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FREE BOUNDARY ELASTIC-PLASTIC PROBLEM

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A succesive approximations by an iteration processes will be introduced for the one-dimensional elastic-plastic problem. We hope that the ideas of these processes can be useful for the two-dimensional problem too.

1. The one-dimensional problem is summarized as follows: The solutions of the elastic and the plastic problems, u^e and u^p respectively, satisfy the following equations with the boundary conditions [1]:

(1)
$$Au^e = -\frac{d}{dr}\left(\frac{1}{r^3}\frac{du^e}{dr}\right) = 0, \quad r \in (0, \bar{r}),$$

(2)
$$\frac{1}{r^3}\left|\frac{du^p}{dr}\right| = k, \quad r \in (\bar{r}, R),$$

(3)
$$u^e(0) = 0, \quad u^p(R) = T,$$

(4)
$$u^e(\bar{r}) = u^p(\bar{r}), \quad \frac{du^e}{dr}(\bar{r}) = \frac{du^p}{dr}(\bar{r}),$$

where $\bar{r}$ is the unknown free boundary between the elastic and the plastic subregions.

Remark I. From (1-4), the known exact solutions of $u^e, u^p, \bar{r}$ are:

(5)
$$u^e(r) = \frac{k}{4\bar{r}} r^4,$$

(6)
$$u^p(r) = T - \frac{k}{3}(R^3 - r^3),$$

(7)
$$\bar{r} = \sqrt[3]{4R^3 - \frac{12}{k}T}$$

Remark II. From (7) $\bar{r}$ depends upon the value of the coefficient k, where k is supposed that $\bar{r} \in (0, R)$.

2. The first iteration process. It is shown, that u^p satisfies the Cauchy-problem of the first order differential equation (2) and it can be solved independently from u^e.

Let $\tilde{r}$ be the approximated free boundary. Now by using the condition $u^e(\tilde{r}) = u^p(\tilde{r})$, we can construct the solution of the elastic problem, which is the second order boundary value problem, and so

(8)
$$\tilde{u}^e(r) = \frac{r^4}{\tilde{r}^4}(T - \frac{k}{3}(R^3 - \tilde{r}^3)).$$

The next iteration for the free boundary can be obtained from the difference between the first derivatives:

(9)
$$\xi = \frac{du^p}{dr}(\tilde{r}) - \frac{du^e}{dr}(\tilde{r}).$$

Let $\delta = \tilde{r} - \bar{r}$, where δ is supposed to be small. Therefore, from (5-7), (8), we get

(10)
$$\xi = -k\tilde{r}\delta + k\delta^2 + 0(\delta^3),$$

and so
(11)
$$\delta \simeq \tilde{\delta} = -\frac{1}{k\tilde{r}}\left(\frac{du^p}{dr} - \frac{d\tilde{u}^e}{dr}\right)_{\tilde{r}}.$$

Then, we can construct the iteration procedure as follows
(12)
$$\tilde{r}_i = \tilde{r}_{i-1} - \tilde{\delta}_{i-1}.$$

where $\tilde{\delta}_i$ can be obtained from (11) in every step of the iteration process.

From (10) and (11), it can be shown that
(13)
$$\delta_i = \frac{\delta_{i-1}^2}{\tilde{r}_{i-1}} + 0(\delta_{i-1}^3),$$

or
(14)
$$|\tilde{r}_i - \bar{r}| = 0(|\tilde{r}_{i-1} - \bar{r}|^2).$$

Formula (14) indicates that the iteration process (12) is convergent.

3. The second iteration process. It is obvious that if $\tilde{r} \neq \bar{r}$, then the curves of the solutions, $\tilde{u}_e$ and u^p, have two points of intersection: $\tilde{r}$ and $\tilde{\tilde{r}}$: $\tilde{r} < \bar{r} < \tilde{\tilde{r}}$ or $\tilde{\tilde{r}} < \bar{r} < \tilde{r}$. It is shown that if $\tilde{r} - \bar{r} = \delta$, $\bar{r} - \tilde{\tilde{r}} = \epsilon$ and $u^p(\tilde{\tilde{r}}) = \tilde{u}^e(\tilde{\tilde{r}})$, then
(15)
$$\epsilon = \delta - \frac{8}{3\tilde{r}}\delta^2 + 0(\delta^3).$$

After that, as for the next step in the iteraion process, let
(16)
$$\tilde{r}_{i+1} = \frac{\tilde{r}_i + \tilde{\tilde{r}}_i}{2}.$$

From (15) it is obtained that

(17)
$$\delta_{i+1} = \frac{4}{3\tilde{r}}\delta_i^2 + 0(\delta_i^3),$$

therefore, the iteration process (16) is convergent.

We should like to notify that we intend to use the ideas introduced in this paper in constructing an iteration process for the two-dimensional elastic-plastic problem in our forthcoming work.

References

[1] **Gryer, C.W.** Numerical Methods for P.D.E. Academic Press (New York) London, 1979.