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A COMPUTER-ASSISTED NUMBER THEORETICAL CONSTRUCTION OF (3,K)-RAMSEY GRAPHS

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Abstract : The concept of maximal triangle-free circular graphs is introduced and their relation to sum-free bases for finite intervals of integers is studied, leading to the construction of an infinite sequence of (3, k)-Ramsey graphs if $k \geq 61$.

1. Introduction

Throughout, graphs are undirected, with no loops or multiple edges. The number of vertices will be denoted by n, the degree of a vertex v by $d(v)$, the distance between vertices u, v by $d(u, v)$. A set of vertices is independent (or stable) if the vertices are pairwise non-adjacent. The maximum size of such a set in a graph G is called the independence (or stability) number and will be denoted by $\alpha\ (G)$. For a vertex v, $N(v)$ will denote the set of its neighbours. graph is triangle-free if it does not contain three mutually adjacent vertices.

We shall use the symbol $gcd(a_1, a_2, \ldots)$ for the greatest common divisor of the integers $a_1, a_2, \ldots$. $[n]$ will denote the integer part of n.

The Ramsey number $R(3, k)$ is the smallest integer n such that any graph with n vertices either contains a triangle K_3 or an independent set of size k. The asymptotic bounds

$$\frac{ck^2}{log^2 k} < R(3,k) < \frac{ck^2}{log k}$$

are known since many years, the lower bound is due to Erdős [1] and the upper bound to Ajtai, Komlós and Szemerédi [2] with c=100. This constant was improved to 2.4 by Griggs [3, 4]. A graph is called a *(3, k)-Ramsey graph* if it contains neither a triangle, nor a $k-$ *element* independent set. Constructing such graphs can lead to improve lower bounds for $R(3,k)$.

2. Circular and maximal triangle-free circular graphs

Let $1 \leq j_1 < j_2 \ldots < j_k < [\frac{n}{2}]$ be integers. We define $G = G(n; j_1, j_2, \ldots, j_k)$ as a simple undirected graph with n vertices where two vertices are adjacent if and only if the difference of their indices equals one of $j_1, j_2, \ldots, j_k$ (modulo n). Such graphs are called circular. For example, $G(5;1)$ is a circuit of length 5.

A maximal triangle-free circular graph (**MTC**) is a triangle-free circular graph G with $G+e$ containing triangles for any edge e not belonging to G.

An infinite sequence $G_3, G_4, \ldots$ of **MTC**'s can be obtained if G_3 is a circuit of length 5 and G_{r+1} is obtained from G_r by associating a new vertex x' with each vertex x of G_r and joining it to all neighbours of x in G_r; moreover, by adding a new vertex y and joining it to the new vertices x', see [5].

One can easily prove the following statements:

(1). Circular graphs are regular.

(2). If $G(n; j_1, j_2, \ldots, j_k)$ with $k \geq 2$ is a triangle-free circular graph then its girth is 4.

(3). Let $k = gcd(n, j)$ and $h = \frac{n}{k}$ for a circular graph $G(n, j)$. Then the graph consists of k circuits of length h [7].

(4). If $gcd(n, j_1, j_2, \ldots, j_k) = 1$ for a circular graph $G(n; j_1, j_2, \ldots, j_k)$ then the graph is connected [7].

(5). A triangle-free graph G is maximal with respect to this property if and only if the distance of any two vertices of G is at most 2.

(6). If G is a maximal triangle-free graph and v is a vertex of G with maximum degree Δ then $N(v)$ is a maximal independent set of G.

(7). If G is a **MTC** then $N(v)$ is a maximal independent set of G for every vertex v of G.

(8). A circular graph $G(n; j_1, j_2, \ldots, j_k)$ is triangle-free if and only if

(i). $j_{i_1} + j_{i_2} + j_{i_3} \neq n$ for any three, not necessarily distinct integers $j_{i_1}, j_{i_2}, j_{i_3} \in \{j_1, j_2, \ldots, j_k\}$;

(ii). $j_{i_1} + j_{i_2} \neq j_{i_3}$ for any three integers $j_{i_1}, j_{i_2}, j_{i_3} \in \{j_1, j_2, \ldots, j_k\}$ (where j_{i_1} and j_{i_2} can be equal).

(The necessity of these conditions is obvious. For the sufficiency suppose that the graph has a triangle T. Without loss of generality we may suppose that the vertices of T contain the n^{th} vertex. Let the subscript of the other two vertices be h and k with $h < k$. If $k \leq [\frac{n}{2}]$ then the choice $h = j_{i_1}$ and $k = j_{i_2}$ leads to a contradiction with (ii). Otherwise put $h = j_{i_1}$ and $k - h = j_{i_2}$ for a contradiction with (i).)

(9). Let $G(n; j_1, j_2, \ldots, j_k)$ be a triangle-free circular graph, let $S = \{j_1, j_2, \ldots, j_k\}$. Suppose that S is a base for the interval $[1, [\frac{n}{2}]]$ and that for any $j \in S$ even, there are some $j_{i_1}, j_{i_2} \in S$ so that $j_{i_1} + j_{i_2} = \frac{j}{2}$ or $|j_{i_1} - j_{i_2}| = \frac{j}{2}$. Then the graph is **MTC**.

