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# NONLINEAR ELLIPTIC SYSTEMS IN UNBOUNDED DOMAIN WITH QUADRATIC GROWTH CONDITIONS

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## 1. Introduction and assumption

The aim of this paper is to give some existence and perturbation results for following system of nonlinear elliptic equations with quadratic growth:

(1.1)
$$\mathcal{A}(u) + f^{\nu}(x,u,Du) = 0 \quad in \quad \Omega$$
$$u = 0 \quad on \quad \partial\Omega.$$

Here $\Omega$ may be unbounded open set of $R^n$ with boundary $\partial\Omega$ and

$$\mathcal{A}(u) := -\sum_{i=1}^{n} D_i[A_i^{\nu}(x,u,Du)] + a_0^{\nu}(x)u^{\nu}, \nu = 1,\dots,M$$

$$u = (u^1,\dots,u^M), Du = Du^1,\dots,Du^M), Du^{\nu} = (\frac{\partial u^1}{\partial x_1},\dots,\frac{\partial u^M}{\partial x_n})$$

For boundend domains such results have been proved in [2]. Also for a single equation in unbounded domains, a similar type of result has been proved in [6].

We assume that the coefficients $a_i^\nu : \Omega x R^M \cdot R^{Mn} \to R, i = 1, \ldots, n$, satisfy the Caratheodory conditions, i.e. $a_i^\nu$ are measureble in $x$ for all $\eta, \xi$ belonging to $R^M \cdot R^{Mn}$ and continuous in $\eta, \xi$ for almost all $x$ in $\Omega$. Further, let us assume that

$(A_1)\exists \alpha > 0, \alpha_1 > 0$ *and* $\alpha_0 > 0$ such that for a.e. $x$ in $\Omega$, $\eta \in R^M, \xi \in R^{Mn}$,

$$\sum_{i=1}^{n} a_i^\nu(x, \eta, \xi)\xi_i^\nu \geq \alpha \mid \xi^\nu \mid^2, \alpha_0 \leq a_0^\nu(x) \leq \alpha_1.$$

$(A_2)\exists C_1 > 0$, and $K_1 \in L^2(\Omega)$ such that for a.e. $x$ in $\Omega$, $\eta \in R^M, \xi \in R^{Mn}$,

$$\mid a_i^\nu(x, \eta, \xi) \mid \leq C_1[\mid \eta \mid + \mid \eta \mid] + K_1(x).$$

$(A_3)$For all $x$ in $\Omega, \eta \in R^M, \xi, \xi^{(1)} \in R^{Mn}$ with $\xi \neq \xi^{(1)}$ we have

$$\sum_{\nu=1}^{m} \sum_{i=1}^{n} [a_i^\nu(x, \eta, \xi) - a_i^\nu(x, \eta, \xi^{(1)},][\xi_i^\nu - \xi_i^{\nu(1)}] > 0.$$

$(A_4)a_i^\nu(x, \eta, \xi)$ may be written as

$$a_i^\nu(x, \eta, \xi) = \tilde{a}_a^\nu(x, \eta)\xi^\nu + q_i^\nu(x, \eta, \xi),$$

where $\tilde{a}_i^\nu$ and $q_i^\nu$ satisfy Caratheodory conditions, $\tilde{a}_i^\nu$ is bounded for $\mid \eta \mid$ and $\mid q_i^\nu(x, \eta, \xi) \mid \leq \tilde{q}_i^\nu(x)$, where $\tilde{q}_i^\nu \in L^2(\Omega)$.

$(A_5)$ Let us assume that

$$A(u) = P^\nu u^\nu + Q^\nu(x, u, Du),$$

where $P^\nu$ is an operator having constant-coefficients, and $Q^\nu(x,\eta,\xi)=0$ if $x$ is out of $K$, a compact subset of $\Omega$.

$(A_6)$ Further, we assume that if $h\in L^\infty(\Omega)$ then every weak solution of $P^\nu u^\nu+Q^\nu(u)=h$ belongs to $L^\infty_{loc}(\Omega)$.

The second part $f^\nu:\Omega\cdot R^M\cdot R^{Mn}\to R$ also satisfy Caratheodory conditionss. We shall assume that $f^\nu$ satisfies following assumption:

$(A_7)$ For a.e.x in $\Omega, \eta\in R^M$ and $\xi\in R^{Mn}$,
(i) $|f^\nu(x,\eta,\xi)|_R\le C_0(x)+b(|\eta|_{R}\ M)\,|\xi^\nu|^2_R\ n$,
(ii) $f^\nu(x,\eta,\xi)\eta^\nu\ge 0$, for all $(\eta,\xi)\in R^M x R^{Mn}$,
where $C_0\in L^\infty(\Omega)$ and $b$ is a positive nondecreasing function.

Remark For linear case results for $(A_6)$ can be found in [1] and [4]. The reader is also referred to papers [9,10].

## 2. The existence theorem

In the following steps we prove an existence theorem by giving $L^\infty$ - and $H^1$ - estimates with the help of assumptions $(A_1)-(A_7)$.

