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AN INTEGRAL EQUATION FOR THE ELECTRO-MAGNETIC VECTOR POTENTIAL

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Dedicated to Prof. I. Kátai
on the occasion of his fiftieth birthday

1. About 1975, a research work in the mathematical modelling of the propagation of electromagnetic waves, and its application in geophysic, started at the Department of Numerical Analysis and Computer Science of the Eötvös Loránd University about in 1975. Later, this theme was inscribed in the plan of teamworks with the Lomonosov University, Moscow, and already a Collection of papers [1] was published in 1980, edited by V. I. Dmitriev and I. Kátai. In this work the Hungarian Eötvös Loránd Geophysical Institute also took part.

The topicality and importance of these researches, generally and mainly for finding and exploring ore deposits (first of all bauxite) in Hungary, has been explained in the paper by Tihonov, Dmitriev, Kátai and Szabadváry [2], the first work of the Collection of papers [1]. Now these researches are in progress at Computer Center of the Eötvös Loránd University too. The present paper is the prosecution of these researches.

In [3] an integral equation is derived for the electric field in stratified media containing inhomogeneous local bodies. The object of this paper is an integral equation for the vector potential of the electro-magnetic field in the same media. The solution of an integral equation for vector potential has some advantage in comparison with that of an integral equation for electric field.

2. It is known that if the time-dependence of a source inducing an electro-magnetic field is expressed in the form $\exp(-i\omega t)$, then Maxwell's equations can be rewritten by means of an integral transformation into a system of two vector equations of elliptic type. The vectors of the electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) fields can be expressed in terms of the vector potential $\mathbf{A}$ [4, 5]. If we suppose that the magnetic susceptibility μ is a constant, one obtains

$$\text{(1)} \qquad \begin{aligned} \mathbf{E} &= \frac{i\omega}{k^2}\left(\nabla \times (k^2\mathbf{A})\right), \\ \mathbf{H} &= \frac{k^2}{\mu}\mathbf{A} + \frac{1}{\mu}\nabla \operatorname{div}\mathbf{A}, \end{aligned}$$

where $k^2 = i\omega\mu\sigma$ ($\Re e\, ik < 0$), σ is the electric conductivity which may be a discontinuous function of the co-ordinates. If $\mathbf{f}$ is the density function of the magnetic sources in the medium then

$$\text{(2)} \quad L(k^2)\mathbf{A} \stackrel{\text{def}}{=} -\nabla \times \left(\frac{1}{k^2}\left(\nabla \times (k^2\mathbf{A})\right)\right) + \nabla \operatorname{div}\mathbf{A} + k^2\mathbf{A} = \mathbf{f}.$$

The function $\mathbf{A}$ satisfies the radiation conditions [5] at infinity. On the surfaces of discontinuity the conditions for $\mathbf{A}$ can be expressed in such a way that the tangential components of the vectors $\mathbf{E}$ and $\mathbf{H}$ are continuous.

Let the origin of the co-ordinate system be at some point of the surface of discontinuity of k, and Oz the direction of the normal to the surface. Now the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are as follows:

(3)
$$
\begin{aligned}
E_x &= \frac{i\omega}{k^2}\left(\frac{\partial(k^2A_z)}{\partial y} - \frac{\partial(k^2A_y)}{\partial z}\right),\\
E_y &= \frac{i\omega}{k^2}\left(\frac{\partial(k^2A_x)}{\partial z} - \frac{\partial(k^2A_z)}{\partial x}\right),\\
H_x &= \frac{k^2}{\mu}A_x + \frac{1}{\mu}\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right),\\
H_y &= \frac{k^2}{\mu}A_y + \frac{1}{\mu}\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right).
\end{aligned}
$$

