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AN ITERATIVE METHOD FOR NONSELFADJOINT ELLIPTIC PROBLEMS ON REGIONS PARTITIONED INTO SUBSTRUCTURES

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Abstract. A new preconditioned conjugate gradient (PCG)-based domain decomposition method is given for the solution of linear equations arising in the finite element method applied to a nonselfadjoint elliptic problem. The domain under consideration is broken into subdomains and the preconditioner is defined such that it require only the solution of matrix problems on the subdomains. Analitic estimates are given which under appropriate hypotheses garantee the geometric convergence of the preconditioned iterative procedure. The rate of convergence is independent of the number of unknowns.

1. Introduction

In the paper we describe a PCG-based domain decoposition method for the solution of linear equations which arises from the discretization of second-order nonselfadjoint uniformly elliptic boundary value problem in a bounded region Ω via finite element method. For the sake of exposition we will assume that $\Omega \subset R^2$ and it is partitioned into two subregion Ω_1 and Ω_2.

Domain decomposition methods for elliptic problems are described in many papers see [1-2], [5], [7-8] and the references given there in. Our method is based on technique explained in [1], [5] and [7]. Here we give an extension of method of decomposition for selfadjoint elliptic problems proposed in [1] for the nonselfadjoint elliptic problems.

The nonselfadjoint case is studied for example in [5] using the minimum residual method which is less effective than the method proposed here.

2. Differential and Discrete Problems

We consider as a modell problem the weak form of the following Dirichlet problem for the second order elliptic equation.

For $f \in L^2(\Omega)$ find a function $u \in H_0^1(\Omega)$ such that

$$a(u,v) = Q(v) \qquad \forall v \in H_0^1(\Omega) \tag{2.1}$$

where Ω is a bounded region in R^2 and

$$a(u,v) = \int_\Omega \{ \sum_{i,j=1}^{2} a_{ij}(x)\partial_i u \partial_j v +$$

$$\sum_{i=1}^{2} b_i(x)\partial_i uv + c(x)uv\} dx$$

$$Q(v) = \int_\Omega fvdx, \quad \forall v \in H_0^1(\Omega), x = (x_1, x_2) \in \Omega$$

and $a_{ij}, b_i, c \in L^\infty(\Omega)$.

We assume that the bilinear form $a(u,v)$ satisfies the condition: there exist positive constants ν_1 and ν_2 such that

$$\nu_1 \| u \|^2_{H_0^1(\Omega)} \leq a(u,u), \quad \forall u \in H_0^1(\Omega) \tag{2.2}$$

and

$$a(u,v) \leq \nu_2 \| u \|^2_{H_0^1(\Omega)} \cdot \| v \|^2_{H_0^1(\Omega)}, \forall u, v \in H_0^1(\Omega)$$

This problem has an unique solution if the boundary $\partial\Omega$ of the region Ω is piecewise Lipschitz-continuous; see for example [3].

We solve the problem (2.1) by the finite element method with triangular elements and piecewise linear approximation. For simplicity, we assume that the boundary $\partial\Omega$ of Ω is a polygonal line, but a similar analysis applies for the general form of $\partial\Omega$. The region Ω is partitioned into triangles e_i such that the intersection of two different triangles is either empty or consist of exactly one vertex of exactly one side. The element e_i is characterized by its greates side h_i and we denote $\max_i h_i$ by h.

For a given partition we define the finite element space

$$v_h(\Omega) = \{v \in C(\Omega) \mid v|_{e_i} \in P_1(x); v(x) = 0, \forall x \in \partial\Omega\}$$

Where $P_1(x)$ is the set of linear polynomials. A function v on e_i is represented by its values at vertices (nodes) of e_i. The approximation problem for (2.1) in the space $V_h(\Omega)$ is the following.

Find a function $u_h \in V_h(\Omega)$ such that

$$a(u_h, v) = Q(v), \quad \forall v \in V_h(\Omega) \tag{2.3}$$

This problem has a unique solution. It the partition of Ω is regular (see for exaple [3], p.132) and $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then for $s = 0, 1$

$$\| u - u_h \|_{H^s} \leq M h^{2-s} \| u \|_{H^2}$$

where u and u_h are the solution of the problems (2.1) and (2.3) respectively, and M is a postitive constant independent of h.

