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CONNECTION BETWEEN THE BMO- AND THE $\mathcal{K}_\Phi$-SPACES

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1. Introduction

It is well-known that the dual of the Hardy space $\mathcal{H}_1$ is the BMO-space [4]. The dual space of the Hardy space $\mathcal{H}_p$, where $1 \le p \le 2$ has been investigated by Garsia, A. M. [5]. Mogyoródi, J. [7] and the author of the present note [2] have investigated the dual space of the Hardy space generated by Young functions Φ. This paper consists of two parts. In the first part we show that the dual of the space $\mathcal{H}_\Phi$ is the space $\mathcal{H}_\Psi$, where (Φ, Ψ) is a pair of conjugate Young functions, Φ has finite power and has the form $\Phi(x) = \Phi_1(x^2)$, where Φ_1 itself is also a Young function. In the second part we prove that the BMO-space can be approximated by the class of the so called $\mathcal{K}_\Phi$-spaces. Theorem 3.2. is a generalization of some results of [1] and [7].

2. Basic notations and definitions

Let $\varphi(t)$ be a nondecreasing and left-continuous function defined on $[0, +\infty)$ such that $\varphi(0) = 0$ and $\lim\limits_{t \to +\infty} \varphi(t) = +\infty$. For $x \ge 0$ define

$$\Phi(x) = \int_0^x \varphi(t)dt \ .$$

Then Φ is a convex, continuous and increasing function. Φ is called Young function. The power of the Young function Φ is defined by the formula

$$p = \sup_{x>0}[x\varphi(x)/\Phi(x)] .$$

The generalized inverse function ψ of the function φ is defined as follows

$$\psi(u) = \begin{cases} 0, & \text{if } u = 0 \\ \sup(s > 0 : \varphi(s) < u), & \text{if } u > 0 \end{cases}$$

It is easy to see that ψ is also nondecreasing, left-continuous and $\lim_{u\to+\infty} \psi(u) = +\infty$. Then the Young function

$$\Psi(x) = \int_0^x \psi(u)du$$

is called the conjugate function of the Young function Φ. Its power is defined as that of Φ by the formula

$$q = \sup_{x>0}[x\psi(x)/\Psi(x)]$$

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space. Denote

$$L_0(\Omega, \mathcal{F}, P) = \{X : X : \Omega \to R; X \text{ if } \mathcal{F}\text{−measurable}\} .$$

DEFINITION 2.1. Let Φ be a Young function. The Orlicz space generated by Φ is defined as follows

$$L_\Phi = L_\Phi(\Omega, \mathcal{F}, P) = \{X \in L_0 : \exists a > 0 : E\Phi(a^{-1} \mid X \mid) \leq 1\}$$

If $X \in L_\Phi$ then the Luxemburg norm of X is defined by

$$\|X\|_\Phi = \inf\{a > 0 : E\Phi(a^{-1} \mid X \mid)) \leq 1\} .$$

It is well-known that L_Φ equipped with the norm $\|\cdot\|_\Phi$ is a Banach-space. Further,

$$\|X\|_\Phi \leq \sup_{E\Psi(|Y|)\leq 1} E(XY) \leq 2\|X\|_\Phi \ .$$

where Ψ is the conjugate Young function of the function Φ (see [8]). The quantity $\sup_{E\Psi(|Y|)\leq 1} E(XY)$ is called the Orlicz-norm of X.

We refer to Krasnoselskii and Rutickii [6] for a complete treatment of the theory of Young functions.

Let $\mathcal{F}_0 \subset \mathcal{F}_1 \ldots$ be an increasing sequence of subsigma-fields of F. We suppose that $\mathcal{F} = \sigma(\cup_{n=1}^{\infty} \mathcal{F}_n)$. Consider the random variable $X \in L_1(\Omega, \mathcal{F}, P)$ and the martingale $X_n = E(X \mid \mathcal{F}_n)$, $n \geq 0$, where for the sake of commodity we suppose that $X_0 = 0$ a.s. Denote $d_i = X_i - X_{i-1}$, $i \geq 1$, $d_0 = 0$.

