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AN IDENTIFICATION PROBLEM OF DISTRIBUTED PARAMETER SYSTEMS

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In [1] we have investigated the identification of an age-specific population growth model. For the model see also [2]. In paper [3] the identification with respect to f of a more general system

$$\partial_t u + \langle \partial_x u, f \rangle + gu = h,$$

$$u(t_0, x) = k(x)$$

with a Volterra type observation has been studied.

In this paper we shall consider the general case ,i. e. we shall deal with the simultaneous identification of the system with respect to f and g. For the exact formulation of the problem let $n \in \mathbf{N}, T \in \mathbf{R}_+, a, b \in \mathbf{R} \quad (a < 0 < b)$and consider the sets

$$\begin{aligned} \Omega &:= \{(t, \tau, x, \xi) : t \in [0,T], \quad \tau \in [0,T], \\ &\quad x \in [a,b], \xi \in [0,x]\}, \\ \Omega_0 &:= [0,T] \times [a,b]. \end{aligned}$$

For $\ell = 0,1,2\ldots$ denote by $C^\ell(\Omega), C^\ell(\Omega_0)$ and $C^\ell[0,T]$ the space of ℓ times deiiferentiable real-valued functions on the respective domains. For the unknow function

$$(t,x) \to u(t,x) \quad ((t,x) \in \Omega_0)$$

we consider the equations

$$(1)\qquad \begin{aligned} \partial_t u + f\partial_x u + gu &= h \\ u(\bullet, x) = k(x) \qquad &(x \in [a,b]) \end{aligned}$$

where $f, g, h \in C^2(\Omega_0), k \in C^2[0,T]$ are given.

The solutions of the system (1) will be observed by the observation operator $\mathcal{O}$ defined as fallows:

$$(2)\qquad \begin{aligned} (\mathcal{O}_u)(t,x) := \int_0^t \int_0^x K(t.\tau, x, \xi) u(\tau, \xi) d\xi d\tau + \\ + K_0(t,x) u(t,x), \end{aligned}$$

where the functions

$$K : \Omega \to \mathbf{R} \quad , K_0 : \Omega \to \mathbf{R}$$

and their derivatives :

$$\partial_1 \partial_3 K : \Omega \to \mathbf{R}, \quad \partial_1 \partial_2 K_0 : \Omega_0 \to \mathbf{R}$$

are also continuous.

Our objective is the simultaneous identification of the system (1) with respect to f and g. The inhomogeneous part h of the equation (1) can be changed. Thus we can suppose that the functions $h_1, h_2, \dots, h_n$ are given. Then the corresponding solutions $u_1, u_2, \dots, u_n$ have known observations

$$o_i := \mathcal{O} u_i$$

By the Cauchy formula we can deduce that the solution of (1) is twice differentiable, however, the problem can not be solved by that formula because the latter implicitely involves the knowledge of f. For this reason, we must solve our problem directly, without using the mentioned formula.

From (2), by partial integration, using (1) we get

$$o_i(t,x) = (Ou_i)(t,x) = \int_0^x \int_0^t K(t,s,x,\xi)ds u_i(t,\xi)d\xi -$$

$$- \int_0^t \int_0^x \int_0^\tau K(t,s,x,\xi)ds \partial_\tau u_i(\tau,\xi)d\xi d\tau + K_0(t,x)u_i(\tau,x) =$$

$$= \int_0^x \int_0^t K(t,s,x,\xi)ds u_i(t,\xi)d\xi -$$

$$- \int_0^t \int_0^x \int_0^\tau K(t,s,x,\xi)ds h_i(\tau,\xi)d\xi d\tau +$$

$$+ \int_0^t \int_0^x \int_0^\tau K(t,s,x,\xi)ds(\partial_x u_i(\tau,\xi)f(\tau,\xi) +$$

$$+ u_i(\tau,\xi)g(\tau,\xi))d\tau d\xi + K_0(t,x)u_i(t,x).$$

Now, for $i = 1,\dots,n$ we define functions

$$p : \Omega_0 \to \mathbf{R}^1, \quad L : \Omega \to \mathbf{R}^{n \times 2}$$

by

$$p_i(t,x) := o_i(t,x) - K_0(t,x)u_i(t,x) -$$

$$- \int_0^x \int_0^t K(t,s,x,\xi)ds u_i(t,\xi)d\xi +$$

$$+ \int_0^t \int_0^x \int_0^\tau K(t,s,x,\xi)ds h_i(\tau,\xi)d\xi d\tau,$$

$$L_{i1}(t,\tau,x,\xi) := \int_0^\tau K(t,s,x,\xi)ds\partial_x u_i(\tau,\xi)$$

$$L_{i2}(t,\tau,x,\xi) := \int_0^\tau K(t,s,x,\xi)dsu_i(\tau,\xi),$$

$$(i = 1,\ldots,n).$$

Then the observation equation has the following form

$$\int_0^t \int_0^x L(t,\tau,x,\xi)\ell(\tau,\xi)d\tau d\xi = p(t,x) \tag{3}$$

where

$$\ell(\tau,\xi) := \begin{bmatrix} f(\tau,\xi) \\ g(\tau,\xi) \end{bmatrix}.$$

REMARKS. Unfortunately, (3) is Volterra equation of the first kind and p, L involve terms u_i resp. $\partial_x u_i$ which are unknown. From the original observation equations the functions u_i can be ontained equations the functions u_i can be obtained via Volterra equations of second kind :

$$u_i(t,x) + \int_0^t \int_0^x \frac{K(t,\tau,x,xi)}{K_0(t,x)} u_i(\tau,\xi)d\xi d\tau = \frac{o_i(t,x)}{K_0(t,x)}$$

provided that K_0 does not vanish anywhere. Clearly, if $\partial_x o_i$, $\partial_x K$, $\partial_x K_0$ are continuously differentiable then the functions $\partial_x u_i$ have the some property.