(The statement easily follows from statement (8). The conditions are also necessary. For example, $G(10;2,3)$ and $G(10;1,4)$ are **MTC** while $G(10;2,5)$ is not.)

We associate a $(2k+1) \times 2k$ matrix M_v to each vertex v of a circular graph $G(n; j_1, j_2, \dots, j_k)$ as follows. The first row $A_v = (a^v_{1,1}, a^v_{1,2}, \dots, a^v_{1,2k})$ consists of the elements $a^v_{1,i} \in N(v)$ in circular order (starting with $v + j_1$, ending with $v + (n - j_1)$ and increasing mod n). The elements of $N(a^v_{1,i})$ in the same order are arranged as column vectors $b_1, b_2, \dots, b_{2k}$ for every $i = 1, 2, ..., 2k$. These vectors form a $2k \times 2k$ block B_v which is placed under A_v.

For example, if the vertices of the **MTC** graph $G(65; 7, 10, 11, 12, 13, 15, 16)$ are numbered from 0 to 64 then M_0 is shown on the next page.

The following statements are again straightforward:

(10). For any vertex v of a circular graph $G(n; j_1, j_2, \dots, j_k)$, the matrix B_v has the following properties:

(i). B_v is symmetric;

(ii). $b_{i,i} = 2a^v_{1,i} \pmod{n} - v$;

(iii). $b_{1,2k} = b_{2,2k-1} = \dots = b_{2k,1} = v$;

(iv). $b_{i,j} + b_{2k-j+1,2k-i+1} = 2v \pmod{n}$ for every i, j =1, 2, ... , 2k.

(11). For the first vertex of $G(n; j_1, j_2, \dots, j_k)$, i.e. for that with label 0,

(i). $A_0 = (j_1, j_2, \dots, j_k, n - j_k, \dots, n - j_1)$;

(ii). $b^T_i = A_{j_i}$ and $b^T_{k+i} = A_{n-j_{k+1-i}}$ for every i =1, 2, ... , k;

(12). Let the vertices of the circular graph $G(n; j_1, j_2, \dots, j_k)$ be labeled from 0 to n-1. If there are subsets $S_1 \subseteq N(0)$ and

$S_2 \subseteq N(k)$ where $k \in N(0)$ so that $S = \{N(0) - S_1\} \cup S_2$ is also an independent set then $|S_2| \leq |S_1|$, i.e. $|S| \leq |N(0)|$.

MAT M_0

MAT A_0 07...10...11...12...13...15...16...49...50...52...53...54...55...58

MAT B_0 14...17...18...19...20...22...23...56...57...59...60...61...62...00

17...20...21...22...23...25...26...59...60...62...63...64...00...03

18...21...22...23...24...26...27...60...61...63...64...00...01...04

19...22...23...24...25...27...28...61...62...64...00...01...02...05

20...23...24...25...26...28...29...62...63...00...01...02...03...06

22...25...26...27...28...30...31...64...00...02...03...04...05...08

23...26...27...28...29...31...32...00...01...03...04...05...06...09

56...59...60...61...62...64...00...33...34...36...37...38...39...42

57...60...61...62...63...00...01...34...35...37...38...39...40...43

59...62...63...64...00...02...03...36...37...39...40...41...42...45

60...63...64...00...01...03...04...37...38...40...41...42...43...46

61...64...00...01...02...04...05...38...39...41...42...43...44...47

62...00...01...02...03...05...06...39...40...42...43...44...45...48

00...03...04...05...06...08...09...42...43...45...46...47...48...51

3. Sum-free bases and MTC graphs

Let S be a set of integers. $S + S$ and $S - S$ denote the sets of integers arising as sums, or differences, respectively, of two elements of S. S is called sum-free if $S \cap (S+S) = S \cap (S-S) = \phi$.

A subset S of $\{1, 2, \ldots, n\}$ is called a base if $S \cup (S+S) \cup (S-$

$S) = \{1, 2, \ldots, n\}$. The minimum cardinality c_n of such a base was proved to be $c_n \leq \sqrt{3.6}\sqrt{n}$, see [8]; this was later improved to be $c_n \leq \sqrt{3.5}\sqrt{n} + o(\sqrt{n})$, see [9].

The constructions in [8] and [9] are not suitable for our purposes since these bases are not sum-free. In this section we construct a sum-free base (with a somewhat larger cardinality).

Theorem 1: Let $t \geq 4$ and $n = 10t^2 + 19t + 3$. The subset $S \subseteq [1, 2, \ldots, n]$ of integers be the union of the following seven arithmetic progressions:

I. $\alpha_1 = 2t + 1$, $\alpha_2 = 3t + 2$, ... , $\alpha_{3t} = 3t^2 + 4t$; here the difference of the consecutive terms is $d_1 = t + 1$.