DEFINITION 2.2. (see [7])

We say that the random variable $X \in L_1$ belongs to the Hardy-space $\mathcal{H}_\Phi$ generated by the Young function Φ if the quadratic variation $S = (\sum_{i=1}^{\infty} d_i^2)^{1/2}$ belongs to L_Φ. We define

$$\|X\|_{\mathcal{H}_\Phi} = \|S\|_\Phi$$

The space $\mathcal{H}_\Phi$ equipped with the norm $\|\cdot\|_{\mathcal{H}_\Phi}$ is a Banach space.

3. The dual of the Hardy space $\mathcal{H}_\Phi$

In [2] we proved that the dual of $\mathcal{H}_\Phi$ is $\mathcal{H}_\Psi$ in the case when both Φ and Ψ have finite power (in this case the authors of [1] have shown that the $\mathcal{H}_\Psi$-and the K_Ψ-spaces coincide).
In this paper we shall prove that under some additional condition imposed on the function Φ, our result remains valid without the condition that Ψ, the conjugate of Φ, has finite power

We begin this section by proving the following

Lemma 3.1. *Let (Φ, Ψ) be a pair of conjugate Young functions. Suppose that Φ has the form $\Phi(x) = \Phi_1(x^2)$ where Φ_1 itself is a Young function and Φ_1 has finite power p.Let*

$$\delta\mathcal{H}_\Phi = \{\Theta = (\Theta_n)_{n\geq 1} : \Theta_n \in L_\Phi(\Omega, \mathcal{F}_n, P); \|\Theta\|_{\delta\mathcal{H}_\Phi} = \\ = \|(\sum_{n=1}^{\infty} \Theta_n^2)^{1/2}\|_\Phi < +\infty\}$$

and let $\Lambda(\Theta)$ be a bounded and linear functional on $\delta\mathcal{H}_\Phi$, i.e.

$$|\Lambda(\Theta)| \leq B \cdot \|\Theta\|_{\delta\mathcal{H}_\Phi}, \quad \forall\Theta \in \delta\mathcal{H}_\Phi,$$

where $0 < B < +\infty$ is a finite constant. Then there exists $\sigma \in \delta\mathcal{H}_\Psi$ satisfying the conditions

$$\text{(3.1.)} \quad \begin{aligned} &\|(\sum_{n=1}^{\infty} \sigma_n^2)^{1/2}\|_\Psi \leq \sqrt{p}B \\ &\|\Big[\sum_{n=1}^{\infty} E^2(\sigma_n \mid \mathcal{F}_{n-1})\Big]^{1/2}\|_\Psi \leq (\sqrt{p}+1)B, \end{aligned}$$

such that

$$\Lambda(\Theta) = \sum_{n=1}^{\infty} E(\Theta_n \cdot \sigma_n), \quad \forall\Theta \in \delta\mathcal{H}_\Phi.$$

Proof. First we remark that Φ has finite power if and only if so does Φ_1. Now for any fixed integer $n \geq 1$ and any $X \in L_\Phi(\Omega, \mathcal{F}_n, P)$ let $\Theta_X = (\Theta_1^X, \Theta_2^X, \ldots)$, where $\Theta_i^X = X$, if $i = n$ and $\Theta_i^X = 0$, if $i \neq n$. Obviously, $\Theta_X \in \delta\mathcal{H}_\Phi$ and $\|\Theta_X\|_{\delta\mathcal{H}_\Phi} = \|X\|_\Phi$. Then

$$|\Lambda(\Theta_X)| \leq B\|\Theta_X\|_{\delta\mathcal{H}_\Phi} = B\|X\|_\Phi$$

Since Φ has finite power, there exists $\sigma_n \in L_\Psi(\Omega, \mathcal{F}_n, P)$ such that

$$\Lambda(\Theta_X) = E(X \cdot \sigma_n), \quad \forall X \in L_\Phi(\Omega, \mathcal{F}_n, P) .$$

