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BLOCK-PULSE FUNCTIONS SERIES SOLUTION OF TIME-VARYING LINEAR SYSTEMS OF HIGHER ORDER

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1. Introduction

Block-pulse function /b.p.f./ and Walsh functions are closely related. As basis functions in an approximation the two sets of functions lead to the same results. The results are piecewise constant with minimal mean square error. Walsh series approximation has been successfully used to analyze, optimize and identify linear systems by Chen and Hsiao [1] - [4], and Corrington [6]. With the introduction of Walsh product matrix, Walsh series approximation is extended to the analysis and optimization of time-varying linear systems [8].

This paper establishes a procedure for solving linear systems of high order differential equations with variable coefficients via the b.p.f. technique. First the b.p.f. and the operational matrix are introduced and their properties briefly summarized [5], [7]. Then the block-pulse product matrix is defined and the operational property of product matrix is proved.

2. Block-pulse functions and the operational matrix

A set of b.p.f. on a unit interval [0,1] is defined as follows: for each integer i, $0 \leq i < m$ and $m \in P = \{1, 2, \ldots\}$ the function φ_i is given by

(1)
$$\varphi_i(t) = \begin{cases} 1 & \text{for } \frac{i}{m} \le t < \frac{i+1}{m} \\ 0 & \text{otherwise} \end{cases}$$

This set of functions can be concisely described by an m-vector $\Phi_{(m)}$ with φ_i as its ith component. $\Phi_{(m)}$ is called the block-pulse vector. It is well known that a function f which is integrable in [0,1] can be approximated as

(2)
$$f \simeq \sum_{i=0}^{m-1} y_i \varphi_i$$

where the coefficients y_i are determined such that

$$\int_0^1 [f(t) - \sum_{i=0}^{m-1} y_i \varphi_i(t)]^2 \, dt$$

is minimized. In fact, y_i is given by

(3)
$$y_i = m \int_{\frac{i}{m}}^{\frac{i+1}{m}} f(t) dt, \qquad 0 \le i < m.$$

The coefficient vector y of (2) is

(4)
$$y = (y_0, y_1, \ldots, y_{m-1})^T.$$

The b.p.f. saatisfies the properties

(5)

$$\varphi_i\varphi_j = \delta_{ij}\varphi_i$$

and

(6)

$$\int_0^1 \varphi_i(t)\varphi_j(t)dt = \frac{1}{m}\delta_{ij}$$

where δ_{ij} denotes the Kronecker δ symbol.

The function $t^k, t \in [0,1], k \in N = \{0,1,2,\ldots\}$ can be approximated as a b.p.f. series of size m. Indeed, from (2) and (3), we have

$$t^k \simeq \sum_{i=0}^{m-1} t_k(i)\varphi_i(t), \qquad k \in N,$$

where

$$t_k(i) = m\int_{\frac{i}{m}}^{\frac{i+1}{m}} t^k dt = \frac{m}{k+1}[(\frac{i+1}{m})^{k+1} - (\frac{i}{m})^{k+1}].$$

Then,

(7)

$$t^k \simeq \frac{m}{k+1}\sum_{i=0}^{m-1}[(\frac{i+1}{m})^{k+1} - (\frac{i}{m})^{k+1}]\varphi_i(t)$$

and in matrix form

(8)
$$t^k \simeq \frac{m}{k+1} T_k^T \Phi_{(m)}(t)$$

where

(9)
$$T_k^T = ((\frac{1}{m})^{k+1}, (\frac{2}{m})^{k+1} - (\frac{1}{m})^{k+1}, \dots, 1 - (\frac{m-1}{m})^{k+1}).$$

The first integration of b.p.f. can be expressed by b.p.f. Indeed, from (1), we have

$$\int_0^t \varphi_i(\lambda) d\lambda = \begin{cases} 0, & 0 \le t < \frac{i}{m} \\ t - \frac{i}{m}, & \frac{i}{m} \le t < \frac{i+1}{m} \\ \frac{1}{m}, & \frac{i+1}{m} \le t < 1 \end{cases}$$

(10)
$$= (t - \frac{i}{m})\varphi_i(t) + \frac{1}{m} \sum_{j=i+1}^{m-1} \varphi_j(t).$$

