{\overline{R}}.$$

Proof: The countable infinite sets are isomorph, so it is enough to show an example for a special concrete A [11]. Let $A = N_0$, $R \subseteq N_0 \times N_0$,and

$$R(a) := \begin{cases} \{b \mid b > 1\}, & \text{if } a = 0 \\ \{a - 1\}, & \text{if } a > 1. \end{cases}$$

$$\mathcal{D}_{\overline{R}} = N_0, \quad \mathcal{D}_{\overline{\overline{R}}} = \{a \mid a \geq 1\}.$$

Hence 0 is not an element of the domain of the bounded closure of R, but it is an element of the domain of the closure of R. □

In the previous proof $|R(0)| = \infty$. If for all $a \in A$: $R(a)$ is finite, then the situation is openly other.

E) Statement : If A is countable infinite, $R \subseteq A \times A$, and $\forall a \in A : R(a)$ is a finite set, then $\overline{R} = \overline{\overline{R}}$.

Proof : Let us apply the sequence of ideas which is found at the proof of Kõnig lemma [12]. We have to construct an infinite oriented graph first (a similar one occuring in [19]) :

Let the number of element of the sets $A_1, A_2, \ldots, A_k, A_{k+1}, \ldots$ be equal to the number of the elements of set A, and $g_1, g_2, \ldots, g_k$, $g_{k+1}, \ldots$ is an infinite sequence of relations. Denote the elements of A_k by $a_{k,0}, a_{k,1}, a_{k,2}, \ldots, a_{k,r}, a_{k,r+1}, \ldots$. We require that

$$\begin{aligned} &\forall k \in N : g_k \subseteq A_k \times A_{k+1} \\ &\qquad \wedge k > 1 \rightarrow ((a_{k,m}, a_{k+1,n}) \in g_k \leftrightarrow (a_m, a_n) \in R) \\ &\qquad \wedge k = 1 \rightarrow ((a_{1,m}, a_{2,n}) \leftrightarrow ((a_m, a_n) \in R \wedge a_m \in \mathcal{D}_{\overline{R}})) \end{aligned}$$

Let the points of the set $(\cup A_k)$ correspond to the vertices of an oriented graph in a one-to-one manner . Denote every vertex by the identifier of its corresponding point of the set $(\cup A_k)$. Let us lead an edge from vertex $a_{k,m}$ to vertex $a_{k+1,n}$, if and only if $(a_{k,m}, a_{k+1,n}) \in g_k$.

Let us investigate the case when the domain of the closure of R is not empty. (If it is empty, then the statement is obviously true.) Let us choose an arbitrary point a_x of the domain of R.

Denote that connected part of graph by RL_x which a path from $a_{1,x}$ leads in. (Remark : RL_x contains the paths corresponding to those point-chains of relation R whose start point is a_x .)

Let us suppose in an indirect way that RL_x contains infinite pieces of verteces. That means $a_{1,x}$ has infinite pieces of descendants ,but since the number of its successors (immediate descendants) is finite, (following the condition with respect to relation R) that is why at least surely one exist among them which inherits that property of $a_{1,x}$ that it has infinite pieces of descendants. By the help of this sequence of ideas we can construct an infinite path starting from $a_{1,x}$.

But a_x is an element of the domain of the closure of the relation R, hence $a_{1,x}$ cannot be a start point of an infinite path! It means that the number of the verteces of RL_x is finite. It is necessary that there is at least one vertex among a finite number of them, whose first index is maximal. The value of this index is an upper bound for the point-chains starting from a_x , thus a_x is an element of the domain of the bounded closure, too. Hence the domain of the closure is a part of the bounded one. On the basis of V.A) and V.B) we can conclude, that the closure and the bunded closure of R is the same. □

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