At first suppose that div **A** has a discontinuity of the first kind:

$$[\text{div } \mathbf{A}]_{z=0} = d, \;\; 0 <| d |< \infty,$$

where

$$[F]_{z_i} \overset{\text{def}}{=} F(z_i + 0) - F(z_i - 0).$$

Then the derivative

$$\frac{\partial}{\partial z} \text{ div } A$$

may be characterized at the point z_0 by the δ-function, so H_z will not be bounded. Therefore $[\text{div } \mathbf{A}] = 0$. Thus by the continuity of H_x and H_y we have $[k^2A_x] = [k^2A_y] = 0$, hence A_x and A_y are bounded. Further, if $[A_z] \neq 0$, its derivative may be characterized by the δ-function and div **A** will not be bounded. Therefore $[A_z] = 0$. By the continuity of E_x and E_y one has

$$\left[\frac{1}{k^2}\frac{\partial(k^2A_x)}{\partial z}\right] = \left[\frac{1}{k^2}\frac{\partial k^2}{\partial x}\right]A_z, \quad \left[\frac{1}{k^2}\frac{\partial(k^2A_y)}{\partial z}\right] = \left[\frac{1}{k^2}\frac{\partial k^2}{\partial y}\right]A_z,$$

and finally

$$\left[\frac{\partial A_z}{\partial z}\right] = -\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial y}\right].$$

If the sources are electric, in [6,7] the expressions for **E** and **H** are given in terms of the vector potential and the type of the conditions on the surfaces of discontinuity.

3. Equation (2) is a three-dimensional vector equation having discontinuous coefficients defined in an infinite domain. It may be possible that by reducing this equation to an integral equation in a finite domain the problem becomes simpler.

Let $\mathbf{A}^n$ (the vector of the normal field) be the solution of (2) with $k = k_0$:

$$L(k_0^2)\mathbf{A}^n = \mathbf{f}. \tag{4}$$

It is supposed that solving equation (4) is easier than solving equation (2). Let $k \neq k_0$ in a local domain V_T only. The Green's function of the equation (4) is the fundamental matrix $\widehat{G}(\mathbf{R}, \mathbf{R}_0)$,

$$L(k_0^2)\widehat{G} = \widehat{\delta}(\mathbf{R} - \mathbf{R}_0),$$

where $\widehat{\delta}$ is a diagonal matrix and for columns of $\widehat{G}$

$$L(k_0^2)\mathbf{G}^x = (\delta(\mathbf{R} - \mathbf{R}_0), 0, 0)^T$$

etc. Then obviously

$$\mathbf{A}^n(\mathbf{R}) = \int \widehat{G}(\mathbf{R}, \mathbf{R}_0)\mathbf{f}(\mathbf{R}_0)dV_0. \tag{5}$$

The difference $\mathbf{A}^a = \mathbf{A} - \mathbf{A}^n$ (which is call by vector of anomalous field) satisfies the equation

$$L(k_0^2)\mathbf{A}^a = L(k_0^2)\mathbf{A} - \mathbf{f} \tag{6}$$

and therefore this equation is the same as (4) with density function of sources $L(k_0^2)\mathbf{A} - \mathbf{f}$. This density function differs from zero only in V_T. Using the fundamental matrix we have

$$\mathbf{A}^a(\mathbf{R}) = \int_{V_T} \widehat{G}(\mathbf{R}, \mathbf{R}_0)\left(L(k_0^2)\mathbf{A}(\mathbf{R}_0) - \mathbf{f}(\mathbf{R}_0)\right) dV_0. \tag{7}$$

If $\mathbf{R} \in V_T$, we have integro-differential equation in the finite domain V_T,

$$\mathbf{A}(\mathbf{R}) - \int_{V_T} \widehat{G}(\mathbf{R}, \mathbf{R}_0) L(k_0^2) \mathbf{A}(\mathbf{R}_0) dV_0 = 0. \tag{8}$$