From (7), we have

(11)
$$t \simeq \frac{m}{2} \sum_{i=0}^{m-1} [(\frac{i+1}{m})^2 - (\frac{i}{m})^2] \varphi_i(t).$$

Substituting (11) into (10) and using (5), we obtain

(12)

$$\int_0^t \varphi_i(\lambda)d\lambda \simeq \frac{1}{2m}\varphi_i(t) + \frac{1}{m}\sum_{j=i+1}^{m-1} \varphi_j(t), \quad 0 \leq i < m.$$

Therefore, we can write the relationship between b.p.f. and their integrals in the following matrix form

$$\begin{bmatrix} \int_0^t \varphi_0(\lambda)d\lambda \\ \int_0^t \varphi_1(\lambda)d\lambda \\ \vdots \\ \int_0^t \varphi_{m-1}(\lambda)d\lambda \end{bmatrix} \simeq \frac{1}{m} \begin{bmatrix} \frac{1}{2} & 1 & 1 & \dots & 1 \\ 0 & \frac{1}{2} & 1 & \dots & 1 \\ 0 & 0 & \frac{1}{2} & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_{m-1}(t) \end{bmatrix}$$

in or compact form

(13)

$$\int_0^t \Phi_{(m)}(\lambda)d\lambda \simeq B\Phi_{(m)}(t).$$

B is called the operational matrix of dimension m which relates b.p.f. and their integrals.

3. Block-pulse product and coefficient matrices

The product of a block-pulse vector $\Phi_{(m)}$ and its transpose $\Phi^T_{(m)}$ is called the block-pulse product matrix and is denoted by $\Phi_{(mxm)}$. That is

(14)

$$\Phi_{(mxm)} = \Phi_{(m)}\Phi^T_{(m)}.$$

Using the disjoint property of the b.p.f. in (5), we obtain
(15)

$$\Phi_{(mxm)} = \begin{bmatrix} \varphi_0 & 0 & 0 & \dots & 0 \\ 0 & \varphi_1 & 0 & \dots & 0 \\ 0 & 0 & \varphi_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \varphi_{m-1} \end{bmatrix}$$

$\Phi_{(mxm)}$ can be represented in the composite matrix notation

(16)

$$\Phi_{(mxm)} = [\Lambda_0^{(m)}\Phi_{(m)}, \Lambda_1^{(m)}\Phi_{(m)}, \dots, \Lambda_{m-1}^{(m)}\Phi_{(m)}]$$

where $\Lambda_k^{(m)}, k = 0, 1, \dots, m-1$ are mxm symmetric matrices and can be obtained from

(17)

$$\Lambda_k^{(m)} = (\delta_{ki}\delta_{ij})_{i,j=0}^{m-1}.$$

For example, when m=4

$$\Lambda_0^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \Lambda_1^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; etc.$$

The block-pulse coefficient matrix $Y_{(mxm)}$ corresponding to the coefficient y or (4) is defined by

(18)

$$Y_{(mxm)} = \begin{bmatrix} y_0 & 0 & 0 & \dots & 0 \\ 0 & y_1 & 0 & \dots & 0 \\ 0 & 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & y_{m-1} \end{bmatrix}.$$

Similarly $Y_{(mxm)}$ can be representedd as

(19)

$$Y_{(mxm)} = [\Lambda_0^{(m)} y, \Lambda_1^{(m)} y, \dots, \Lambda_{m-1}^{(m)} y].$$

Lemma.

(20)

$$\Phi_{(mxm)} y = Y_{(mxm)} \Phi_{(m)}.$$

Proof. From (16) and since the matrices $\Phi(mxm)$ and $\Lambda_k^{(m)}$ are symmetric, then

$$\Phi_{(mxm)} Y = [\Lambda_0^{(m)} \Phi_{(m)}, \Lambda_1^{(m)} \Phi_{(m)}, \dots, \Lambda_{m-1}^{(m)} \Phi_{(m)}]^T y$$

$$= \begin{bmatrix} \Phi_{(m)}^T \Lambda_0^{(m)} y \\ \Phi_{(m)}^T \Lambda_1^{(m)} y \\ \vdots \\ \Phi_{(m)}^T \Lambda_{m-1}^{(m)} y \end{bmatrix}.$$