Theorem 2.1. If the assumptions mentioned above are astisfied, then there exists $u\in(H^1_0(\Omega))^M\cap(L^\infty(\Omega))^M$ such that, for arbitrary test functions $\phi^\nu\in C^\infty_0(\Omega)$

$$\sum_{\nu=1}^{M}\sum_{i=1}^{n}<a_i^\nu(x,u,Du),D_i\phi^\nu>+\sum_{\nu=1}^{M}<a_0^\nu u^\nu,\phi^\nu>= \tag{2.1}$$

$$=-\sum_{\nu=1}^{M}<f^\nu,\phi^\nu>, i.e.$$

$$\sum_{\nu=1}^{M}\sum_{i=1}^{n}\int_\Omega a_i^\nu(x,u,Du)D_i\phi^\nu\,dx+\sum_{\nu=1}^{M}\int_\Omega a_0^\nu u^\nu\phi^\nu\,dx=$$

$$= -\sum_{\nu=1}^{m} \int f^{\nu}(x,u,Du)\phi^{\nu}\,dx.$$

Proof. Let us define for the arbitrary samll $\mu > 0$,

$$f_{\mu}^{\nu}(x,\eta,\xi) := \frac{f^{\nu}(x,\eta,\xi)}{1+\mu \mid f^{\nu}(x,\eta,\xi) \mid} \cdot J(\mu x), \quad \text{where } J \in C_0^{\infty}(R^n)$$

is a fixed function which is equal to 1 in a neighbourhood of zero.

There $\mid f_{\mu}^{\nu}(x,\eta,\xi) \mid \leq \frac{1}{\mu} \cdot J(\mu x)$.

We consider the systems

(2.2)

$$-\sum_{i=1}^{n} D_i[a_i^{\nu}(x,u_{\mu},Du_{\mu})] + a_{\mu}^{\nu}(x)u_{\mu}^{\nu} + f_{\mu}^{\nu}(x,u_{\mu},Du_{\mu}) = 0$$

$u_{\mu} \in (H_0^1(\Omega))^M$ and $\nu = 1\ldots,M$. The problems (2.2) have solutions according to the result [3], which was proved for a single equation. The generalisation of this result has been used for the system of equations (2.2) considered here. The solutions $u_{\mu}$ belong to $L^{\infty}(\Omega))^M$ because of the assumptions $(A_5),(A_6)$, since for the solution of the equations

$$P^{\nu}u^{\nu} + Q^{\nu}(x,u,Du) = h,$$

h be longing to $L^{\infty}(\Omega), u(x) \to 0$ as $\mid x \mid \to \infty$.

Let us consider the function $z_{\mu}^{\nu} = u_{\mu}^{\nu} - \frac{S}{\alpha_0}$, where $S = \sup C_0(x)$ and $z_{\mu}^{\nu} \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$. After substitution in equation (2.2), we obtain

$$-\sum_{i=1}^{n} D_i[a_i^{\nu}(x,u_{\mu},Dz_{\mu}] + a_0^{\nu} z_{\mu}^{\nu} = -f_{\mu}^{\nu}(x,u_{\mu},Du_{\mu}) -$$

(2.3)

$$-a_0^\nu \frac{S}{\alpha_0}.$$

Let $b(\| u_\mu \|_{L^\infty(\Omega)}) = c_2^\mu$, then by $(A_7)$ on the right hand side (r,h.s)

$$-f^\nu(x, u_\mu, Du_\mu)(x) \leq C_0(x) + C_2^\mu \mid Du_\mu^\nu \mid^2 .$$

Therefore by $(A_1)$

$$-f^\nu(x, u_\mu, Du_\mu(x) - a_0^\nu \sup \frac{C_0(x)}{\alpha_0} \leq C_0(x) + C_2^\mu \mid Du_\mu^\nu \mid^2 -$$

$$-a_0^\nu \frac{S}{\alpha_0} \leq c_2^\mu \mid Du_\mu^\nu \mid^2 = c_2^\mu \mid Dz_\mu^\nu \mid^2,$$

where $\sup C_0(x) = S$.

We define test functions by

$$\phi_\mu^\nu := \exp\{\lambda_\mu[(z_\mu^\nu)^+]^2\}(z_\mu^\nu)^+,$$

where $\lambda_\mu = c_2^\mu / 2\alpha^2, e_\mu^\nu = \exp\{\lambda_\mu[(z_\mu^\nu)^+]^2\} \geq 1$.
Then $\phi_\mu^\nu \in H_0^1(\Omega)$, because at $\partial\Omega, (z_\mu^\nu)^+ = 0$, since $z_\mu^\nu = -\frac{S}{\alpha_0} < 0$ on $\partial\Omega$. Multiplying the equation (2.3) by $\phi_\mu^\nu$, and integrating over $\Omega$, we obtain on left hand side (1.h.s.)

$$\sum_{i=1}^{n} \int_\Omega a_i^\nu(x, u_\mu, Dz_\mu)\{e_\mu^\nu(D_i z_\mu^\nu)^+ + 2\lambda_\mu e_\mu^\nu \mid (z_\mu^\nu)^+ \mid^2 (D_i z_\mu^\nu)^+\}dx +$$

$$+ \int_\Omega a_0^\nu z_\mu^\nu e_\mu^\nu (z_\mu^\nu)^+ dx \leq \int_\Omega c_2^\mu \mid Dz_\mu^\nu \mid^2 e_\mu^\nu (z_\mu^\nu)^+ dx.$$

By the use of assumption $(A_1)$ on 1.h.s. and that of Cauchy's inequelity

$$\frac{\alpha}{2}\int_{\Omega} e^{\nu}_{\mu} \mid (Dz^{\nu}_{\mu})^{+} \mid^{2} + \lambda_{\mu}\alpha \int_{\Omega} e^{\nu}_{\mu} \mid (z^{\nu}_{\mu})^{+} \mid^{2} (Dz^{\nu}_{\mu})^{+} \mid^{2} +$$

$$+\alpha_0 \int_{\Omega} e^{\nu}_{\mu} \mid (z^{\nu}_{\mu})^{+} \mid^{2} \leq 0.$$

Since $e^{\nu}_{\mu} \geq 1$, thus $(z^{\nu})^{+} = 0$ and hence $z^{\nu}_{\mu} \leq 0$. This shows that $u^{\nu}_{\mu} - \frac{S}{\alpha_0} \leq 0$.