Consider now an element $\Theta^{(n)} = (\Theta_1, \Theta_2, \ldots)$, for which $\Theta_i = 0$ if $i > n$. By the linearity of the functional Λ we have

$$\Lambda(\Theta^{(n)}) = \sum_{i=1}^{n} E(\Theta_i \sigma_i)$$

where $\sigma_i \in L_\Psi(\Omega, \mathcal{F}_i, P)$, $i = 1, 2, \ldots$. From the boundedness of Λ it follows that

$$| \Lambda(\Theta) - \Lambda(\Theta^{(n)}) | \leq B \| (\sum_{i=n+1}^{\infty} \Theta_i^2)^{1/2} \|_\Phi .$$

The right-hand side of this inequality tends to zero as $n \to +\infty$. Consequently, we deduce

$$\Lambda(\Theta) = \lim_{n \to +\infty} \Lambda(\Theta^{(n)}) = \lim_{n \to +\infty} \sum_{i=1}^{n} E(\Theta_i \sigma_i) = \sum_{i=1}^{\infty} E(\Theta_i \sigma_i) .$$

We prove that

$$\left(\sum_{i=1}^{\infty} \sigma_i^2 \right)^{1/2} \in L_\Psi .$$

For this purpose we observe that the space $\delta \mathcal{H}_\Phi$ can be imbedded into the following larger space:

$$\widehat{\delta} \mathcal{H}_\Phi = \{ \Theta = (\Theta_n)_{n \geq 1} : \Theta_n \in L_\Phi(\Omega, \mathcal{F}, P), \|\Theta\|_{\widehat{\delta} \mathcal{H}_\Phi} = \\ = \| (\sum_{i=1}^{\infty} \Theta_i^2)^{1/2} \|_\Phi < +\infty \}$$

Then the functional Λ can be extended onto $\widehat{\delta} \mathcal{H}_\Phi$ as follows:

$$(3.2)\qquad \Lambda(\Theta)=\sum_{n=1}^{\infty}E[E(\Theta_n \mid \mathcal{F}_n)\sigma_n],\quad \forall\Theta\in\widehat{\delta\mathcal{H}}_\Phi$$

This can be done since we can show that $(E(\Theta_n \mid \mathcal{F}_n))_{n\geq 1}\in \delta\mathcal{H}_\Phi$ if $(\Theta_n)_{n\geq 1}\in\widehat{\delta\mathcal{H}}_\Phi$.
Indeed, denote

$$a=\|(\sum_{n=1}^{\infty}\Theta_n^2)^{1/2}\|_\Phi$$

We can suppose without any restriction that $a>0$. Then

$$(3.3)\qquad E\Big[\Phi_1(a^{-2}\cdot\sum_{n=1}^{\infty}\Theta_n^2)\Big]=E\Big[\Phi\Big(a^{-1}(\sum_{n=1}^{\infty}\Theta_n^2)^{1/2}\Big)\Big]\leq 1\,.$$

So $\sum_{n=1}^{\infty}\Theta_n^2$ belongs to L_{Φ_1} and $\|\sum_{n=1}^{\infty}\Theta_n^2\|_{\Phi_1}\leq a^2$. From the inequality $E^2(\Theta_n \mid \mathcal{F}_n)\leq E(\Theta_n^2 \mid \mathcal{F}_n)$ we deduce that

$$(3.4)\qquad \begin{aligned} &E\Big(\Phi_1\Big[p^{-1}a^{-2}\sum_{n=1}^{\infty}E^2(\Theta_n \mid \mathcal{F}_n)\Big]\Big)\leq\\ &\leq E\Big(\Phi_1\Big[p^{-1}a^{-2}\sum_{n=1}^{\infty}E(\Theta_n^2 \mid \mathcal{F}_n)\Big]\Big)\end{aligned}$$

holds. On the other hand using the convexity inequality of Burkholder-Davis-Gundy (see [3]) we can write (remarking that the power of Φ_1 is finite)

$$(3.5)\quad E\Big(\Phi_1\Big[p^{-1}a^{-2}\sum_{n=1}^{\infty}E(\Theta_n^2 \mid \mathcal{F}_n)\Big]\Big)\leq E\Big[\Phi_1(a^{-2}\sum_{n=1}^{\infty}\Theta_n^2)\Big]\,.$$

Combining (3.3), (3.4) and (3.5) we have

$$E\Big[\Phi_1\Big(p^{-1}a^{-2}\sum_{n=1}^{\infty}[E(\Theta_n \mid \mathcal{F}_n)]^2\Big)\Big] \leq 1$$