Noting that $\Phi_{(m)}^T \Lambda_k^{(m)} y$ is a scalar,

$$\Phi_{(m)}^T \Lambda_k^{(m)} y = (\Phi_{(m)}^T \Lambda_k^{(m)} y)^T = y^T \Lambda_k^{(m)} \Phi_{(m)}$$

for $k = 0, 1, \ldots, m-1$. Hence

$$\Phi_{(mxm)}y = \begin{bmatrix} y^T \Lambda_0^{(m)} \Phi_{(m)} \\ y^T \Lambda_1^{(m)} \Phi_{(m)} \\ \vdots \\ y^T \Lambda_{m-1}^{(m)} \Phi_{(m)} \end{bmatrix} = \begin{bmatrix} y^T \Lambda_0^{(m)} \\ y^T \Lambda_1^{(m)} \\ \\ y^T \Lambda_{m-1}^{(m)} \end{bmatrix} \Phi_{(m)}$$
$$= Y^T_{(mxm)} \Phi_{(m)}.$$

Since $Y_{(mxm)}$ is symmetric, the proof is completed. With the operational property of (14) and (20), the solution and optimization of linear time-varying systems of high order can be easily carried out.

4. Solution of linear time-varying systems of higher order

Consider the following linear system of differential equations of order $r (r \geq 1)$ with variable coefficients

(21)

$$X^{(r)} + \sum_{k=1}^{r} A^{(r-k)} X^{(r-k)} = GU,$$
$$X^{(i)}(0) = X_0^{(i)} \qquad (i = 0, 1, \ldots, r-1)$$

where X is an n-vector

$$X = (x_1, x_2, \ldots, x_n)^T,$$

U a q-vector

$$U = (u_1, u_2, \ldots, u_q)^T,$$

and $A^{(k)}(k = 0,1,\ldots,r-1)$ and G are nxn and nxq matrices with variable coefficients, respectively. Suppose every element of $U, A^{(k)}(k = 0,1,\ldots,r-1)$ and G is integrable in the interval [0,1].

For solving this problem by the b.p.f. series approach, let $a_{ij}^{(k)}$ and g_{ij} be the elements of $A^{(k)}$ and G, respectively. For convenience, let $A_i^{(k)}$ be the i-th column of $A^{(k)}$ and G_i be the i-th column of G. Thus (21) becomes

(22)

$$X^{(r)} + \sum_{k=1}^{r}\sum_{i=1}^{n} A_i^{(r-k)} x_i^{(r-k)} = \sum_{i=1}^{q} G_i u_i .$$

We expand $X^{(r)}$ in b.p.f. series of size m,

(23)

$$X^{(r)} \simeq \sum_{j=0}^{m-1} c_j \varphi_j = C\Phi_{(m)}$$

where

$$c_j = (c_{1j}, c_{2j}, \ldots, c_{nj})^T$$

is the j th column of the nxm matrix C.

Now, integration of (23) from 0 to t, and using (13), we get

$$X^{(r-1)} = CB\Phi_{(m)} + X_0^{(r-1)} .$$

In fact, the k th integration of (23) yields

(24)

$$X_{(t)}^{(r-k)} = CB^k \Phi_{(m)}(t) + \sum_{i=1}^{k} X_0^{(r-i)} \frac{t^{k-1}}{(k-i)!}$$

$$(k = 1, 2, \ldots, r, \quad t \in [0, 1]).$$

From (8) and (9), we have

(25)

$$\frac{t^{k-i}}{(k-i)!} \simeq \frac{m}{(k-i+1)!} T_k^T \Phi_{(m)}(t).$$

Substituting (25) into (24), we get

(26)

$$X^{(r-k)} = [CB^k + Z^{(k)}]\Phi_{(m)} \quad (k = 1, 2, \ldots, r)$$

where

$$Z^{(k)} = m \sum_{i=1}^{k} \frac{i}{(k-i+1)!} X_0^{(r-i)} T_{k-i}^T$$

is nxm constant matrix. Let $V_i^{(k)T}$ be the i th row of the matrix $[CB^k + Z^{(k)}]$. Then (26) can be written as

(27)

$$x_i^{(r-k)} = V_i^{(k)T} \Phi_{(m)} \qquad (i = 1, 2, \ldots, n),$$

since every element of $U, A^{(k)}$ and G is integrable in the interval [0,1]. A b.p.f. series approximation of size m gives