(2.4)

$$\text{or } u^{\nu}_{\mu} \leq \frac{S}{\alpha_0}$$

In a similar manner one can prove

$$u^{\nu}_{\mu} \geq -\frac{S}{\alpha_0}.$$

Next, we show that the estimation

$$\| u_{\mu} \|_{(H^1_0(\Omega))^M} \leq \text{ constant,}$$

holds. From above it may be written that

$$c_3 = b(\sqrt{M}\frac{S}{\alpha_0}) \geq b_1(\| (u^1, \ldots, u^M) \|).$$

Set $\Phi^{\nu}_{\mu} = E^{\nu}_{\mu} u^{\nu}_{\mu} = \exp\{\lambda(u^{\nu}_{\mu})^2\}$ and $\lambda = c_3^2/2\alpha^2$. Then $\phi^{\nu}_{\mu} \in H^1_0(\Omega)$, since $u_{\mu} \in (H^1_0(\Omega))^M \cap (L^{\infty}(\Omega))^M$. Multiplying equation (2.2) by $\phi^{\nu}_{\mu}$ and integrating, we obtain

$$\sum_{i=1_{\Omega}}^{n} \int_{\Omega} a^{\nu}_{i}(x, u_{\mu}, Du_{\mu}) E^{\nu}_{\mu} D_i u^{\nu}_{\mu} +$$

$$\sum_{i=1}^{n} 2\lambda \int_{\Omega} a_i^{\nu}(x, u_\mu, Du_\mu) E_\mu^\nu u_\mu^{\nu 2} D_i u_\mu^\nu +$$

(2.5)

$$+ \int_{\Omega} a_0^\nu u_\mu^\nu E_\mu^\nu U_\mu^\nu = - \int_{\Omega} f_\mu^\nu (x, u_\mu, Du_\mu) E_\mu^\nu U_\mu^\nu .$$

Using the assumption $(A_1)$ and the Cauchy's inequality
(2.6)

$$\frac{\alpha}{2} \int_{\Omega} E_\mu^\nu \mid Du_\mu^\nu \mid^2 + \frac{c_3^2}{2\alpha} \int_{\Omega} E_\mu^\nu (u_\mu^\nu)^2 \mid Du_\mu^\nu \mid^2 +$$

$$+ \alpha_0 \int_{\Omega} E_\mu^\nu (u_\mu^\nu)^2 \leq \text{ constant}$$

Since $E_\mu^\nu \geq 1$, therefore $u_\mu$ is bounded in $(H_0^1(\Omega))^M$ . This shows that (for a subsequence)

$$u_\mu \to u \text{ in } (H_0^1(\Omega))^M \text{weakly,}$$
$$u_\mu \to u \text{ a.e. in } \Omega, \| u \|_{(L^\infty(\Omega))^M} \leq \text{constant.}$$

Next, we show that $u_\mu$ converges strongly in $(H_0^1(\Omega))^M$ . For this, we follow the method of Leary-Lions [8]. Let $\bar{u}_\mu = u_\mu - u$ and substitute this in equation (2.2). Multiply by test function $\Phi_\mu^2 = (\bar{E}_\mu^\nu \bar{u}_\mu^\nu)$, where $\bar{E}_\mu^\nu = \exp\{\bar{\lambda}(u_\mu^\nu)^2\}\bar{\lambda} = 2c_2^3/\alpha^2$. Integration over $\Omega$ gives:

$$\sum_{i=1}^{n} \int_{\Omega} a_i^\nu (x, u_\mu, Du + D\bar{u}_\mu) \bar{E}_\mu^\nu D_i \bar{u}_\mu^\nu dx +$$

$$\sum_{i=1}^{n} 2\bar{\lambda} \int_{\Omega} a_i^\nu (x, u_\mu, Du + D\bar{u}_\mu).$$

$$E_\mu^\nu (u_\mu^\nu)^2 D_i \bar{u}_\mu^\nu dx + \int_\Omega a_0^\nu \bar{u}_\mu^\nu \bar{E}_\mu^\nu \bar{u}_\mu^\nu dx =$$

$$- \int_\Omega f_\mu^\nu (x, u_\mu, Du_\mu) \bar{E}_\mu^\nu \bar{u}_\mu^\nu dx - \int_\Omega a_0^\nu u_\mu^\nu \bar{E}_\mu^\nu \bar{u}_\mu^\nu dx.$$

or

$$\sum_{i=1}^n \int_\Omega a_i^\nu (x, u_\mu, Du + D\bar{u}_\mu) \bar{E}_\mu^\nu D_i \bar{u}_\mu^\nu dx +$$

$$+ \sum_{i=1}^n 2\bar{\lambda} \int_\Omega a_i^\nu (x, u_\mu, Du_\mu) \bar{E}_\mu^\nu (\bar{u}_\mu^\nu)^2 D_i \bar{u}_\mu^\nu dx$$

$$+ \int_\Omega a_0^\nu \bar{u}_\mu^\nu \bar{E}_\mu^\nu \bar{u}_\mu^\nu dx = - \int_\Omega f_\mu^\nu (x, u_\mu, Du_\mu) \bar{E}_\mu^\nu \bar{u}_\mu^\nu dx -$$

$$\int_\Omega a_0^\nu u^\nu \bar{E}_\mu^\nu \bar{u}_\mu^\nu dx - \sum_{i=1}^n \int \tilde{a}_i^\nu (x, u_\mu) Du \bar{E}_\mu^\nu D_i \bar{u}_\mu^\nu dx -$$

$$- \sum_{i=1}^n 2\bar{\lambda} \int_\Omega \tilde{a}_i^\nu (x, u_\mu) Du \bar{E}_\mu^\nu (\bar{u}_\mu^\nu)^2 D_i \bar{u}_\mu^\nu dx - \sum_{i=1}^n <$$

$$< D_i [q_i^\nu (x, u_\mu, Du + D\bar{u}_\mu) - q_i^\nu (x, u, D\bar{u}_\mu)], \bar{E}_\mu^\nu \bar{u}_\mu^\nu >$$