Using the nodal basis of the space $V_h(\Omega)$ we rewrite the problem (2.3) as the linear system

$$Au_h = f_h \tag{2.4}$$

where $(Au_h, v_h)_{R^N} = a(u_h, v_h), (f_h, v_h)_{R^N} = Q(v_h)$ and N is the number of nodes belonging to Ω. Form now on we use notation that makes no distinction between the function v_h belonging to $V_h(\Omega)$ and the grid function (vector) v_h if they are equal at the given nodes.

3. The Preconditioning Algorithm

For the solution of the linear system (2.4) we shall use the preconditioned conjugate gradient method (see [4] or [6]) of the form

$$Bu_h^{k+1} = \alpha_{k+1}(B - \tau_{k+1}A^T B^{-1} A)u_h^k + (1-\alpha_{k+1})Bu_h^{k-1} + \alpha_{k+1}\tau_{k+1}A^T B^{-1} f \qquad (k = 1, 2, \ldots) \tag{3.1}$$

$$Bu_h^1 = (B - \tau_1 A^T B^{-1} A)u_h^0 + \tau_1 A^T B^{-1} f,$$

where $u_0 \in R^N$ is a given vector and the parameters α_{k+1} and τ_{k+1} are equals to

$$\tau_{k+1} = \frac{(r_k B^{-1} r_k)_{R^N}}{(B^{-1}AB^{-1}r_k, AB^{-1}r_k)_{R^N}}, \quad (k = 0, 1, 2, \ldots) \tag{3.2}$$

$$\alpha_{k+1} = \{1 - \frac{\tau_{k+1}}{\tau_k} \frac{(r_k, B^{-1}r_k)_{R^N}}{(r_{k-1}, B^{-1}r_{k-1})_{R^N}} \frac{1}{\alpha_k}\}^{-1},$$

and

$$r_k = A^T B^{-1}(AU_h^k - f).$$

In the above expression by B we denote a symmetric positive definit matrix. The construction of B we give in section 5. We assume that the partition of Ω is regular and the systems involving matrix B is solved by a dirct method. Let $w_1(B)$ denote the cost of the factorization of B, i.e. $B = LU$, by a direct method and $w_2(B)$ - the cost of solving the system with B when the factorization $B = LU$ is given.

On the convergence of iterative method (3.1), (3.2) we can prove the following theorem.

Theorem 1.: With the symmetric positive definit matrix B (given in section 5) the computational work to find solution u_h of (2.4) with the accuracy ϵ, i.e.

$$\| u_h - u_h^k \|_{H^1(\Omega)} \leq \epsilon \tag{3.3}$$

requires of order

$$w_1(B) + w_2(B) ln(\epsilon^{-1}) \tag{3.4}$$

arithmetic operations, where u_h^k is computed by the method (3.1) and (3.2) in k iterations.

4. The Construction of the Preconditioner

In this point we give the construction of the preconditioner B from (3.1) in the form of bilinear form $b(.,.)$. The algorithm given here we use to build up a simple algorithm for solving the equation $Bx = y$ described in the point 5.

Let Γ denote a curve contained in Ω and consisting only the certain sides of the triangular elements. It divides the region Ω into two subregions Ω_1 and Ω_2. Suppose that the partition of Ω is regular and the number of pieces of Γ is finite when h tends to zero (see figure 1.).

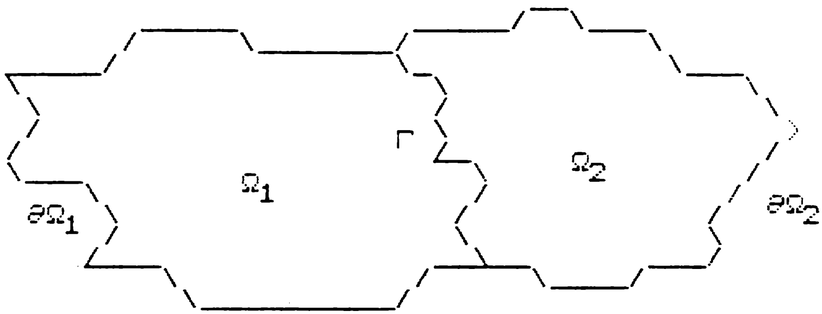

Figure 1.