(28)

$$u_i \simeq \sum_{j=0}^{m-1} u_{ij} \varphi_j = U_i^T \Phi_{(m)} \quad (i = 1, 2, \ldots, q)$$

(29)

$$a_{ij}^{(k)} \simeq \sum_{l=0}^{m-1} a_{ijl}^{(k)} \varphi_l = \alpha_{ij}^{(k)T} \Phi_{(m)} \quad (k = 1,2,\dots,r; i,j = 1,2,\dots,n)$$

and

(30)

$$g_{ij} \simeq \sum_{l=0}^{m-1} g_{ijl} \varphi_l = \beta_{ij}^T \Phi_{(m)} \quad (i = 1,2,\dots,n; j = 1,2,\dots,q)$$

where

(31)
$$U_i^T = (u_{i0}, u_{i1}, \dots, u_{i,m-1}),$$

(32)
$$\alpha_{ij}^{(k)T} = (a_{ij0}^{(k)}, a_{ij1}^{(k)}, \dots, a_{ij,m-1}^{(k)})$$

and

(33)
$$\beta_{ij}^T = (g_{ij0}, g_{ij1}, \dots, g_{ij,m-1}).$$

The components in (31) - (33) can be obtained by applying (3). Indeed,

$$u_{ij} = m \int_{\frac{i}{m}}^{\frac{i+1}{m}} u_i(t) dt,$$

$$a_{ijl}^{(k)} = m \int_{\frac{l}{m}}^{\frac{l+1}{m}} a_{ij}^{(k)}(t)dt$$

and

$$g_{ijl} = m \int_{\frac{l}{m}}^{\frac{l+1}{m}} g_{ij}(t)dt$$

Therefore,

(34)

$$A_i^{(r-k)} = \begin{bmatrix} \alpha_{1i}^{(r-k)T} \\ \alpha_{2i}^{(r-k)T} \\ \vdots \\ \alpha_{ni}^{(r-k)T} \end{bmatrix} \Phi_{(m)} = \alpha_i^{(r-k)} \Phi_{(m)}$$

and

(35)

$$G_i = \begin{bmatrix} \beta_{1i}^T \\ \beta_{2i}^T \\ \vdots \\ \beta_{ni}^T \end{bmatrix} \Phi_{(m)} = \beta_i \Phi_{(m)}.$$

Inserting (23), (27), (28), (34) and (35) into (22) gives

(36)

$$C\Phi_{(m)} + \sum_{k=1}^{r} \sum_{i=1}^{n} \alpha_i^{(r-k)} \Phi_{(m)} V_i^{(k)T} \Phi_{(m)} =$$

$$= \sum_{i=1}^{q} \beta_i \Phi_{(m)} U_i^T \Phi_{(m)}.$$

Noting that $V_i^{(k)T} \Phi_{(m)}$ and $U_i^T \Phi_{(m)}$ are scalars,

$$V_i^{(k)T} \Phi_{(m)} = \Phi_{(m)}^T V_i^{(k)}$$

$$U_i^T \Phi_{(m)} = \Phi_{(m)}^T U_i$$

Hence (36) becomes

(37)

$$C\Phi_{(m)} + \sum_{k=1}^{r} \sum_{i=1}^{n} \alpha_i^{(r-k)} \Phi_{(mxm)} V_i^{(k)} = \sum_{i=1}^{q} \beta_i \Phi_{(mxm)} U_i$$

where

$$\Phi_{(mxm)} = \Phi_{(m)} \Phi_{(m)}^T$$

is the block-pulse product matrix introduced in (14). From lemma 1, we have

(38)

$$\Phi_{(mxm)} V_i^{(k)} = V_{i(mxm)}^{(k)} \Phi_{(m)}$$

(39)

$$\Phi_{(mxm)} U_i = U_{i(mxm)} \Phi_{(m)}$$

where $V_{i(mxm)}^{(k)}$ and $U_{i(mxm)}$ are the coefficient matrices corresponding to the coefficient vectors $V_i^{(k)}$ and U_i, respectively.