Observe that if k and k_0 are piecewise constant functions, then

$$L(k^2)\mathbf{A} = \triangle \mathbf{A} + k^2 \mathbf{A},$$

and so the equation (2) and expressions (1) can be rewritten as

$$\triangle \mathbf{A} + k^2 \mathbf{A} = \mathbf{f}; \tag{9}$$

$$\begin{aligned} \mathbf{E} &= i\omega(\nabla \times \mathbf{A}), \\ \mathbf{H} &= \frac{k^2}{\mu}\mathbf{A} + \frac{1}{\mu}\nabla \operatorname{div} \mathbf{A}. \end{aligned} \tag{10}$$

Equation (9) is the three-dimentional Helmholtz equation. The source in equation (6) is

$$L(k_0^2)\mathbf{A} - \mathbf{f} = \triangle \mathbf{A} + k_0^2 \mathbf{A} - (\triangle \mathbf{A} + k^2 \mathbf{A}) = (k_0^2 - k^2)\mathbf{A}.$$

Therefore one can rewrite equation (6) and expression (7) as

$$\triangle \mathbf{A}^a + k_0^2 \mathbf{A}^a = (k_0^2 - k^2)\mathbf{A}, \tag{11}$$

$$\mathbf{A}^a(\mathbf{R}) = \int_{V_T} \left(k_0^2(\mathbf{R}_0) - k^2(\mathbf{R}_0)\right) \widehat{G}(\mathbf{R}, \mathbf{R}_0) \mathbf{A}(\mathbf{R}_0) dV_0 \tag{12}$$

and from (11) we have the integral equation [8]

$$\begin{aligned} \mathbf{A}(\mathbf{R}) + \int_{V_T} \left(k^2(\mathbf{R}_0) - k_0^2(\mathbf{R}_0)\right) \widehat{G}(\mathbf{R}, \mathbf{R}_0) \mathbf{A}(\mathbf{R}_0) dV_0 = \\ = \mathbf{A}^n(\mathbf{R}). \end{aligned} \tag{13}$$

4. Let the infinite domain be a stratified medium, where $k = k_0(z)$ is a piecewise constant function, and $k_0 = 0$ if $z < 0$. If the source of the electro-magnetic field is a magnetic dipole parallel to Oz and placed at the origin of the co-ordinate system, then we have

$$\triangle \mathbf{A}^n + k_0^2 \mathbf{A}^n = \Big(0,\ 0,\ -\ \mu m \delta(\mathbf{R})\Big)^T. \tag{14}$$

For A_x^n we can write

$$\triangle A_x^n + k_0^2(z) A_x^n = 0,$$
$$[k_0^2 A_x^n]_{z_i} = \left[\frac{\partial A_x}{\partial z}\right]_{z_i} = 0, \quad \lim_{|\mathbf{R}|\to\infty} |\ A_x^n\ | = 0,$$

where the planes $z = z_i$ are the surfaces of discontinuity. It is obvious that $A_x^n \equiv 0$ in the whole infinite domain. Analoguosly $A_y^n \equiv 0$, and so $\mathbf{A}^n = (0,\ 0,\ A_z^n)^T$ and

$$\triangle A_z^n + k_0^2 A_z^n = -\mu m \delta(\mathbf{R}), \tag{15}$$

where $A_z^n = A_z^n(r, z)$, $r = \sqrt{x^2 + y^2}$. Let us use Hankel's transformation

$$A_z^n(r, z) = \frac{\mu m}{2\pi} \int_0^\infty J_0(tr) v_0(t, z) t\ dt, \tag{16}$$

where v_0 satisfies the ordinary differential equation

$$\begin{gathered} \frac{d^2 v_0}{dz^2} - \beta^2 v_0 = -\delta(z),\ \beta^2 = t^2 - k_0^2,\ \Re e\ \beta > 0, \\ [v_0]_{z_i} = 0,\ \left[\frac{dv_0}{dz}\right]_{z_i} = \begin{cases} 0, & \text{if } z_i \neq 0, \\ -1, & \text{if } z_i = 0, \end{cases} \\ \lim_{|z|\to\infty} |\ v_0\ | = 0. \end{gathered} \tag{17}$$

The solution of equation (17) for piecewise constant k_0 $\big(k_0(z) = k_i,\ z_{i-1} < z < z_i\big)$ is given in [6].