(2.7)

$$((H^{-1}(\Omega))^M, (H_0^1(\Omega)^M)$$

where assumption $(A_4)$ has been used.

By the assumption $(A_1)$ we can estimate 1.h.s. and thus we obtain

$$\alpha \int_\Omega \bar{E}^\nu_\mu \mid D\bar{u}^\nu_\mu \mid^2 dx + 2\bar{\lambda}\alpha \int_\Omega \bar{E}]^\nu_\mu (\bar{u}^\nu_\mu)^2 \mid D\bar{u}^\nu_\mu \mid^2 + \alpha_0 \int_\Omega \bar{E}^\nu_\mu (\bar{u}^\nu_\mu)^2 \leq$$

$$\leq - \int_\Omega f^\nu_\mu (x, u_\mu, Du_\mu) \bar{E}^\nu_\mu \bar{u}^\nu_\mu dx - \int_\Omega a^\nu_0 u^\nu \bar{E}^\nu_\mu \bar{u}^\nu_\mu dx -$$

$$- \int_{i=1}^{n} \int_\Omega \tilde{a}^\nu_i (x, u_\mu) Du \bar{E}^\nu_\mu D_i \bar{u}^\nu_\mu dx -$$

$$\sum_{i=1}^{n} 2\bar{\lambda} \int_\Omega \tilde{a}^\nu_i (x, u_\mu) Du \bar{E}^\nu_\mu (\bar{u}^\nu_\mu)^2 D_i \bar{u}^\nu_\mu dx -$$

$$- \sum_{i=1}^{n} < D_i [q^\nu_i (x, u_\mu, Du + D\bar{u}_\mu) - q^\nu_i (x, D_\mu, D\bar{u}_\mu)], \bar{E}^\nu_\mu \bar{u}^\nu_\mu > .$$

$$(((H^{-1}(\Omega))^M, (H^1_0(\Omega))^M)$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, Young's inequality and assumption $(A_7)$ we have

$$\alpha \int_\Omega \bar{E}^\nu_\mu \mid D\bar{u}^\nu_\mu \mid^2 dx + \bar{\lambda}\alpha \int_\Omega E^\nu_\mu (\bar{u}^\nu_\mu)^2 \mid D\bar{u}^\nu_\mu \mid^2 dx +$$

$$+ \alpha_0 \int_\Omega \bar{E}^\nu_\mu (\bar{u}^\nu_\mu)^2 \leq \int \bar{E}^\nu_\mu C_0(x) \mid \bar{u}^\nu_\mu \mid dx$$

$$+ \int_\Omega \bar{E}^\nu_\mu (2c_3) \mid Du \mid^2 \mid \bar{u}^\nu_\mu dx - \int_\Omega a^\nu_0 u^\nu \bar{E}^\nu_\mu \bar{u}^\nu_\mu dx -$$

$$-\sum_{i=1}^{n}\int_{\Omega}\tilde{a}_i^{\nu}(x,u)Du\bar{E}_{\mu}^{\nu}D_i\bar{u}_{\mu}^{\nu}dx-$$

$$-\sum_{i=1}^{n}2\bar{\lambda}\int_{\Omega}\tilde{a}_i^{\nu}(x,u_{\mu})Du\bar{E}_{\mu}^{\nu}(\bar{u}_{\mu}^{\nu})^2D_i\bar{u}_{\mu}^{\nu}dx+$$

$$+\sum_{i=1}^{n}\int_{\Omega}[q_i^{\nu}(x,u_{\mu},Du+D\bar{u}_{\mu})-q_i^{\nu}(x,u_{\mu},D\bar{u}_{\mu})]\bar{E}_{\mu}^{\nu}D_i\bar{u}_{\mu}^{\nu}dx+$$

$$\sum_{i=1}^{n}2\bar{\lambda}\int_{\Omega}[q_i^{\nu}(x,u_{\mu},Du+D\bar{u}_{\mu})-q_i^{\nu}(x,u_{\mu},D\bar{u}_{\mu})]\bar{E}_{\mu}^{\nu}(\bar{u}_{\mu}^{\nu})^2D_i\bar{u}_{\mu}^{\nu})dx.$$

Since $u_{\mu}\rightarrow u$ a.e. in $\Omega$ and $\| u_{\mu} \|^{M}_{L^{\infty}(\Omega))}\leq$ const, then by the use of Lebesques dominated convergence theorem, we have

$$\int_{\Omega}\bar{E}_{\mu}^{\nu}C_0(x)\bar{u}_{\mu}^{\nu}dx+\int_{\Omega}\bar{E}_{m\,u}^{\nu}(2C_{(3)}\mid Du\mid^2\bar{u}_{\mu}^{\nu}dx\rightarrow 0,$$

as we know that $C_0(x)\in L^1(\Omega)$ and $\mid Du\mid^2\in L^1(\Omega)$, also $\bar{E}_{\mu}^{\nu}$ is bounded in $L^{\infty}(\Omega)$.