Substituting (38) and (39) into (37) yields

(40)
$$C+\sum_{k=1}^{r}\sum_{i=1}^{n}\alpha_i^{(r-k)}V_{i(mxm)}^{(k)}=\sum_{i=1}^{q}\beta_i U_{i(mxm)}.$$

Equation (40) is a set of nxm linear algebraic equations which are used to compute the unknown coefficients c_{ij}. Once c_{ij} have been computed, the solution X in b.p.f. series approximation is obtained by substituting C into (26), namely

(41)
$$X=[CB^r+Z^{(r)}]\Phi_{(m)}$$

To increase accuracy of the solution, m should be chosen large.

Now, according to the properties of the operational matrix B, we derive a recursive algorithm to solve C from (40). First, we note that the matrix B is triangular matrix and has the form.

$$B=\frac{1}{2m}\begin{bmatrix} b_1^{(1)} & b_2^{(1)} & b_3^{(1)} & \dots & b_m^{(1)} \\ 0 & b_1^{(1)} & b_2^{(1)} & & b_{m-1}^{(1)} \\ 0 & 0 & b_1^{(1)} & \dots & b_{m-2}^{(1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & b_1^{(1)} \end{bmatrix}$$

where

$$b_1^{(1)}=1 \quad \text{and} \quad b_i^{(1)}=2 \qquad (i=2,3,\dots,m).$$

An elementary calculation show that the powers $B^k\,(k>1)$ has the form

$$B^k = \frac{1}{(2m)^k}\begin{bmatrix} b_1^{(k)} & b_2^{(k)} & b_3^{(k)} & \dots & b_m^{(k)} \\ 0 & b_1^{(k)} & b_2^{(k)} & \dots & b_{m-1}^{(k)} \\ 0 & 0 & b_1^{(k)} & \dots & b_{m-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & b_1^{(k)} \end{bmatrix}$$

where the elements of B^k are determined by the following recursive formulae

$$b_1^{(j)} = 1$$

$$b_i^{(j)} = \sum_{s=1}^{i} b_s^{(j-1)} b_{i-s+1}^{(1)} \quad (j = 2, 3, \dots, k; i = 2, 3, \dots, m).$$

Consequently,

$$CB^k + Z^{(k)} = [\frac{1}{(2m)^k}\sum_{p=0}^{j-1} b_{j-p}^{(k)} c_{ip} + z_{ij}^{(k)}]_{i,j=1}^{n,m}$$

where $y_{ij}^{(k)}$ are the elements of $Z^{(k)}$. Using the general form of the coefficient matrix in (18), we get

(42)

$$V_{i(mxm)}^{(k)} =$$

$$\begin{bmatrix} V_{i0}^k & 0 & \dots & 0 \\ 0 & V_{i1}^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & V_{im-1}^2 \end{bmatrix}$$

with

$$V_{ij}^{k} = \frac{1}{(2m)^{k}} \sum_{p=0}^{j} b_{j+1-p}^{(k)} C_{ip} + z_{ijn}^{(k)}$$

$$(j = 0, 1, \ldots, m-1)$$

and

(43)

$$U_{i(mxm)} = \begin{bmatrix} u_{i0} & 0 & \ldots & 0 \\ 0 & u_{i1} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & u_{i,m-1} \end{bmatrix}.$$

Substituting (34), (35), (42) and (43) into (40), we have

$$c_{ij} + \sum_{k=1}^{r} \frac{1}{(2m)^{k}} \sum_{l=1}^{n} a_{ilj}^{(r-k)} \sum_{p=0}^{j} b_{j-p+1}^{(k)} c_{lp} =$$

$$= -\sum_{k=1}^{r} \sum_{l=1}^{n} a_{ilj} z_{l,j+1} + \sum_{l=1}^{q} g_{ilj} u_{lj}$$

or in simple form

(44)

$$\sum_{p=0}^{j} \sum_{l=1}^{n} f_{il}^{jp} c_{lp} = d_{ij} \quad (j = 0, 1, \ldots, m-1; i = 1, 2, \ldots, n)$$

where

(45)

$$f_{il}^{jp} = \sum_{k=1}^{r} \frac{1}{(2m)^k} a_{ilj}^{(r-k)} b_{j-p+1}^{(k)} + \delta_{il}\delta_{jp}$$

and

(46)

$$d_{ij} = -\sum_{k=1}^{r}\sum_{l=1}^{n} a_{ilj}^{(r-k)} z_{l,j+1}^{(k)} + \sum_{l=1}^{q} q_{ilj} u_{lj}.$$