5. Let us compute now the matrix $\widehat{G}$. For $\mathbf{G}^z$

$$\triangle \mathbf{G}^z + k_0^2 \mathbf{G}^z = \big(0,\ 0,\ \delta(\mathbf{R} - \mathbf{R}_0)\big)^T.$$

Thus, it is clear that this equation differs from (14) in that $\mathbf{R} - \mathbf{R}_0$ is used in the δ-function for $\mathbf{R}$. Therefore, in a similar manner $\mathbf{G}^z = (0,\ 0,\ G_z^z)^T$ and for G_z^z

$$\triangle G_z^z + k_0^2 G_z^z = \delta(\mathbf{R} - \mathbf{R}_0), \tag{18}$$

and by using the Hankel's transformation

$$G_z^z = \frac{1}{2\pi}\int_0^\infty J_0(tr)v(t, z, z_0)t\ dt, \tag{19}$$

where $\mathbf{R}_0 = (x_0, y_0, z_0)$, $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, we have

$$\frac{d^2 v}{dz^2} - \beta^2 v = \delta(z - z_0),$$

$$[v]_{z_i} = 0, \left[\frac{dv}{dz}\right]_{z_i} = \begin{cases} 0, & \text{if } z \neq z_0, \\ 1, & \text{if } z = z_0, \end{cases} \qquad \lim_{|z|\to\infty} |\ v\ | = 0.$$

This equation can be solved analoguosly as (17) (see [7]). If $z_0 \in (z_{m-1}, z_m)$, v can be written in this layer as

$$v = -\frac{1}{2\beta_m} e^{-\beta_m |z - z_0|} + c_v(t) e^{-\beta_m (z + z_0)}, \tag{20}$$

and by using Sommerfeld's integral [9] we get

$$\begin{aligned} G_z^z = &-\frac{1}{4\pi}\frac{e^{ik_m|\mathbf{R}-\mathbf{R}_0|}}{|\ \mathbf{R} - \mathbf{R}_0\ |} + \\ &+ \frac{1}{2\pi}\int_0^\infty J_0(tr)c_v(t)e^{-\beta_m(z+z_0)}t\ dt. \end{aligned} \tag{21}$$

Let us pass to the computing of $\mathbf{G}^x$:

$$\triangle \mathbf{G}^x + k_0^2 \mathbf{G}^x = (\delta(\mathbf{R} - \mathbf{R}_0),\ 0,\ 0)^T.$$

As in the case of $\mathbf{G}^z$, similarly $G_y^x \equiv 0$. For G_x^x we can write

$$\triangle G_x^x + k_0^2 G_x^x = \delta(\mathbf{R} - \mathbf{R}_0),$$

$$[k_0^2 G_x^x] = \left[\frac{\partial G_x^x}{\partial z}\right] = 0.$$

It is shown that $G_x^x = G_x^x(r, z, z_0)$ and so

$$G_x^x = \frac{1}{2\pi}\int_0^\infty J_0(tr)u(t, z, z_0)t\, dt \tag{22}$$

$$\frac{d^2u}{dz^2} - \beta^2 u = \delta(z - z_0),$$

$$[k_0^2 u]_{z_i} = 0, \quad \left[\frac{du}{dz}\right] = \begin{cases} 0, & \text{if } z_i \neq z_0, \\ 1, & \text{if } z_i = z_0, \end{cases} \quad \lim_{|z|\to\infty} | u |= 0.$$

The solution of this problem has the same structure as (20):

$$u = -\frac{1}{2\beta_m} e^{-\beta_m |z - z_0|} + c_u e^{-\beta_m (z + z_0)},$$

and for G_x^x we have

$$G_x^x = -\frac{1}{4\pi}\frac{e^{ik_m|\mathbf{R}-\mathbf{R}_0|}}{|\mathbf{R} - \mathbf{R}_0|} + \frac{1}{2\pi}\int_0^\infty J_0(tr)c_u(t)e^{-\beta_m(z+z_0)}t\, dt. \tag{23}$$