The term $\int_{\Omega}a_0^{\nu}u^{\nu}\bar{E}_{\mu}^{\nu}\bar{u}_{\mu}^{\nu}\rightarrow 0$, because $a_{\nu}^{0}u^{\nu}$ belongs to $L^2(\Omega)$, $u_{\mu}\rightarrow u$ in $(H_0^1(\Omega))^M$ weakly which implies that $\bar{u}_{\mu}\rightarrow 0$ in $(L^2(\Omega))^M$ weakly. Thus we may prove in a similar manner that other terms tend to zero. Therefore left hand side of the last inequality also tends to zero. We know that $E_{\mu}^{\nu}\geq 1$. This in turn gives that $\bar{u}_{\mu}$ tends 0 in $(H_0^1(\Omega))^M$ strongly.

$$u_{\mu}\rightarrow u \text{ in } (H_0^1(\Omega))^M \quad \text{strongly implies that}$$
$$Du_{\mu}\rightarrow Du \quad \text{a.e.in } \Omega$$

for a subsequence.

By $(A_7)$

$$| f^{\nu}_{\mu}(x, u_{\mu}, Du_{\mu}) | \leq C_0(x) + c_3 \, | Du^{\nu}_{\mu}(x) |^2,$$

where $| Du^{\nu}_{\mu} |^2$ converges in $L^1(\Omega)$ - norm. Thus we can apply Vitali's convergence theorem to the sequence

$$f^{\nu}_{\mu}(x, u_{\mu}, Du_{\mu}) \to f^{\nu}(x, u_{\mu}, Du_{\mu}) \qquad \text{a.e. in} \Omega,$$

and so we have that this sequence is converging also strongly in $L^1(\Omega)$ - norm.

Further, since $a^{\nu}_i$ satisfy Caratheodory conditions thus,

$$a^{\nu}_i(x, u_{\mu}, Du_{\mu}) \to a^{\nu}_i(x, u, Du) \qquad \text{a.e. in } \Omega.$$

Consequently, $(A_2)$ and Vitali's convergence theorem imply that

$$a^{\nu}_i(x, u_{\mu}, Du_{\mu}) \to a^{\nu}_i(x, u, Du) \qquad \text{in } L^2(\Omega),$$

as $\nu \to 0$. Therefore applying (2.2) to fixed test functions $\phi^{\nu}_{\mu} \in C^{\infty}_0(\Omega)$, as $\nu \to 0$ we obtain theorem 2.1.

## 3. A theorem on perturbations

3.1. Formulation and assumptions. Consider for $u^k = (u^{1k}, \ldots, u^{Mk}) \in (H^1_0(\Omega))^M \cap (L^{\infty}(\Omega))^M$ the following system of equations $(k = 1, 2 \ldots)$ :

(3.1)

$$-\sum_{i=1}^{n} D_i[a^{\nu k}_i(x, u^k, Du^k)] + a^{\nu k}_0(x) u^{\nu k} + f^{\nu k}(x, u^k, Du^k) = 0,$$

where $\nu = 1, \ldots, M$. We assume that the functions $a^{\nu k}_i$ and $f^{\nu k}$ satisfy Caratheodory conditions, also $a^{\nu k}_0(x)$ are measurable.

Furthermore, we suppose that:

$$(A_1)' \begin{cases} \exists \alpha > 0, \alpha_1 > 0 \text{ and } \alpha_0 > 0 \text{ such that} \\ \sum\limits_{i=1}^{n} a_i^{\nu k}(x,\mu,\xi)\xi_i^{\nu k} \geq \alpha \mid \xi^\nu \mid^2, \\ \qquad \alpha_0 \leq a_0^{\nu k}(x) \leq \alpha_1; \end{cases}$$

$$(A_2)' \begin{cases} \exists c_1 > 0, \text{ a constant and a function } K_1 \in L^2(\Omega) \\ \text{such that} \\ \mid a_i^{\nu k}(x,\mu,\xi) \mid \leq c_1[\mid \mu \mid + \mid \xi \mid] + K_1(x); \end{cases}$$

$$(A_3)' \quad \sum_{\nu=1}^{m}\sum_{i=1}^{n}[a_i^{\nu k}(x,\mu,\xi) - a_i^{\nu k}(x,\mu,\xi^{(1)})][\xi_i^\nu - \xi_i^{\nu(1)}] > 0$$

for $x \in \Omega$ and $\xi \neq \xi^{(1)}$;

$$(A_4)' \quad a_i^{\nu k}(x,\mu,\xi) = \tilde{a}_i^{\nu k}(x,\mu)\xi + q_i^{\nu k}(x,y,\xi),$$

where $\mid q_i^{\nu k}(x,\mu,\xi) \mid \leq \tilde{q}_i^{\nu k}(x), \tilde{a}_i^{\nu k}(x), q_i^{\nu k}$ satisfy Caratheodory conditions, $\tilde{a}_i^{\nu k}$ is uniformly bounded if $\mid f \mid$ is bounded and also we have $\tilde{q}_i^{\nu k} \in L^2(\Omega)$.

$(A_5)'$ The first terms of equation (3.1) may be written as

$$-\sum_{i=1}^{n} D_i[a_i^{\nu k}(x,u,Du)] + a_0^{\nu k}(x)u^\nu = p^{\nu k}u^\nu + Q^{\nu k}(u),$$

where $P^{\nu k}$ is an operator with constant coefficients and

$$Q^{\nu k}(u) = -\sum_{i=1}^{n} D_i b_i^{\nu k}(x,u,Du) + b_0^{\nu k}(x)u^\nu,$$

$b_i^{\nu k}(x,\mu,\xi), b_0^{\nu k}(x)$ are zero if x is out of a compact subset of $\Omega$.