Equation (44) can be written in the matrix form

(47)

$$\sum_{p=0}^{j} F_{jp} c_p = D_j \quad (j = 0, 1, \ldots, m-1)$$

where

$$F_{jp} = (f_{il}^{jp})_{i,l=1}^{n}$$

are nxn matrices

Take $j = 0$ in (47), we get

$$F_{00} c_0 = D_0$$

Then the first column c_0 of the solution C is obtained

$$c_0 = F_{00}^{-1} D_0.$$

Take $j = 1$ in (47), we get

$$F_{10}c_0 + F_{11}c_1 = D_1.$$

Then by substituting c_0 we obtain c_1

$$c_1 = F_{11}^{-1}(D_1 - F_{10}c_0).$$

In general, all the vectors $c_j, j = 0, 1, \dots, m-1$ are found by the following recursive formulae

$$c_0 = F_{00}^{-1}D_0$$

$$c_j = F_{jj}^{-1}(D_j - \sum_{p=0}^{j-1} F_{jp}c_p) \quad (j = 1, 2, \dots, m-1).$$

In the above algorithm, we always sork with matrices F of nxn. Therefore there is no need to operate with larger matrices whatever m is chosen. The method described in this paper is easy to implement on computer. A program has been written in Basic language for this method.

Example

A simple example is chosen for illustration:

$$x'' - 2tx' + 6x = 0; \qquad x(0) = 0, x'(0) = 3,$$

The exact solution is

$$x(t) = 3t - 2t^3.$$

Comparison of the exact solution and the b.p.f. series solution, for $m = 8$, is shown in table I and drawn in Figure I

Range	Eight-term approximation	Exact value at mid-interval	Error
$[0, \frac{1}{8}]$	.184615385	.187011719	2.39633414E-03
$[\frac{1}{8}, \frac{2}{8}]$	.542307692	.549316406	7.0087139E-03
$[\frac{2}{8}, \frac{3}{8}]$	.865384615	.876464844	.0110802283
$[\frac{3}{8}, \frac{4}{8}]$	1.13076923	1.14501953	.0142503008
$[\frac{4}{8}, \frac{5}{8}]$	1.31538462	1.33154297	.0161583531
$[\frac{5}{8}, \frac{6}{8}]$	1.39615385	1.41259766	.0164438095
$[\frac{6}{8}, \frac{7}{8}]$	1.35	1.36474609	.0147460923
$[\frac{7}{8}, 1]$	1.15384615	1.16455078	.0107046324

Table I

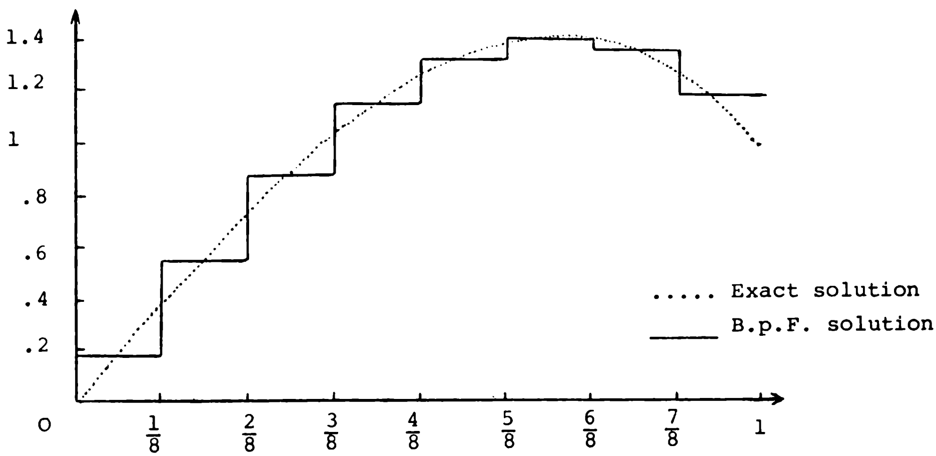

Figure I

The agreement between the b.p.f. series solution and the exact solution is very satisfactory considering that the series was truncated after the eight term. A better result would be obtained if large m is used.

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