In order to construct the preconditioner, we have to define three finite element spaces, related to $V_h(\Omega)$. Let $V_h(\Omega_1)$ and $V_h(\Omega_2)$ be the restriction of elements in $V_h(\Omega)$ which vanish in Ω_2 and Ω_1 respectively and in particular on Γ and let $V_h(\Gamma)$ consist of those element in $V_h(\Omega)$ which vanish in Ω/Γ. Let

$$\tilde{a}(u,v) = \int_\Omega \{ \sum_{i,j=1}^{2} \tilde{a}_{ij}(x)\partial_i u \partial_j v + \tilde{c}(x)uv\} dx$$

$$\forall u, v \in V_h(\Omega) \tag{4.1}$$

where $\tilde{a}_{ij}, \tilde{c} \in L^\infty(\Omega)$, a suitably choiched selfadjoint uniform elliptic bilinear form. That is it satisfies the following conditions:

$$\tilde{a}(u,v) = \tilde{a}(v,u), \forall u, v \in V_h(\Omega) \tag{4.2}$$

$$\tilde{\nu}_1 \parallel u \parallel^2_{H^1_0(\Omega)} \leq \tilde{a}(u,u), \forall u \in V_h(\Omega)$$

and

$$\tilde{a}(u,v) \leq \tilde{\nu}_2 \parallel u \parallel_{H^1_0(\Omega)} \cdot \parallel v \parallel_{H^1_0(\Omega)}, \forall u, v \in V_h(\Omega)$$

where $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are positive constants.

Set

$$\tilde{a}(u,v) = \tilde{a}_1(u,v) + \tilde{a}_2(u,v), \forall u,v \in V_h(\Omega) \tag{4.3}$$

on $V_h(\Omega)$, where

$$\tilde{a}_k(u,v) = \int_{\Omega_k} \{\sum_{i,j=1}^{2} \tilde{a}_{ij}(x)\partial_i u \partial_j v + \tilde{c}(x)uv\}dx$$

$$\forall u,v \in V_h(\Omega) \qquad (k=1,2)$$

Let us now consider an arbitrary function $w \in V_h(\Omega)$. We decompose w on Ω, as follows. Let $w = w_p + w_H$ where $w_p \in V_h(\Omega)$ and satisfies

$$\tilde{a}_2(w_p, v) = \tilde{a}_2(w,v) \qquad \forall v \in V_h(\Omega_2)$$

Notice that w_p is determined on Ω_2 by the values of w on Ω_2 and

$$a_2(w_H, v) = 0 \qquad \forall v \in Vh(\Omega_2)$$

Thus on $\Omega_2 w$ is decomposed into a function w_H which satisfies the above homogeneous equations.

We now can define our preconditioner in the form of

$$b(u,v) = \tilde{a}_1(u,v) + \tilde{a}_2(u_p, v_p) \quad \forall u,v \in V_h(\Omega) \tag{4.4}$$

On the basis of (4.1) and (4.2) easy to prove, that this bilinear form is also selfadjoint and uniform elliptic.

We shall show that the equations

$$b(w,v) = Q_g(v) \qquad \forall v \in V_h(\Omega) \tag{4.5}$$

where

$$Q_g(v) - \int_\Omega gvdx \quad \forall v \in V_h(\Omega) \text{ and } g \in L^2(\Omega)$$

given in the above form we can solve by solving related Galjorkin equations on Ω_1 and Ω_2 separately.

The algorithm for the solution of (4.5) can be defined in the following four steps:

(i) Consider $v \in V_h(\Omega_2)$. Then (4.5) reduces to

$$\tilde{a}_2(w_p, v) = Q_g(v) \qquad \forall v \in V_h(\Omega_2) \tag{4.6}$$

Since $w_p \in V_h(\Omega_2)$ this is just the solution of a discrete Dirichlet problem on Ω_2.

(ii) With w_p known, we can write (4.5) as

$$\begin{aligned} \tilde{a}_1(w, v) &= Q_g(v) - \tilde{a}_2(w_p, v_p) = \\ &= Q_g(v) - \tilde{a}_2(w_p, v_p) \quad \forall v \in V_h(\Omega) \end{aligned} \tag{4.7}$$

The last equality is true because $\tilde{a}_2(w_p, v_H) = 0$. The equations (4.7) uniquely determine w on Ω_1. In fact $w_{|\Omega_1} (\in V_h(\Omega_1))$ is the discrete solution of a mixed Neumann-Dirichlet problem on Ω_1. So we have w on Ω_1 and, in particular on Γ.