$(A_6)'$ Further we suppose that all the solutions of the equation

$$P^{\nu k}u^\nu + Q^{\nu k}(u) = h, \qquad h \in L^\infty(\Omega)$$

belong to $L^\infty_{loc}(\Omega)$.

For the second part $f^{\nu k}(x,u^k,Du^k)$, it has been assumed that is satisfies the sign condition

$(A_7)'$

$$f^{\nu k}(x,\mu,\xi)\mu^{\nu k} \geq 0;$$

$(A_8)'$

$$| f^{\nu k}(x,\mu,\xi) | \leq C_0(x) + b(| \mu |) | \xi^\nu |^2$$

where $C_0 \in L^\infty(\Omega) \cap L^1(\Omega)$ and b is a nonnegative continous function.

$(A_9)'$

If $\xi^k \to \xi^0, \mu^k \to \mu^0$ then for a.e.$x \in \Omega$

$a_i^{\nu k}(x,\mu^k,\xi^k) \to a_i^{\nu 0}(x,\mu^0,\xi^0)$,

$a_i^{\nu k}(x) \to a_i^{\nu 0}(x)$

*and*

$f^{\nu k}(x,\mu^k,\xi^k) \to f^{\nu 0}(x,\mu^0,\xi^0)$, as $k \to \infty$.

We shall now prove following result: if $u^k$ is a sequence of solutions of (3.1) belonging to $(H_0^1(\Omega))^M \cap (L^\infty(\Omega))^M$, $(k = 1,2,\ldots)$. then there exists a subsequence $\tilde{u}^k$ such that $\tilde{u}^k$ converges strongly to $u^0$ in $(H_0^1(\Omega))^M$, where $u^0$ is the solution of (3.1) with $k = 0$.

Multiplying equation (3.1) by $\Phi^{\nu k}$, integrating over $\Omega$, we obtain

(3.4)

$$\sum_{i=1}^{n} \int_{\Omega} a_i^{\nu k}(x, u^k, Du^k) E^{\nu k} D_i u^{\nu k} dx +$$

$$+ \sum_{i=1}^{n} 2\lambda \int_{\Omega} a_i^{\nu k}(x, u^k, Du^k) E^{\nu k} (u^{\nu k})^2 D_i u^{\nu k} dx +$$

$$+ \int_{\Omega} a_0^{\nu k}(x) E^{\nu k} (u^{\nu k})^2 dx = - \int_{\Omega} f^{\nu k}(x, u^k, Du^k) E^{\nu k} u^{\nu k} dx.$$

By the assumption $(A_1)'$, l.h.s. is greater than or equal to noindent(3.5)

$$\alpha \int_{\Omega} E^{\nu k} \mid Du^{\nu k} \mid^2 dx + 2\lambda\alpha \int_{\Omega} E^{\nu k} \mid u^{\nu k} \mid^2 \mid Du^{\nu k} \mid^2 dx +$$

$$\alpha_0 \int_{\Omega} E^{\nu k} \mid u^{\nu k} \mid^2 dx$$

and for r.h.s. we have

$$-f^{\nu k}(x, u^k, Du^k) E^{\nu k} u^{\nu k} dx \leq \int_{\Omega} \mid f^{\nu k}(x, u^k, Du^k) \mid\mid E^{\nu k} u^{\nu k} \mid dx$$

$$\leq \frac{S}{\alpha_0} \sup E^{\nu k} \int_{\Omega} C_0(x) dx + \frac{\alpha}{2} \int_{\Omega} E^{\nu k} \mid Du^{\nu k} \mid^2 +$$

$$\frac{1}{2\alpha} \int_{\Omega} (C_2)^2 E^{\nu k} (u^{\nu k})^2 \mid Du^{\nu k} \mid^2 dx$$

Thus,

(3.6)

$$\frac{\alpha}{2}\int\limits_{\Omega} E^{\nu k} \mid Du^{\nu k} \mid^2 dx + \lambda\alpha \int\limits_{\Omega} E^{\nu k} \mid u^{\nu k} \mid^2 \mid Du^{\nu k} \mid^2 dx +$$

$$+\alpha_0 \int\limits_{\Omega} E^{\nu k} \mid u^{\nu k} \mid^2 dx \leq \text{constant.}$$

or

(3.7)

$$\parallel u^k \parallel_{(H_0^1(\Omega))^M} \leq \text{constant,}$$

$$\text{since} \quad E^{\nu k} \geq 1.$$

The boundedness of $u^k$ in $(H_0^1(\Omega))^M$ implies that there exists a subsequence of $u^k$ (dnoted again by $u^k$) such that $u^k \to u^0$ weakly in $(H_0^1(\Omega))^M$ and $u^k \to u^0$ a.e. in $\Omega$.

Theorem 3.4. $u^k \to u^0$ strongly in $(H_0^1(\Omega))^M$ as $k \to \infty$.