For G_z^x the following are the equation and the conditions to be satisfied on the surfaces of discontinuity

$$\triangle G_z^x + k_0^2 G_z^x = 0, \tag{24}$$

$$(25) \qquad [G_z^x] = 0, \quad \left[\frac{\partial G_z^x}{\partial z}\right]_{z_i} = -\left[\frac{\partial G_x^x}{\partial x}\right]_{z_i}.$$

We notice that since $G_x^x = G_x^x(r, z, z_0)$, G_z^x is a function of $x - x_0$, $y - y_0$, z and z_0.

Let us define the function B as the solution of the following problem:

$$(26) \qquad \begin{gathered} \triangle B + k_0^2 B = 0, \\ [B]_{z_i} = 0, \quad \left[\frac{\partial B}{\partial z}\right]_{z_i} = -[G_x^x]_{z_i}, \quad \lim_{|\mathbf{R}| \to \infty} | B | = 0. \end{gathered}$$

It is obvious that $B = B(r, z, z_0)$, and then Hankel's transformation can be used:

$$B = \frac{1}{2\pi} \int_0^\infty J_0(tr) w(t, z, z_0) t \, dt,$$

$$(27) \qquad \begin{gathered} \frac{d^2 w}{dz^2} - \beta^2 w = 0, \\ [w]_{z_i} = 0, \quad \left[\frac{dw}{dz}\right]_{z_i} = -[u]_{z_i}, \quad \lim_{|z| \to \infty} | w | = 0. \end{gathered}$$

It is not difficult to solve this equation. If $z_0 \in (z_{m-1}, z_m)$, in this layer

$$(28) \qquad w = c_w(t) e^{-\beta_m (z + z_0)}.$$

We remark that if the number of layers in the stratified medium is not large, the functions v_0, v, u and w can be written in explicit form.

We show that

$$G_z^x = \frac{\partial B}{\partial x}, \tag{29}$$

because G_z^x (29) satisfies equation (24) and the conditions on the discontinuous surfaces as well.

The computation of the vector $\mathbf{G}^y$ can be done analoguosly. It is obvious that $G_y^x = G_x^x = G$, $G_x^y \equiv 0$ and $G_z^y = \partial B/\partial y$, where B is given by (27). So

$$\widehat{G} = \begin{pmatrix} G & 0 & 0 \\ 0 & G & 0 \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} & G_z^z \end{pmatrix} \tag{30}$$

i.e. $\widehat{G}$ has four different nonzero elements.

6. We now determine a vector potential $\mathbf{A}$ from the integral equation (13). Suppose that the anomalous body V_T with constant electric conductivity σ_T is placed in one of the layers (z_{m-1}, z_m). Then

$$k(\mathbf{R}) = \begin{cases} k_0(z), & \text{if } \mathbf{R} \notin V_T, \\ k_T, & \text{if } \mathbf{R} \in V_T. \end{cases}$$

For the numerical solution of (13), let us divide V_T into elementary subdomains V_i:

$$V_T = \bigcup_{i=1}^{M} V_i \tag{31}$$

so that in each V_i $\mathbf{A}$ can be taken as constant and this constant value should be equal to the value at some point inside the subdomain

$$\mathbf{A}(\mathbf{R}) = \mathbf{A}(\mathbf{R}_i), \ \mathbf{R}, \mathbf{R}_i \in V_i.$$

Now, the integral equation (13) has been reduced to the system of algebraic vector equations (see [3,8])

$$\tag{32} \begin{aligned} &\mathbf{A}(\mathbf{R}_i)+\sum_{j=1}^{M}\widehat{\alpha}_{ij}\mathbf{A}(\mathbf{R}_j) = \mathbf{A}^n(\mathbf{R}_i),\ i=1,\dots,M\ ,\\ &\widehat{\alpha}_{ij} = (k_T^2-k_m^2)\int\limits_{V_j}\widehat{G}(\mathbf{R}_i,\mathbf{R}_0)dV_0. \end{aligned}$$

The elements of the matrices $\widehat{\alpha}_{ij}$ can be computed by the method proposed in [10, 11, 12] for the integral equation of the electric field vector.