In the language of Φ this can be written in the form

$$E\Big[\Phi\Big(p^{-1/2}a^{-1}\Big[\sum_{n=1}^{\infty} E^2(\Theta_n \mid \mathcal{F}_n)\Big]^{1/2}\Big)\Big] =$$

$$= E\Big(\Phi_1\Big[p^{-1}a^{-2}\sum_{n=1}^{\infty} E^2(\Theta_n \mid \mathcal{F}_n)\Big]\Big) \leq 1.$$

This means that

$$\Big[\sum_{n=1}^{\infty} E^2(\Theta_n \mid \mathcal{F}_n)\Big]^{1/2} \in L_\Phi$$

and

$$\|\Big[\sum_{n=1}^{\infty} E^2(\Theta_n \mid \mathcal{F}_n)\Big]^{1/2}\|_\Phi \leq \sqrt{p}\cdot a = \sqrt{p}\|(\sum_{n=1}^{\infty}\Theta_n^2)^{1/2}\|_\Phi .$$

Then we can estimate (3.2) as follows

$$\text{(3.6)} \qquad |\Lambda(\Theta)| \leq \sqrt{p}\cdot B\|(\sum_{n=1}^{\infty}\Theta_n^2)^{1/2}\|_\Phi, \ \forall\Theta \in \hat{\delta}\mathcal{H}_\Phi .$$

We use the Orlicz norm to estimate the L_Ψ-norm of $(\sum_{n=1}^{\infty}\sigma_n^2)^{1/2}$.

Denote $\gamma^2 = \sum_{n=1}^{\infty}\sigma_n^2$. Then

$$\|\gamma\|_\Psi = \sup_{\substack{E\Phi(|X|)\leq 1 \\ X\in L_\Phi(\Omega,\mathcal{F},P)}} E(X\gamma)$$

Define

$$Y = \begin{cases} X\gamma^{-1}, & \text{if } \gamma \neq 0 \\ 0, & \text{if } \gamma = 0. \end{cases}$$

Consider the element $\Theta^{(n)} = (Y\sigma_1, \ldots, Y\sigma_n, 0, 0, \ldots)$. Obviously, $\Theta^{(n)} \in \widehat{\delta \mathcal{H}}$ and $\|\Theta^{(n)}\|_{\widehat{\delta \mathcal{H}}_\Phi} \leq 1$, since

$$E\Big(\Phi\Big[(\sum_{i=1}^{n} Y^2\sigma_i^2)^{1/2}\Big]\Big) \leq E[\Phi(| X |)] \leq 1.$$

We can write

$$\begin{aligned} E(X\gamma) = & E(Y\gamma^2) = \sum_{i=1}^{n} E(Y\sigma_i^2) = \\ & = \sum_{i=1}^{n} E\Big[E(Y\sigma_i \mid \mathcal{F}_i)\sigma_i\Big] = \Lambda(\Theta^{(n)}). \end{aligned}$$

Using (3.6) we have

$$| E(X\gamma) | = | \Lambda(\Theta^{(n)}) | \leq \sqrt{p} \cdot B \|\Theta^{(n)}\|_{\widehat{\delta \mathcal{H}}_\Phi} \leq \sqrt{p} \cdot B.$$

Consequently,

$$\|\gamma\|_\Psi \leq \sqrt{p} \cdot B$$

and, letting $n \to +\infty$, finally we have

$$\|(\sum_{n=1}^{\infty} \sigma_n^2)^{1/2}\|_\Psi \leq \sqrt{p} \cdot B.$$

For arbitrary $\Theta = (\Theta_1, \Theta_2, \ldots) \in \delta\mathcal{H}_\Phi$ define the element

$$\bar{\Theta} = (0, \Theta_1, \Theta_2, \ldots) \in \delta\mathcal{H}_\Phi$$

and consider the functional $\bar{\Lambda}$ defined by the formula

$$\bar{\Lambda}(\Theta) = \Lambda(\bar{\Theta}), \quad \forall \Theta \in \delta\mathcal{H}_\Phi.$$