(iii) We can now determine w_H on Ω_2 as the solution of a homogeneous Dirichlet problem with values $w_{|\Gamma}$ on Γ and zero on the rest of $\partial\Omega_2$. That is by solution of the equation

$$\tilde{a}_2(w_H, v) = 0 \qquad \forall v \in V_h(\Omega_2). \tag{4.8}$$

(iv) Let

$$w = w_p + w_H \tag{4.9}$$

5. The Block Matrix Representation of the Preconditioner

In this section we will describe the preconditioner B in the terms of block matrices. It will be shown that B - the matrix representation of the bilinear form $b(.,.)$ - has a special structure and the process for solving (4.5) previously described may also be seen as a block Gauss elimination process. We shall suppose that we have the usual nodal basis for $V_h(\Omega)$ and the nodes are partitioned into three subset correspondingly to those in Γ, Ω_1 and Ω_2. We order the vectors as follows

$$v = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \tag{5.1}$$

where v_0, v_1 and v_2 corresponding to the nodes on Γ, Ω_1 and Ω_2, respectively. So the equations (4.5) can be rewrite into the following linear system
(5.2)

$$\begin{bmatrix} B_{00} + B_{02}B_{22}^{-1}B_{01}^T & B_{01} & B_{02} \\ B_{01}^T & B_{11} & 0 \\ B_{02}^T & 0 & B_{22} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} Q_{g0} \\ Q_{g1} \\ Q_{g2} \end{bmatrix}$$

where the submatrices and the Q_g vector are

$$\begin{aligned} \begin{bmatrix} B_{00} & B_{01} \\ B_{01}^T & B_{11} \end{bmatrix}_{i,j} &= \tilde{a}_1(\varphi_i, \varphi_j), & \varphi_i, \varphi_j \in V_h(\Omega_1) \\ (B_{22})_{i,j} &= \tilde{a}_2(\varphi_i, \varphi_j), & \varphi_i, \varphi_j \in V_h(\Omega_2) \\ (B_{02})_{i,j} &= \tilde{a}_1(\varphi_i, \varphi_j), & \varphi_i \in V_h(\Gamma), \varphi_j \in V_h(\Omega_2) \\ (Q_g)_i &= Q_g(\varphi_i), & \varphi_i \in V_h(\Omega) \end{aligned}$$

In the expressions by φ_i we denote the elements of the nodal basis $V_h(\Omega)$.

Using (4.6) - (4.9) the algorithm for the solution of (5.2) we can define in the following four steps.

(i) Solve the system

$$B_{22}(w_p)_2 = Q_{g_2} \tag{5.3}$$

This corresponds to the reformulation of the system (5.2) into

$$\begin{bmatrix} B_{00} & B_{01} & 0 \\ B_{01}^T & B_{11} & 0 \\ B_{02}^T & 0 & B_{22} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} Q_{g_0} - B_{02}B_{22}^{-1}Q_{g_2} \\ Q_{g_1} \\ Q_{g_2} \end{bmatrix} \tag{5.4}$$

(ii) Solve the system

$$\begin{bmatrix} B_{00} & B_{01} \\ B_{01} & B_{11} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} Q_{g_0} - B_{02}B_{22}^{-1}Q_{g_2} \\ Q_{g_1} \end{bmatrix} \tag{5.5}$$

(iii) Solve the system

$$B_{22}(w_H)_2 = -B_{02}^T w_0 \tag{5.6}$$

because w_0 and w_1 are known.

(iv) Let

$$w = w_H + w_p. \tag{5.7}$$

6. Proof of Theorem 1

To prove the Theorem 1 from section 3. we need a few auxiliary lemmas. Let N denote the number of the nodes belonging to Ω.

Lemma 1: Let (2.2) satisfies and suppose that the partition of Ω is regular and the number of pieces of Γ (see section 4) is finite when h tends to zero. Then

$$\delta_1(Bv,v)_{R^N} \le (Av,v)_{R^N} \le \delta_2(Bv,v)_{R^N}$$
$$\forall v \in R^N \tag{6.1}$$

where δ_1 and δ_2 are positive constants independent of h, and A and B matrices from (2.3) and (5.2), respectively.