One can notice the difference between the computation of the $\widehat{\alpha}_{ij}$ here and the papers cited above. Here, for $i=j$, the matrix $\widehat{\alpha}_{ii}$ has no nonintegrable singularity of $O(|\mathbf{R}-\mathbf{R}_0|^{-3})$ if $|\mathbf{R}-\mathbf{R}_0|\to 0$, but only a singularity of $O(|\mathbf{R}-\mathbf{R}_0|^{-1})$. Such a singularity is integrable and it vanishes if, for example, a spherical co-ordinate system is used. Further, in the cited papers the fundamental matrix can have nine different elements, while here it can have four different nonzero elements, and its structure is simpler too.

The common property is that the main parts of the integrals G and G_z^z containing singularities of $O(|\mathbf{R}-\mathbf{R}_0|^{-1})$ can be separated (see (18) and (23)). Likewise, as G_z^x and G_z^y, the remaining parts of G and G_z^z have no singularities. So the main parts of the integrals from the diagonal elements of $\widehat{G}$ are triple integrals on local domain. The remaining parts of the integrals from diagonal elements of $\widehat{G}$ as well as the integrals $\alpha_{ij,z}^x$ and $\alpha_{ij,z}^y$ are quadruple integrals but their absolute value is much smaller than that of above integrals. Since the integrands are smooth and contain no singularities, in their integration with respect to x_0 and y_0 quadrature formulae having no high powers can be used. Integration with respect to z_0 can be done analitically. It is advantageous to use the quadrature proposed in [13] for integration with respect to t.

After the coefficient matrices $\hat{\alpha}_{ij}$ have been computed, the values of the vector potential **A** at the points $\mathbf{R}_i \in V_T$ can be obtained as the solution of the system of algebraic equations (32).

If the source of the electro-magnetic field is a magnetic dipole parallel to Oz then in the normal vector potential $\mathbf{A}^n$ there is only one nonzero component A_z^n. Because of the same structure of the matrices $\hat{\alpha}_{ij}$ and the fundamental matrix $\hat{G}$ it is seen that in this case

$$A_x(\mathbf{R}_i) = A_y(\mathbf{R}_i) = 0, \; i = 1, \ldots, M, \tag{33}$$

and so for $A_z(\mathbf{R}_i)$ only $\alpha_{ij,z}^z$ are used.

7. After the vector potential **A** has been obtained at every point $\mathbf{R}_i \in V_T$, formula (12) can be used for the computation of the anomalous field $\mathbf{A}^a$ throughout the domain. It should be noticed however that division (31) of the domain V_T and supposition that **A** is constant in V_i are equivalent if the density function of sources $(k_0^2 - k^2)\mathbf{A}(\mathbf{R})$ is changed by a set of dipoles with space-charge density $(k_0^2-k^2)\mathbf{A}(\mathbf{R}_i)V_i$ placed at the points $\mathbf{R}_i$. Since $\mathbf{A}^a$ satisfies the linear equation (11), it can be solved for each of the dipoles and after this the results can be summed up. In addition the fundamental matrix $\hat{G}$ can be used

$$\mathbf{A}^a(\mathbf{R}) = \sum_{i=1}^{M}(k_m^2 - k_T^2)V_i\hat{G}(\mathbf{R},\mathbf{R}_i)\mathbf{A}(\mathbf{R}_i). \tag{34}$$

This vector together with vector of the normal field $\mathbf{A}^n$ (5) will give the vector potential **A**. For obtaining the vectors of the electric (**E**) and magnetic (**H**) fields, formula (10) is useful.