Then $\bar{\Lambda}$ is also a bounded and linear functional on $\delta\mathcal{H}_\Phi$ since

$$|\bar{\Lambda}(\Theta)| = |\Lambda(\bar{\Theta})| \leq B\|\bar{\Theta}\|_{\delta\mathcal{H}_\Phi} = B\|\Theta\|_{\delta\mathcal{H}_\Phi}$$

As we have shown above, there exists $\mu = (\mu_n)_{n\geq 1} \in \delta\mathcal{H}_\Psi$ satisfying $\|(\sum_{n=1}^{\infty} \mu_n^2)^{1/2}\|_\Psi \leq \sqrt{p}\cdot B$ such that

$$\bar{\Lambda}(\Theta) = \sum_{n=1}^{\infty} E(\Theta_n \mu_n). \tag{3.7}$$

But

$$\begin{aligned}\bar{\Lambda}(\Theta) = \Lambda(\bar{\Theta}) &= \sum_{n=1}^{\infty} E(\Theta_n \sigma_{n+1}) = \\ &= \sum_{n=1}^{\infty} E\Big[\Theta_n \cdot E(\sigma_{n+1} \mid \mathcal{F}_n)\Big].\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we deduce that

$$E(\sigma_{n+1} \mid \mathcal{F}_n) = \mu_n \quad \text{a.s.}$$

This implies

$$\begin{aligned}\|\Big[\sum_{n=0}^{\infty}(\sigma_{n+1} \mid \mathcal{F}_n)\Big]^{1/2}\|_\Psi \leq \| \mid E(\sigma_1 \mid \mathcal{F}_0) \mid + \\ \Big[\sum_{n=1}^{\infty} E^2(\sigma_{n+1} \mid \mathcal{F}_n)\Big]^{1/2}\|_\Psi \leq B + \sqrt{p}\cdot B = (\sqrt{p}+1)B.\end{aligned}$$

Therefore, the proof is complete.

Now we present the main result which we formulate in

Theorem 3.2. *Let* (Φ, Ψ) *be a pair of conjugate Young functions.*

a/ *For every* $X \in \mathcal{H}_\Phi, Y \in \mathcal{H}_\Psi$ *we have for arbitrary* $n \geq 1$

$$| E(X_n Y_n) | \leq 2\|X_n\|_{\mathcal{H}_\Phi} \|Y_n\|_{\mathcal{H}_\Psi}.$$

Further, $\lim_{n\to+\infty} E(X_n, Y_n)$ *exists, it is finite and*

$$| \lim_{n\to+\infty} E(X_n Y_n) | \leq 2\|X\|_{\mathcal{H}_\Phi} \|Y\|_{\mathcal{H}_\Psi}.$$

b/ *Suppose that* Φ *has the form* $\Phi(x) = \Phi_1(x^2)$, *where* $\Phi_1(x)$ *itself is a Young function having finite power p. If F is a bounded and linear functional on* $\mathcal{H}_\Phi$, *i.e.*

$$| F(x) | \leq B\|X\|_{\mathcal{H}_\Phi} \qquad (B > 0 \text{ constant})$$

then there exists $Y \in \mathcal{H}_\Psi$ *such that* $\|Y\|_{\mathcal{H}_\Psi} \leq \sqrt{2}(2\sqrt{p}+1)B$ *and*

$$\lim_{n\to+\infty} E(X_n Y_n) = F(X), \quad \forall x \in \mathcal{H}_\Phi.$$

Proof. a/ For any $X \in \mathcal{H}_\Phi$, $Y \in \mathcal{H}_\Psi$ denote
$\triangle X_i = X_i - X_{i-1}$, $i = 1, 2, \ldots$; $\triangle Y_j = Y_j - Y_{j-1}$, $j = 1, 2, \ldots$
The Cauchy-Schwartz inequality gives

$$\Big(\sum_{i=1}^n \triangle X_i \triangle Y_i\Big)^2 \leq \Big[\sum_{i=1}^n (\triangle X_i)^2\Big]\Big[\sum_{j=1}^n (\triangle Y_j)^2\Big]$$

and from this by Hölder's inequality for conjugate Young functions

$$| E(X_n Y_n) | = | \sum_{i=1}^n E(\triangle X_i \triangle Y_i) | = | E(\sum_{i=1}^n \triangle X_i \triangle Y_i) | \leq$$

$$\leq E\Big([\sum_{i=1}^n (\triangle X_i)^2]^{1/2} [\sum_{j=1}^n (\triangle Y_j)^2]^{1/2}\Big) \leq$$

$$\leq 2\| [\sum_{i=1}^n (\triangle X_i)^2]^{1/2} \|_\Phi \| [\sum_{j=1}^n (\triangle Y_j)^2]^{1/2} \|_\Psi.$$