II. $\beta_1 = 3t^2 + 4t + 2$, $\beta_2 = 3t^2 + 4t + 3$, ... , $\beta_{t-2} = 3t^2 + 5t - 1$; $d_2 = 1$.

III. $\gamma_1 = 3t^2 + 5t + 1$, $\gamma_2 = 3t^2 + 5t + 2$, ... , $\gamma_t = 3t^2 + 6t$; $d_3 = 1$.

IV. $\delta_1 = 3t^2 + 6t + 2$.

V. $\epsilon_1 = 6t^2 + 12t + 3$, $\epsilon_2 = 6t^2 + 13t + 3$, ... , $\epsilon_{t+2} = 7t^2 + 13t + 3$; $d_5 = t$.

VI. $\eta_1 = 6t^2 + 13t + 4$, $\eta_2 = 6t^2 + 14t + 5$, ... , $\eta_{t-4} = 7t^2 + 9t - 1$; $d_6 = t + 1$.

VII. $\lambda_1 = 7t^2 + 11t + 1$.

Then

(1). S has cardinality $7t - 2$;

(2). $S \cup (S + S) \cup (S - S)$ covers every integer between 1 and n except $7t^2 + 10t$ and $7t^2 + 12t + 2$;

(3). S is sum-free;

(4). $a+b+c \neq 2n$ for any $a, b, c \in S$.

The first statement is obvious. (2) can be verified by the following sequence of statements:

1. 1 and 2 can be covered easily.

2. $\{\gamma_i - \beta_j\} \supseteq [3, t+1]$.

3. $\{\alpha_i\} \cup \{\gamma_i - \alpha_j\} \cup \{\delta_1 - \alpha_i\} \supseteq [t+2, 3t^2+4t+1]$.

4. $\{\beta_i\} \supseteq [3t^2+4t+2, 3t^2+5t-1]$.

5. $\alpha_1 + \alpha_{3t-1} = 3t^2+5t$.

6. $\{\gamma_i\} \supseteq [3t^2+5t+1, 3t^2+6t]$.

7. $\alpha_1 + \alpha_{3t} = 3t^2+6t+1$.

8. $\delta_1 = 3t^2+6t+2$.

9. $\{\epsilon_i - \gamma_j\} \supseteq [3t^2+6t+3, 4t^2+8t+2]$.

10. $\{\epsilon_i\} \cup \{\epsilon_i - \alpha_j\} \cup \{\eta_i\} \cup \{\alpha_i + \epsilon_j\} \supseteq [4t^2+8t+3, 7t^2+10t+3]$, except $7t^2+10t$.

11. $\{\alpha_i + \epsilon_j\} \supseteq [7t^2+10t+4, 7t^2+11t]$.

12. $\lambda_1 = 7t^2+11t+1$.

13. $\epsilon_{t+2} - \alpha_1 = 7t^2+11t+2$.

14. $\{\epsilon_i\} \cup \{\alpha_i + \epsilon_j\} \supseteq [7t^2+11t+3, 9t^2+18t+3]$, except $7t^2+12t+2$.

15. $\{\gamma_i + \epsilon_j\} \supseteq [9t^2+18t+3, 10t^2+19t+3]$.

(In order to see item no. 10, observe that if $t=4$, $\{\epsilon_i\} \cup \{\epsilon_i - \alpha_j\} \supseteq [4t^2+8t+3, 7t^2+10t+3]$ except $7t^2+10t$ and that in case of $t \geq 5$ we also need $\{\alpha_i + \epsilon_j\}$, i, j =1, 2, ..., t-3, for the numbers of form $[7t^2 + (10 - i)t + 3 + j]$, i =1, 2, ... , t-4; j =1, 2, ... , (t-4)+1-i, and then only $7t^2 + 10t$ and the η_i's are left. The other items are obvious.)

The proofs of (3) and (4) require a large number of technical steps. The interested reader is referred to [12]; details in English are available from the author.

Using the above sets $S = \{j_1, j_2, \ldots, j_k\}$ we obtain an infinite sequence of **MTC** graphs $G(n; j_1, j_2, \ldots, j_k)$, i.e. an infinite sequence of $(3, k) -$ ***Ramsey*** graphs.

4. Computational results and some remarks

Using some ideas of Balas and Chang [10] we developed a **FORTRAN program** for finding a maximum independent set in a circular graph, and determined the first eight **MTC** graphs of the above sequence (for t =4, 5, 6, 7, 8, 9, 10, and 11). The main parameters of these graphs are as follows:

t	n	Δ	α
4	478	52	60
5	696	66	82
6	954	80	102
7	1252	94	160
8	1590	108	187
9	1968	122	255
10	2386	136	239
11	2844	150	300

The explicit lists of S and a maximum independent set for each graph can be found in [12] and are available from the author.

Some open problems arise from these considerations. For example, we conjecture that $\frac{\alpha}{n}$ is increasing if t is odd and decreasing if t is even for the above sequence of **MTC** graphs.

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