By differentiation of (3) we obtain a Volterra equation of the second kind : First differentiate with respect to t,

$$\int_0^x L(t,t,x,\xi)\ell(t,\xi)d\xi + \int_0^t \int_0^x \partial_t L(t,\tau,x,\xi)\ell(\tau,\xi)d\tau d\xi =$$

$$= \partial_t p(t,x).$$

Then differantiate with respect to x,

$$L(t,t,x,x)\ell(t,x) + \int_0^x \partial_x L(t,t,x,\xi)\ell(t,\xi)d\xi +$$

$$+ \int_0^t \partial_t L(t,\tau,x,x)\ell(\tau,\xi)d\tau +$$

$$+ \int_0^t \int_0^x \partial_x \partial_t L(t,\tau,x,\xi)\ell(\tau,\xi)d\tau d\xi = \partial_x \partial_t p(t,x).$$

In the following, we suppose that the term $\ell(t,x)$ can be found explicitely, that is, the rank of the matrix L equals 2 everywhere, or in other words, for the system $h_i, o_i (i = 1,2,\dots,n)$ the corresponding solutions $u_i (i = 1,2,\dots,n)$ of (1) satisfy the following condition : the vectors

$$u(t,x) := \begin{bmatrix} u_1(t,x) \\ u_n(t,x) \end{bmatrix}, \quad \partial_x u(t,x) = \begin{bmatrix} \partial_x u_1(t,x) \\ \partial_x u_n(t,x) \end{bmatrix}$$

are independent everywhere. In this case we say that the system of functions $h_1,\dots,h_n$ is *admissible* with respect to observations $o_1,\dots,o_n$.

Now we find $\ell(t,x)$:

The left multiplication by $((L^* L)^{-1} L^*)(t,t,x,x)$ provides

$$\ell(t,x) + \int_0^x (L^*(t,t,x,x)L(t,t,x,x)^{-1} L^*(t,t,x,x)\partial_x L(t,t,x,\xi)\cdot$$

$$\cdot \ell(t,\xi)d\xi + \int_0^t (L^*(\tau,t,x,x)L(t,t,x,x)^{-1} L^*(t,t,x,x)\cdot$$

$$\cdot\, \partial_t L(t,\tau,x,x)\ell(\tau,x)d\tau + \int_0^t \int_0^x (L^*(t,t,x,x)L(t,t,x,x)^{-1}\cdot$$
$$\cdot\, L^*(t,t,x,x)\partial_x \partial_t L(t,\tau,x,\xi)\ell(\tau,\xi)d\tau d\xi =$$
$$= (L^*(t,t,x,x)L(t,t,x,x)^{-1} L^*(t,t,x,x)\partial_x \partial_t p(t,x) =$$
$$=: q(t,x).$$

Denote by $M_1(t,x,\xi)$, $M_2(t,\tau,x)$ and $M_3(t,\tau,x,\xi)$ the kernels in the first, second and third integral respectively. Then we get equation

$$\ell(t,x) + \int_0^x M_1(t,x,\xi)\ell(t,\xi)d\xi + \int_0^t M_2(t,\tau,x)\cdot$$
$$\cdot\, \ell(\tau,x)d\tau + \int_0^t \int_0^x M_3(t,\tau,x,\xi)\ell(\tau,\xi)d\tau d\xi = \tag{4}$$
$$= q(t,x)$$

which is a Volterra equation of the second kind.

The three operators defined by the kernels M_1, M_2 and M_3 have 0 as their spectral radius thus, the equation can be solved uniquely for $\updownarrow$.

Theorem. *Let* $k \in C^2[a,b], o_i \in C(\Omega_0)$, $h_i \in C^2(\Omega_0)$ $(i = 1,2,\dots,n)$ *and* $K : \Omega \to \mathbf{R}$, $K_0 : \Omega_0 \to \mathbf{R}$ *be continuous functions such that*

a.)$\partial_1 \partial_2 o_i \in C(\Omega_0)$ $(i = 1,2,\dots,n)$,

b.)*the systems* h_i $(i = 1,2,\dots,n)$ *is admissible with respect to the observations* $o_i (i = 1,2,\dots,n)$,

c.)$\partial_1 \partial_3 K \in C(\Omega), \partial_1 \partial_3 K_0 \in C(\Omega_0)$, K_0 *does not vanish.*

Then the system (1) is identifiable simultaneously with respect to f and g.

REMARKS.

1. A similar theorem can be obtained for the following multidimensional case:

$$\partial_t u + \sum_{i=1}^{m} \partial_{x_i} u f_i + ug = h,$$

$$u(\cdot, x) = k(x).$$

2. By the rank condition, we have that $n \geq 2$. In the general case we obtain $n \geq m+1$. This means that we have $m+1$ functions to identify so we need at least $m+1$ observation equations.

References

[1] A.S.A. AL-JANABI, Identification of an age specific Population Growth Model, Conference on Mathematical System Theory, held in Salgótarján, Hungary, May 24-26, 1982.

[2] Z.VARGA and P. Juhász-Nagy, Some problems of Ecosystem Reserch, Conference on Mathematical System Theory, held in Salgótarján, Hungary, May 26-28, 1980. K.Marx Univ. of Economics Publ., 1981,pp. 52-57.

[3] A.S.A. AL-JANABI, Identification of Distributed parameter Observation Systems via Volterra Equations, Annales Univ. Sci. Budapest, Sectio Mathematica /to appear/

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