It is easy to see that $[E(X_n Y_n)]_{n\geq 1}$ is a Cauchy sequence, since for $m \geq n$ we have

$$|E(X_m Y_m) - E(X_n Y_n)| = \left| E\Big(\sum_{i=n+1}^{m} \triangle X_i \triangle Y_i\Big)\right| \leq$$

$$\leq 2\Big\|\Big[\sum_{i=n+1}^{m} (\triangle X_i)^2\Big]^{1/2}\Big\|_\Phi \Big\|\Big[\sum_{j=n+1}^{m} (\triangle Y_j)^2\Big]^{1/2}\Big\|_\Psi \to 0$$

as $n \to +\infty$. Therefore $\lim_{n\to+\infty} E(X_n Y_n)$ exists and

$$\Big|\lim_{n\to+\infty} E(X_n Y_n)\Big| \leq 2\|X\|_{\mathcal{H}_\Phi}\|Y\|_{\mathcal{H}_\Psi} .$$

b/ Suppose that F is a linear functional on $\mathcal{H}_\Phi$ such that

$$|F(X)| \leq B\|X\|_{\mathcal{H}_\Phi}, \quad \forall X \in \mathcal{H}_\Phi .$$

Then $\mathcal{H}_\Phi$ can be inbedded in the space $\delta\mathcal{H}_\Phi$ and the definition of F can be extended onto $\delta\mathcal{H}_\Phi$ with the same bound as that of F. By Lemma 3.1. there exists $\sigma = (\sigma_n)_{n\geq 1} \in \delta\mathcal{H}_\Psi$ such that

$$F(X) = \sum_{n=1}^{\infty} E(\triangle X_n \sigma_n).$$

Consider the martingale

$$Y_n = \sum_{i=1}^{n}\Big[\sigma_i - E(\sigma_i \mid \mathcal{F}_{i-1})\Big].$$

Then $(Y_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}_\Psi$, since

$$\|Y_m - Y_n\|_{\mathcal{H}_\Psi} = \Big\|\Big[\sum_{i=n+1}^{m} (\triangle Y_i)^2\Big]^{1/2}\Big\|_\Psi \leq$$

$$\leq \sqrt{2}\Big(\Big\|\Big[\sum_{i=n+1}^{m} \sigma_i^2\Big]^{1/2}\Big\|_\Psi + \Big\|\Big[\sum_{i=n+1}^{m} E^2(\sigma_i \mid \mathcal{F}_{i-1})\Big]^{1/2}\Big\|_\Psi\Big) \to 0$$

So, Y_n must converge to

$$Y = \sum_{i=1}^{\infty}[\sigma_i - E(\sigma_i \mid \mathcal{F}_{i-1})]$$

belonging to $\mathcal{H}_\Psi$. Finally,

$$\|Y\|_{\mathcal{H}_\Phi} \leq \sqrt{2}\Big(\|(\sum_{i=1}^{\infty}\sigma_i^2)^{1/2}\|_\Psi + \|[\sum_{i=1}^{\infty}E^2(\sigma_i \mid \mathcal{F}_{i-1})]^{1/2}\|_\Psi\Big) \leq$$
$$\leq \sqrt{2}[\sqrt{p}B + (\sqrt{p}+1)B] = \sqrt{2}(2\sqrt{p}+1)B.$$

This completes the proof.

The BMO-space and the $\mathcal{K}_\Phi$-spaces

In this section we present the connection between the BMO-space and the $\mathcal{K}_\Phi$-spaces. We show that the BMO-space can be approximated in some sense by the class of the $\mathcal{K}_\Phi$-spaces. We recall some definitions.