Proof. Put $\bar{u}^k = u^k - u^0$ in equation (3.1) and multiply by $\bar{\phi}^{\nu k} = \bar{E}^{\nu k}\bar{u}^{\nu k} = \exp\{\bar{\lambda}(\bar{u}^{\nu k})^2\}\bar{u}^{\nu k}$, where $\bar{\lambda} = 2C_2^2/\alpha^2$. Then we have

(3.8)

$$\sum_{i=1}^{n} \int\limits_{\Omega} a_i^{\nu k}(x, u^k, D\bar{u}^k)\bar{E}^{\nu k} D_i \bar{u}^{\nu k} dx +$$

$$+\sum_{i=1}^{n} 2\bar{\lambda} \int\limits_{\Omega} a_i^{\nu k}(x, u^k, D\bar{u}^k)\bar{E}^{\nu k}(\bar{u}^{\nu k})^2 D_i \bar{u}^{\nu k} dx +$$

$$+\int\limits_{\Omega} a_0^{\nu k}(x)\bar{u}^{\nu k}\bar{E}^{\nu k}\bar{u}^{\nu k} dx = -\int\limits_{\Omega} f^{\nu k}(x, u^k, Du^k)\bar{E}^{\nu k}\bar{u}^{\nu k} dx -$$

$$-\int_\Omega a_0^{\nu k}(x)u^{\nu 0}\bar{E}^{\nu k}\bar{u}^{\nu k}dx-\sum_{i=1}^{n}\int_\Omega \tilde{a}_i^{\nu k}(x,u^k,Du^{\nu 0}\bar{E}^{\nu k}D_i\bar{u}^{\nu k}dx-$$

$$-\sum_{i=1}^{n}2\bar{\lambda}\int_\Omega \tilde{a}_i^{\nu k}(x,u^k)Du^{\nu 0}\bar{E}^{\nu k}(u^{\nu k})^2D_iu^{\nu k}dx-$$

$$-\sum_{i=1}^{n}< D_i[q_i^{\nu k}(x,u^k,D\bar{u}^k+Du^0)-$$

$$-q_i^{\nu k}(x,u^k,D\bar{u}^k)],\bar{E}^{\nu k}\bar{u}^{\nu k}>((H^{-1}(\Omega))^M,(H_0^1(\Omega))^M)$$

by making use of assumption $(A_4)'$. Also by the use of assumption $(A_1)'$, l.h.s. is greater than or equal to
(3.9)

$$\alpha\int_\Omega E^{\nu k}\mid D\bar{u}^{\nu k}\mid^2dx+2\bar{\lambda}\alpha\int_\Omega \bar{E}^{\nu k}(\bar{u}^{\nu k})^2\mid D\bar{u}^{\nu k}\mid^2dx+$$

$$+\alpha_0\int_\Omega \bar{E}^{\nu k}(\bar{u}^{\nu k})^2dx.$$

On r.h.s. the term
(3.10)

$$-\int_\Omega f^{\nu k}(x,u^k,Du^k)\bar{E}^{\nu k}\bar{u}^{\nu k}dx\le\int_\Omega C_0(x)\bar{E}^{\nu k}\mid\bar{u}^{\nu k}\mid dx+$$

$$+\int_\Omega(2C_2)\mid Du^{\nu 0}\mid^2\bar{E}^{\nu k}\mid\bar{u}^{\nu k}\mid dx+\frac{\alpha}{2}\int_\Omega \bar{E}^{\nu k}\mid D\bar{u}^{\nu k}\mid^2dx+$$

$$+\frac{1}{2\alpha}\int_\Omega (2C_2)^2 \bar{E}^{\nu k}(\bar{u}s^{\nu k})^2 \mid D\bar{u}^{\nu k}\mid^2 dx,$$

by the use of Young's inequality and the inequality

$$(a+b)^2 \leq 2a^2+2b^2.$$

From equations (3.9), (3.10) we obtain

$$\frac{\alpha}{2}\int_\Omega \bar{E}^{\nu k}\mid D\bar{u}^{\nu k}\mid^2 dx+\bar{\lambda}\alpha\int_\Omega \bar{E}^{\nu k}(\bar{u}^{\nu k})^2\mid D\bar{u}^{\nu k}\mid^2 dx+$$

$$+\alpha_0\int_\Omega \bar{E}^{\nu k}(\bar{u}^{\nu k})^2 dx \leq \int_\Omega C_0(x)\bar{E}^{\nu k}\mid \bar{u}^{\nu k}\mid dx+$$

$$+\int_\Omega (2C_2)\mid Du^{\nu 0}\mid^2 \bar{E}^{\nu k}\mid \bar{u}^{\nu k}\mid dx-\int_\Omega a_0^{\nu k}(x)u^{\nu 0}E^{\nu k}u^{\nu k}dx-$$

$$\sum_{i=1}^n\int_\Omega a_i^{\nu k}(x,u^k)Du^{\nu 0}E^{\nu k}D_i u^{\nu k}dx-$$

$$-\sum_{i=1}^n 2\lambda\int_\Omega a_i^{\nu k}(x,u^k)Du^{\nu 0}\bar{E}^{\nu k}(\bar{u}^{\nu k})^2 D_i u^{\nu k}dx+$$

$$+\sum_{i=1}^n\int_\Omega [q_i^{\nu k}(x,u^k,Du^0+Du^k)-q_i^{\nu k}(x,u^k,Du^k,)]E^{\nu k}D_i u^{\nu k}dx+$$

$$+\sum_{i=1}^n 2\lambda\int_\Omega [q_i^{\nu k}(x,u^k,Du^0+Du^k)-$$

$$-q_i^{\nu k}(x,u^k,Du^k)]E^{\nu k}(u^{\nu k})^2 D_i u^{\nu k}dx.$$

Lebesgue,s dominated convergence theorem gives,

$$\int_\Omega C_0(x)E^{\nu k} \mid u^{\nu k} \mid dx + \int_\Omega (2C_2) \mid Du^{\nu 0} \mid^2 E^{\nu k} \mid u^{\nu k} \mid \to 0,$$