Proof: By definition of A and B easy to see that (6.1) is equivalent with the

$$\delta_1 b(v,v) \le a(v,v) \le \delta_2 b(v,v), \quad \forall v \in V_h(\Omega) \tag{6.2}$$

inequalities. From the definition of $\tilde{a}(.,.)$ it follows that

$$c_1\tilde{a}(v,v) \le a(v,v) \le c_2\tilde{a}(v,v), \quad \forall v \in V_h(\Omega)$$

where c_1 and c_2 are positive constants independent of h.
Since

$$c_1 b(v,v) \le c_1\tilde{a}(v,v) \le a(v,v), \quad \forall \in V_h(\Omega) \tag{6.3}$$

we can choice $\delta_1 = c_1$. Thus we need to prove that

$$a(v,v) \le \delta_2 b(v,v), \qquad \forall v \in V_h(\Omega)$$

i.e.

$$c_2\tilde{a}(v,v) \le \delta_2 b(v,v), \qquad \forall v \in V_h(\Omega) \tag{6.4}$$

which follows from the inequality

$$\tilde{a}_2(v_H, v_H) \le c_3\tilde{a}_1(v,v), \qquad \forall v \in V_h(\Omega) \tag{6.5}$$

where c_3 independent of h. The inequality (6.5) is proved as follows: Let w an arbitrary function from $V_h(\Omega)$. Then the restriction of w_H to Ω_2 by definition is the solution of the homogeneous discrete Dirichlet problem

$$w_H = w \quad \text{on} \quad \Gamma$$

and

$$\tilde{a}_2(w_H, v) = 0, \qquad \forall v \in V_h(\Omega_2)$$

From this we have that

$$\tilde{a}_2(w_H, w_H) \leq \tilde{a}_2(v, v),$$
$$\forall v \in \{u \in V_h(\Omega) : u_{|\Gamma} = w_{|\Gamma}\} \tag{6.6}$$

Since

$$\tilde{a}_2(v, v) =$$
$$= \tilde{a}_2(v - w_H + w_H, v - w_H + w_H) =$$
$$= \tilde{a}_2(v - w_H, v - w_H) + 2\tilde{a}_2(w_H, w_H - v) + \tilde{a}_2(w_H, w_H) \geq$$
$$\geq \tilde{a}_2(w_H, w_H)$$

where we used that $\tilde{a}_2(.,.)$ is a selfadjoint form and $v - w_H \in V_h(\Omega_2)$. Let choice $\tilde{w}$ as a finite element extension of $w_{|\Omega_1}$ to Ω (see [7] Theorem 1). In this case the following estimation must hold

$$\| \tilde{w} \|_{H_0^1(\Omega)} \leq c_4 \| w \|_{H^1(\Omega_1)}$$

where c_4 is a constant independent of h and w.
So using (6.6) and the ellipticity of $\tilde{a}(.,.)$, we get

$$\tilde{a}_2(w_H, w_H) \leq \tilde{a}_2(\tilde{w}, \tilde{w}) \leq \tilde{\nu}_2 \| \tilde{w} \|^2_{H^1(\Omega_2)} \leq$$
$$\leq \tilde{\nu}_2 c_4 \| \tilde{w} \|^2_{H^1(\Omega_1)} =$$
$$= \tilde{\nu}_2 c_4 \| w \|_{H^1(\Omega_1)} \leq \tilde{\nu}_1 \tilde{\nu}_2 c_4 \tilde{a}_1(w, w)$$

From this the inequality (6.5) follows with $c_3 = \tilde{\nu}_1 \tilde{\nu}_2 c_4$ and with $\delta_2 = c_2(1 + c_3)$ the inequality (6.4) satisfies too.

Lemma 2: Let the assumption of Lemma 1 is satisfied. Then

$$\delta_3 (Bv, v)_{R^N} \leq (B^{-1}Av, Av)_{R^N} \leq \delta_4 (Bv, v)_{R^N}, \quad \forall v \in R^N \tag{6.7}$$

where δ_3 and δ_4 are positive constants independent of h.