DEFINITION 4.1.

a/ Let $X \in L_1$. Consider the martingale $X_n = E(X \mid \mathcal{F}_n)$, $n \geq 0$, $x_0 = 0$ a.s. We say that X belongs to BMO iff

$$\sup_{n \geq 1} \|E(| X - X_{n-1} | \mid \mathcal{F}_n)\|_\infty < +\infty.$$

b/ Let Φ be a Young function and let $X \in L_1$. Consider the set

$$\Gamma_X^\Phi = \{\gamma : \gamma \in L_\Phi,\ E(| X - X_{n-1} \mid \mathcal{F}_n) \leq E(\gamma \mid \mathcal{F}_n) \text{ a.s.}, \forall n \geq 1\}.$$

We say that $X \in \mathcal{K}_\Phi$ iff Γ_X^Φ is non empty and in this case we set

$$\|X\|_{\mathcal{K}_\Phi} = \inf_{\gamma \in \Gamma_X^\Phi} \|\gamma\|_\Phi$$

The following theorem shows that the K_Φ-space is the direct generalization of BMO.

Theorem 4.1. *Let (Φ, Ψ) be a pair of conjugate Young functions. Suppose that Ψ has finite power q. Then $X \in K_\Phi$ if and only if $\beta^* = \sup_{n\geq 1} E(|X - X_{n-1}| \mid \mathcal{F}_n) \in L_\Phi$ and*

$$\|X\|_{K_\Phi} \leq \|\beta^*\|_\Phi \leq q\|X\|_{K_\Phi}.$$

Proof. If $\beta^* \in L_\Phi$ then $E(|X - X_{n-1}| \mid \mathcal{F}_n) \leq E(\beta^* \mid \mathcal{F}_n)$ a.s. Consequently, $X \in K_\Phi$ and $\|X\|_{K_\Phi} \leq \|\beta^*\|_\Phi$.

Conversely, suppose that $X \in K_\Phi$. Then for all $n \geq 1$ we have

$$E(|X - X_{n-1}| \mid \mathcal{F}_n) \leq E(\gamma \mid \mathcal{F}_n) \text{ a.s.},$$

where $\gamma \in \Gamma_X^\Phi$ is arbitrary. Let $\gamma^* = \sup_{n\geq 1} E(\gamma \mid \mathcal{F}_n)$. Then by the maximal lemma (see [9]) $\gamma^* \in L_\Phi$ and

$$\|\gamma^*\|_\Phi \leq q \sup_{n\geq 1} \|E(\gamma \mid \mathcal{F}_n)\|_\Phi \leq q\|\gamma\|_\Phi.$$

On the other hand

$$\beta^* = \sup_{n\geq 1} E(|X - X_{n-1}| \mid \mathcal{F}_n) \leq \gamma^*$$

This implies that $\beta^* \in L_\Phi$ and

$$\|\beta^*\|_\Phi \leq \|\gamma^*\|_\Phi \leq q\|\gamma\|_\Phi$$

Since the above inequality holds for every γ belonging to Γ_X^Φ we deduce that

$$\|\beta^*\|_\Phi \leq q\|X\|_{K_\Phi}.$$

This completes the proof.

Theorem 4.2. *Let $\mathcal{Y}$ denote the class of all Young functions Ψ whose conjugate Young function Φ has finite power. Then*

$$\mathrm{BMO} = \bigcap_{\Psi \in \mathcal{Y}} \mathcal{K}_\Psi .$$

Proof. Obviously, $\mathrm{BMO} \subset \bigcap_{\Psi \in \mathcal{Y}} \mathcal{K}_\Psi$. Now we suppose that $Y \notin$BMO. THis means that $\beta^* = \sup_{n \geq 1} E(|Y - Y_{n-1}| \,|\, \mathcal{F}_n) \notin L_\infty$. We can suppose that $P(\beta^* < +\infty) = 1$, since in the contrary case $\beta^* \notin L_\Psi$ whatever be the Young function Ψ. Denote $a_n = P(\beta^* \geq n)$. Then $a_n > 0$ and $a_n \downarrow 0$ as $n \to +\infty$. We define the function