since $C_0(x) \in L^1(\Omega), \mid Du^{\nu 0} \mid^2 \in L^1(\Omega)$ and $E^{\nu k}, u^{\nu k}$ are bounded in $L^\infty(\Omega)$. Also $u^{\nu k} \to u^{\nu 0}$ in $H_0^1(\Omega)$ imlies that $D_i u^{\nu k} \to 0$ in $(L^2(\Omega))^M$ weakly. By the assumption $(A_4)'$, $a_i^{\nu k}(x, u^k)$ is bounded in $L^\infty(\Omega)$. Thus the third and fourth terms in the right side will tend to zero. In a similar manner, it can be shown that all other terms in the right side, and consequently all terms in the left side will tend to zero, as $k \to \infty$.
Since $E^{\nu k} \geq 1$, thus

$$u^{\nu k} \to u^{\nu 0} \text{ strongly in } H_0^1(\Omega).$$

Our main objective is to justify the passage to the limit $k \to \infty$ in (3.1) which yields perturbation result. The strong convergence of $u^{\nu k}$ in $H_0^1(\Omega)$ implies that

$$Du^{\nu k} \to Du^{\nu 0} \qquad \text{a.e. in } \Omega,$$

for a subsequence.

Therefore, by $(A_9)'$

$$f^{\nu k}(x, u^k, Du^k) \to f^{\nu 0}(x, u^0, Du^0) \text{ a.e. in } \Omega.$$

Also, by the assumption $(A_7)$

$$\mid f^{\nu k}(x, u^k, Du^k) \mid^2 \geq C_0(x) + b(\mid u^k(x) \mid) \mid (Du^{\nu k}(x) \mid^2,$$

where $C_0(x) \in L^1(\Omega), b(\mid u^k(x) \mid) \leq C_2$ and $\mid Du^{\nu k}(x) \mid^2$ is convergent in $L^1$-norm. Therefore using the Vitali's convergence theorem

$$f^{\nu k}(x,u^k,Du^k) \to f^{\nu 0}(x,u^0,Du^0) \text{ in } L^1(\Omega)$$

strongly. Further, assumption $(Ag)'$ implies that

$$a_i^{\nu k}(x,u^k,Du^k) \to a_i^{\nu 0}(x,u^0,Du^0)$$

for a.e. $x \in \Omega$, and

$$a_0^{\nu k}(x) \to a_0^{\nu 0}(x)$$

for a.e. $x \in \Omega$.
Vitali's convergence theorem and the assumption $(A_2)'$ imply that

$$a_i^{\nu k}(x,u^k,Du^k) \to a_i^{\nu 0}(x,u^0,Du^0)$$

in the norm of $L^2(\Omega)$. Further, since $\alpha_0 \le a_0^{\nu k} \le \alpha_1$, thus

$$a_0^{\nu k}(x)u^{\nu k} \to a_0^{\nu 0}(x)u^{\nu 0} \text{ in } L^2(\Omega).$$

Therefore as $k \to \infty, u^{\nu 0}$ satisfies equation (3.1) for any test function $\phi^\nu \in C_0^\infty(\Omega)$. We have thus proved following theorem.

Theorem 3.5. If the assumptions $(A_1)' - (A_9)'$ are valid and $u^k$ are solutions of system (3.1) belonging to $(H_0^1(\Omega))^M \cap ((L^\infty(\Omega))^M$, then there is a subsequence of $u^k$ (denoted again by $u^k$) such that $u^k \to u^0$ strongly in $(H_0^1(\Omega))^M$ and $u^0 \in (H_0^1(\Omega))^M \cap (L^\infty(\Omega))^M$ is a solution of equation (3.1) for $k = 0$.

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## References

[1] **AGMON, S., DOUGLIS, A. and NIRENBERG, L.**: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary value conditions I., Comm. Pure Appl. Math., 12, 1959.

[2] **BOCCARDO, L., MURAT, F. and PUEL J.P.**: Existence de solutions faibles pour des equations elliptiques quasi-lineaires a croissance quadratique. Nonlinear partial differential equations and their applications. College de France Seminar, Vol. IV. ed. by H. Brezis, J.L. Lions, research Notes in Math. No 84. Pitman, London, 1983.

[3] **BROWDER, F.E.**: Pseudomonotone operators and nonliear elliptic boundary value problems on unbounded domains. Proc. Mat. Acad. Sci. USA, 74, No 7. 1977.

[4] **CHELKAK, S.I. and KOSHELEV, A.I.**: Regularity of solutions of quasilinear elliptic systems. Teubner Texte zur Mathematik. Band 77, Leipzig, 1955.

[5] **COMPANATO, S.**: "Hölder Continuity of the solutions of some nonlinear elliptic systems". Advance in Maths. 48. 1983.

[6] **DONATO, P. and GIACHETTI, D.**: Quadratic growth in unbounded domains. Nonlinear analysis. Theory and Applications, vol. 10., No 8. 1986.

[7] **KINDERLEHRER, D. and STAMPACCHIA, G.**: An introduction to variational inequalities and their applications. A.P.N.Y. London, 1980.

[8] **LEARAY, J. and LIONS, J.L.**: Quelques resultats de Visik sur les problemes elliptiques nonlineaires par la methodes de Minty-Browder. Bull. Soc. Hat. France, 93. 1965.

[9] **NECAS, T.**: On the regularity of weak solutions to

nonlinear elliptic systems of partial differential equations. Lecture Scuole Norm. Sup. Pisa, 1979.

[ 10 ] **NECAS, T.**: Introduction to the theory of nonlinear elliptic equations. Serie Teubner Texte zur Mathematik, Band 52, Leipzig, 1983.

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