

ON DOMAINS OF ATTRACTION OF EXTREME VALUE DISTRIBUTIONS VIA GENERALIZED CONCAVITY - CONVEXITY

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1. Introduction

Consider a sequence of independent, identically distributed random variables $X_1, X_2, X_3 \dots$ with common distribution function (d.f) $F(x)$. For $n \geq 1$, let

$$Z_n = \max(X_1, X_2, \dots, X_n).$$

A d.f. F belongs to the domain of attraction of a nondegenerate d.f. H , if there exist sequences of constants $\{a_n, b_n\}_{n=1}^{\infty}$ with $b_n > 0$, such that

$$(1.1) \quad \lim_{n \rightarrow \infty} P(Z_n < a_n + b_n x) = H(x)$$

holds at all continuity points of H . In the sequel the relation (1.1) will be denoted by $F \in D(H)$. We employ the notation

$H_{1,\gamma}(x) = \exp(-x^{-\gamma}), x > 0, H_{2,\gamma}(x) = \exp(-(-x)^{\gamma}), x < 0$ and $H_{3,0}(x) = \exp(-e^{-x}), x \in R$, where γ is a positive parameter.

A d.f. F can belong only to the domain of attraction of one of the three types $H_{1,\gamma}, H_{2,\gamma}$ and $H_{3,0}$ (see e.g. [1]). Gnedenko gave characterizations of domains of attraction of the three types in [2]. De Haan presented a unified approach involving only the d.f. F itself (see [3]).

The aim of the present paper is to characterize the domains of attraction of extreme value distributions in terms of the concavity-convexity index of the tail of an appropriate d.f. G , or equivalently: in terms of ultimately concavity (convexity) properties of the function $[1 - G(x)]^q$ ($q \in R, q \neq 0$).

In section 2 we give the concepts of r -concave (r -convex), ultimately r -concave (ultimately r -convex) functions and we investigate the main properties of these functions (from our point of view). Moreover, we define concavity, convexity and concavity-convexity indexes of a function g , denoted by $cv(g)$, $cx(g)$ and $c(g)$, respectively. In the case of twice differentiable g , we can give an alternative definition of $c(g)$.

This last result leads us (in Section 3) to the above-mentioned characterizations being supported by the paper [3].

2. Generalized concave and convex functions

The notion of convexity and concavity undoubtedly plays a dominant role in great many branches of mathematics. There exist some different generalizations of these concepts. The definition of r -concave (r -convex) functions is due to Martos (see [4]) and generalizes the usual concavity (convexity) by letting the weighted arithmetic mean of function values be replaced by a more general mean.

Let u, v be positive real numbers, $\lambda \in (0, 1)$ and let $r \neq 0$ be a real number. The λ -weighted r -meand of u, v is defined by

(2.1)

$$M_{\lambda}^r(u, v) = [\lambda u^r + (1 - \lambda)v^r]^{1/r}.$$

Applying continuity argumentation, one can easily see that

$$\begin{aligned} M_{\lambda}^0(u, v) &= u^{\lambda} v^{1-\lambda}, \\ M_{\lambda}^{+\infty}(u, v) &= \max(u, v), \\ M_{\lambda}^{-\infty}(u, v) &= \min(u, v). \end{aligned}$$

A positive real function $g(x)$ defined on some interval (a,b) is said to be an r -concave function on (a,b) , if for any pair $x_1, x_2 \in (a,b)$ and $\lambda \in (0,1)$ we have the following inequality

(2.2)

$$g(\lambda x_1 + (1 - \lambda)x_2) \geq M_\lambda^r(g(x_1), g(x_2)),$$

where $-\infty \leq a < b \leq +\infty$.

Note that r -convex functions are defined in a similar way, with the opposite sign inequality.

A simple characterization of r -concave (r -convex) functions can be obtained by the familiar notion of ordinary concavity and convexity:

a.) $g(x)$ is r -concave (r -convex) if and only if

- (i) $g^r(x)$ is concave (convex) when $r > 0$;
- (ii) $g^r(x)$ is convex (concave) when $r < 0$;
- (iii) $\log g(x)$ is concave (convex) when $r = 0$.

b.) If $g(x)$ is r -concave (r -convex), then it is also q -concave (q -convex) for every $q < r$ ($q > r$).

In case of differentiable functions alternative definitions of r -concave and r -convex functions can be given. Let $g(x)$ be a twice differentiable positive function on an open interval (a,b) . Denote by $g'(x), g''(x)$ the first and second derivatives of $g(x)$, respectively. Then $g(x)$ is r -concave on (a,b) if and only if the inequality

(2.3)

$$(r - 1)[g'(x)]^2 + g(x)g''(x) \leq 0$$

holds for every $x \in (a,b)$.

We note that inequality (2.3) is reversed when r -convex functions are characterized.

Assume that $g(x)$ is a real, non-negative function on some interval (a,b) , where $-\infty \leq a < b \leq +\infty$, and let

$$\omega(g) = \sup\{x : g(x) > 0\}.$$

$g(x)$ is said to be an ultimately r -concave (ultimately r -convex) function if for the given $r \in R$ there exists an x_r such that $x_r < \omega(g)$ and $g(x)$ is r -concave (r -convex) on the interval $(x_r, \omega(g))$.