$$\Psi(x) = \int_0^x \psi(t)dt$$

as follows: let $\psi(0) = 0$, $\psi(1) = 2/a_{2^2}, \ldots, \psi(n) = 2/a_{(n+1)^2}, \ldots$. Further, let $\psi(t)$ be linear in every interval $[n, n+1]$. Consequently, $\Psi(x)$ is a Young function. We shall show that $\beta^* \notin L_\Psi$. For each positive integer $k \geq 1$ we have

$$\begin{aligned} E[\Psi(k^{-1}\beta^*)] &\geq E[\Psi(k^{-1}\beta^*)\chi(\beta^* \geq k^2)] \geq \\ &\geq \Psi(k)P(\beta^* \geq k^2) = \Psi(k)a_{k^2} \end{aligned}$$

By the definition of Ψ we can write

$$\Psi(k) = \int_0^k \psi(t)dt \geq \int_{k-1}^k \psi(t)dt \geq \psi(k-1) = 2/a_{k^2} .$$

Therefore,

$$E[\Psi(k^{-1}\beta^*)] \geq \Psi(k)a_{k^2} \geq (2/a_{k^2})a_{k^2} = 2.$$

This means that $\beta^* \notin L_\Psi$ and by Theorem 4.1. $Y \notin \mathcal{K}_\Psi$ provided that Φ has finite power.

Now, if the conjugate Young function Φ of the function Ψ has infinite power, then remarking that $\sup\limits_{0<x\leq 1} [\varphi(2x)/\varphi(x)] = a < +\infty$ we can define another function φ_1 as follows:

$$\varphi_1(x) = \begin{cases} \varphi(x) & \text{if } 0 \leq x \leq 1. \\ \varphi(x) & \text{if } \varphi(x) \leq a\varphi(x/2) \text{ and } 2^i < x \leq 2^{i+1}, \\ \varphi(x/2) & \text{if } \varphi(x) > a\varphi(x/2) \text{ and } 2^i < x \leq 2^{i+1}, \end{cases}$$

$i = 0, 1, \ldots$ and

$$\Phi_1(x) = \int\limits_0^x \varphi_1(t)dt \ .$$

Obviously, Φ_1 has finite power and $\Phi_1(x) \leq \Phi(x)$, since $\varphi_1(x) \leq \varphi(x)$. This also implies that $\Psi_1(x) \geq \Psi(x)$, where $\Psi_1(x)$ is the conjugate function of the function $\Phi_1(x)$. Consequently, $\beta^* \notin L_{\Psi_1}$ which means that $Y \notin \bigcap\limits_{\Psi \in \mathcal{Y}} K_\Psi$

This proves the assertion.

References

[1] BASSILY, N. L. and MOGYORÓDI, J., On the K_Φ-spaces with general Young function Φ. Annales Univ.Sci. Budapest, Sectio Mathematica, **27**(1985), 205-214.

[2] DAM, B. K., The dual space of the martingale Hardy space with general Young function. Analysis Mathematica. Publishing House of the Hungarian Academy of Science (to appear).

[3] BURKHOLDER, D. L., DAVIS, B. J. and GUNDY, R. F., Integral inequalities for convex functions of operators on martingales. Proc.Sixth Berkeley Symposium on Math.Stat. and Probability, Univ. of California Press (1972), 223-240.

[4] FEFFERMAN, C., Characterizations of bounded mean oscillation. *Bulletin Amer.Math.Soc.* **77**(1971), 587-588.

[5] GARSIA, A. M., Martingale inequalities. Benjamin, Reading, Massachusetts, 1973.

[6] KRASNOSELSKII, M. K. and RUTICKII, Y. B., Convex functions and Orlicz spaces (translated from Russian by L.F. Boron), Noordhoff, Greningen, 1961.

[7] MOGYORÓDI, J., Linear functionals on Hardy spaces, *Annales Univ.Sci.* Budapest, Sectio Mathematica, **26**(1983), 161-174.

[8] NEVEU, J., Discrete parameter martingales, North Holland, Amsterdam, 1975.

[9] MOGYORÓDI, J. and MÓRI, T. F., Necessary and sufficient condition for the maximal inequality of convex Young functions. *Acta Sci.Math.* Szeged, **45**(1983), 325-332.

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