Proof: (i) The first inequality of (6.7) we can prove using the Cauchy - Schwarz inequality in the following way

$$0 \leq (Bv, v)_{R^N} \leq \frac{1}{\delta_1}(Av, v)_{R^N} \leq \frac{1}{\delta_1}(B^{1/2}Av, B^{1/2}v)_{R^N} \leq$$

$$\leq \frac{1}{\delta_1}(B^{1/2}Av, B^{1/2}Av)^{1/2}_{R^N} \cdot (B^{1/2}v, B^{1/2}v, B^{1/2}v)^{1/2}_{R^N}, \forall v \in R^N$$

i.e

$$(Bv, v)_{R^N} \leq \frac{1}{\delta_1} 2(B^{-1}Av, Av)_{R^N}$$

From this we have that with $\delta_3 = \delta_1^2$ first inequality of (6.7) holds.

(ii) The B matrix is symmetric and positive definit thus

$$(B^{-1}Av, Av)_{R^N} = \| B^{-1/2}Av \|^2_{R^N}, \quad \forall v \in R^N$$

and

$$\| B^{-1/2}Av \|_{R^N} = \max_{z \in R^N} \frac{(B^{-1/2}Av, z)_{R^N}}{\| z \|_{R^N}} =$$

$$= \max_{g \in R^N} \frac{(Av, g)_{R^N}}{\| B^{1/2} g \|_{R^N}}$$

where $g = B^{-1/2}z$. Using the Lemma 1 and the fact that the bilinear form $a(.,.)$ is uniform elliptic we have

$$\max_{g\in R^N} \frac{(Av,g)_{R^N}}{\| B^{1/2}g \|_{R^N}} \leq \max_{g\in R^N} \frac{\nu_2 \| v \|_{H_0^1(\Omega)} \cdot \| g \|_{H_0^1(\Omega)}}{\| B^{1/2}g \|_{R^N}} \leq$$

$$\nu_1\nu_2 \max_{g\in R^N} \frac{Av,v)_{R^N}^{1/2} \cdot (Ag,g)_{R^N}^{1/2}}{\| B^{1/2}g \|_{R^N}} \leq \nu_1\nu_2\delta_2^{1/2} \cdot (Av,v)_{R^N}^{1/2}$$

$$\leq \nu_1\nu_2\delta_2 \cdot (Bv,v)_{R^N}^{1/2}$$

From this with $\delta_4 := \nu_1^2\nu_2^2\delta_2^2$ we get the second inequality of (6.7).

Proof (of Theorem 1): The convergence of method (3.1), (3.2) and satisfaction of (3.4) follow from the uniform ellipticity of bilinear form $\tilde{a}(.,.)$ and the relations (6.7) (see [4], [6]).

Remark: Our method can be generalized easily for the higher order elliptic operators and for the higher order finite elements too (see [7]).

References

[1] **Bramble, J.H., Paciak, J.E., Schatz, A.H.:** And iterative method for elliptic problems on regions partitioned into substructures. Math. Coput. 46,361-369 (1986)

[2] **Bramble, J.H., Pasciak, J.E., Schatz, A.H.:** The construction of preconditioners for elliptic problems by substructuring. I, Math Comput. 47,103-134 (1986)

[3] **Ciarlet, P.G.:** The finite element method for elliptic problems. North-Holland: Amsterdam-New York 1977

[4] **Concus, P., Golub, G.H., O'Leary, D.P.:** A generalized conjugate gradient method for the numerical solution of elliptic PDE. In: Sparse matrix computations (Bunch, J.R., Rose, D.J. eds.), pp.309-332. New York: Academic Press (1976)

[5] **Dryja, M.:** A finite element-capacitance matrix method for elliptic problems on regions partitioned into subregions. Numer. Math. 44, 153-168 (1984)

[6] **Samarskii, A.A., Nikolaev, E.S.:** Methods of solving the difference equations. Moscow: Mir 1978 (in Russian)

[7] **Widlund, O.:** An extension theorem for finite element spaces with three application. Technical Report 233, Dept. of Computer Science, Courant Inst., 1986

[8] **Widlund, O.:** Iterative substructuring methods: Algorithm and theory for elliptic problems in the plane. Technical Report 287, Dept. of Computer Science, Courant Inst., 1987

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