Let us define the concavity and convexity indexes of $g(x)$ by
(2.4)

$$cv(g) = \sup \{r \in R : g(x) \text{ is ultimately } r - \text{concave}\}$$

and

(2.5)

$$cx(g) = \inf \{r \in R : g(x) \text{ is ultimately } r - \text{convex}\},$$

respectively.

These terms are well-defined because of the part b.) of the above-mentioned characterization.

If for a given $g(x)$ we have the equality $cv(g) = cx(g)$ then let the concavity-convexity index (or briefly : cc-index) of g be $c(g) = cv(g) = cx(g)$. The following result is fundamental from our point of view.

Proposition. *Assume that for a given positive function $g(x)$ there exists an $\underline{x} < \omega(g)$ such that g is twice differentiable and $g'(x) \neq 0$ on the interval $(\underline{x}, \omega(g))$. Then $cv(g) = cx(g) = q$ if and only if the following relation holds:*

(2.6)

$$\lim_{x \rightarrow \omega} \left[\frac{g(x)}{g'(x)} \right]' = q \cdot \square$$

Proof. Assume that $c(g) = q$. This means that for every $\epsilon > 0$ $g(x)$ is ultimately $(q-\epsilon)$ -concave and is ultimately $(q+\epsilon)$ -convex. It follows from the inequality (2.3) that we have

(2.7)

$$q - \epsilon \leq 1 - \frac{g(x)g''(x)}{[g(x)]^2} \leq q + \epsilon$$

on some interval $(x(\epsilon), \omega(g))$, whence we get the relation (2.6).

The converse is obvious. Thus the theorem is proved. $\square$

3. Domains of attraction results

Let $F(x)$ be a d.f.. Assume that there exists an $x_F < \omega = \omega(1 - F)$ such that

(3.1)

$$\int_{x_F}^{\omega} [1 - F(t)] dt < +\infty.$$

In this case let us define the d.f.'s $F_1(x)$ and $F_2(x)$ by

(3.2)

$$F_1(x) = \max\{0, \int_x^{\omega} [1 - F(t)] dt\}$$

(3.3)

$$F_2(x) = \max\{0, \int_x^{\omega} [1 - F_1(t)] dt\}.$$

We note that (3.1) holds trivially when $\omega < +\infty$. If the integral in (3.1) is not bounded above we can define a new d.f. $F^*(x)$ by

(3.4)

$$F^*(x) = 1 - \frac{1 - F(x)}{x^3}, x \geq 1.$$

De Haan proved in [3] that (3.1) is valid with $F^*(x)$ instead of $F(x)$. So in the case $\omega = +\infty$ let F_1^* and F_2^* be defined by (3.2) and (3.3), respectively, with F^* instead of F .

Now we are able to draw up our main results.

Theorem.

a.) $F \in D(H_{1,\gamma})$ if and only if $\omega = \infty$ and there exists an r_1 such that $-1 < r_1 < 0$ and the function $[1 - F_2^*(x)]^r$ is ultimately concave for $r_1 < r < 0$ and ultimately convex for $r < r_1$ or $r > 0$.

Moreover, $\gamma = -1/r_1 - 1$.

b.) $F \in D(H_{2,\gamma})$ if and only if $\omega < \infty$ and there exists an r_0 such that $0 < r_0 < 1/2$ and the function $[1 - F_2(x)]^r$ is ultimately concave for $0 < r < r_0$ and is ultimately convex for $r < 0$ or $r > r_0$.

Moreover, $\gamma = 1/r_0 - 2$.

c.) $F \in D(H_{3,0})$ if and only if the function $[1 - F_2(x)]^r$ is ultimately convex for every $r \in R, r \neq 0$.

Proof. De Haan proved that $F \in D(H_{1,\gamma})$ iff $\omega = \infty$ and

$$\lim_{x \rightarrow \infty} \frac{[1 - F^*(x)][1 - F_2^*(x)]}{[1 - F_1^*(x)]^2} = d \quad \text{with } 1 < d < 2$$

$$\text{and } \gamma = \frac{2-d}{d-1} \quad (\text{see Theorem 12 in [3]}).$$

It is easy to see by our Proposition that this is equivalent to $c(1 - F_2^*) = 1 - d$, i.e., $1 - F_2^*(x)$ is ultimately r -concave for $r < r_1 = 1 - d$ and ultimately s -convex for $s > r_1$. The proof is complete if we apply the characterizations by ordinary concavity (convexity).

b.) In the above-mentioned paper of de Haan [3] the Theorem 11 asserts that $F \in D(H_{2,\gamma})$ iff $\omega < \infty$ and

(3.5)

$$\lim_{x \uparrow \omega} \frac{[1 - F(x)][1 - F_2(x)]}{[1 - F_1(x)]^2} = d \quad \text{with } 1/2 < d < 1$$

and

$$\gamma = \frac{2d-1}{1-d},$$

and by the proposition, (3.5) is true iff $c(1-F_2) = 1-d, 0 < 1-d < 1/2$, i.e., $1-F_2(x)$ is ultimately r -concave for $r < r_0 = 1-d$ and ultimately s -convex for $s > 0$. One may now finish the proof as in part a.).

c.) In [3] it was proved also that $F \in D(H_{3,0})$ iff (3.5) holds with $d=1$ (see Theorem 10 in [3]). This is equivalent to $c(1-F_2) = 0$, i.e., $1-F_2(x)$ is ultimately r -concave for every $r < 0$ and ultimately s -convex for $s > 0$. Thus our theorem is proved.

References

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Received: June 24, 1987