If the normal and anomalous parts of **E** and **H** are separated, $\mathbf{E}^n$ and $\mathbf{H}^n$ can be obtained by differentiating the integral (5). If the source is a dipole, the integral (16) should be used. For $\mathbf{E}^a$ and

$\mathbf{H}^a$ we have to differentiate the sum (34) in which only $\widehat{G}$ depends on $\mathbf{R}$. Using the structure of the matrix $\widehat{G}$ it can be shown that

$$E_x = i\omega(k_m^2 - k_T^2)\sum_i V_i\left(\frac{\partial^2 B}{\partial x \partial y}A_x + \right.$$

$$\left. + \left(\frac{\partial^2 B}{\partial y^2} - \frac{\partial G}{\partial z}\right)A_y + \frac{\partial G_z^z}{\partial y}A_z\right),$$

$$E_y = i\omega(k_m^2 - k_T^2)\sum_i V_i\left(\left(\frac{\partial G}{\partial z} - \frac{\partial^2 B}{\partial x^2}\right)A_x - \right.$$

$$\left. - \frac{\partial^2 B}{\partial x \partial y}A_y - \frac{\partial G_z^z}{\partial y}A_z\right),$$

$$E_z = i\omega(k_m^2 - k_T^2)\sum_i V_i\left(\frac{\partial G}{\partial x}A_y - \frac{\partial G}{\partial y}A_x\right),$$

$$\mu H_x = (k_m^2 - k_T^2)\sum_i\left(\left(k_0^2 G + \frac{\partial^3 B}{\partial x^2 \partial z}\right)A_x + \right.$$

$$\left. + \frac{\partial^3 B}{\partial x \partial y \partial z}A_y + \mathbf{A}\frac{\partial}{\partial x}\nabla \mathbf{G}\right),$$

$$\mu H_y = (k_m^2 - k_T^2)\sum_i\left(\frac{\partial^3 B}{\partial x \partial y \partial z}A_x + \right.$$

$$\left. + \left(k^2 G + \frac{\partial^3 B}{\partial y^2 \partial z}\right)A_y + \mathbf{A}\frac{\partial}{\partial y}\nabla \mathbf{G}\right),$$

$$\mu H_z = (k_m^2 - k_T^2)\sum_i\left(\left(k_0^2\frac{\partial B}{\partial x} + \frac{\partial^3 B}{\partial x \partial z^2}\right)A_x + \right.$$

$$\left. + \left(k_0^2\frac{\partial B}{\partial y} + \frac{\partial^3 B}{\partial y \partial z^2}\right)A_y + + k_0^2 G_z^z A_z + \mathbf{A}\frac{\partial}{\partial z}\nabla \mathbf{G}\right),$$

where $\mathbf{G} = (G, G, G_z^z)^T$. Here the assumption $\mathbf{R} \notin V_T$ has been used and so $k = k_0(\mathbf{R})$.

If the source is a magnetic dipole parallel to Oz then from (33) $\mathbf{A}(\mathbf{R}_i) = (0,\ 0,\ A_z(\mathbf{R}_i))^T$ and

$$E_x = i\omega(k_m^2 - k_T^2)\sum_i V_i \frac{\partial G_z^z}{\partial y} A_z,$$

$$E_y = i\omega(k_m^2 - k_T^2)\sum_i \left(-V_i \frac{\partial G_z^z}{\partial x} A_z\right),$$

$$E_z = 0,$$

$$\mu\mathbf{H} = (k_m^2 - k_T^2)\sum_i V_i \left(k_0^2 G_z^z \mathbf{A} + A_z \nabla \frac{\partial G_z^z}{\partial z}\right).$$

Finally we would like to remark that an analogous investigation can be done if the source of the electro-magnetic field is an electric dipole perpendicular to Oz. Furthermore, this method can be used for some local bodies in stratified media as well [14].

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