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ON WALD-TYPE INEQUALITIES

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Dedicated to Professor Imre Kátai on his 50^{th} birthday

Abstract. We deduce estimates for the supremum of the stopped partial sums of i.i.d. random variables with zero mean when the stopping time is a.s. finite.If the order of the desired moment of the supremum is not less than 2 the obtained inequalities are two-sided and exact up to a constant. For the absolute moments less than 2 it is not possible to obtain two-sided inequalities in general (see [8, Example 8.2. and page 281]). In this case we give an upper estimate for the desired moment of the supremum. These estimates are given in terms of the corresponding moment of the stopping time and of the i.i.d. random variables. These yield estimates for the corresponding moment of the a.s. limit of the stopped partial sums. In case of nonzero mean i.i.d. random variables the absolute moments of the stopped sums are also estimated by the corresponding moment of the stopping time and of the i.i.d. summands. These are given in the form of necessary and sufficient conditions. The obtained results improve a part of the results of paper [9].

1. Introduction

Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with finite expectation $E(Y_1) = a$. Let $S_0 = 0$ and $S_n = Y_1 + \ldots + Y_n$, $n \geq 1$, be the corresponding random walk.

Consider the σ-field $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$, $n \geq 1$, and let $\mathcal{F}_0 = (0, \Omega)$ be the trivial σ-field. Then the sequence $\{S_n - an\}_{n=0}^{\infty}$ is a martingale with respect to the increasing sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$ of σ-fields.

Given two real numbers a and b we introduce the notation

$$\min(a, b) = a \wedge b.$$

If ν is a stopping time with respect to $\{\mathcal{F}_n\}_{n=1}^{\infty}$ such that $P(\nu < +\infty) = 1$, then the sequence

$$\{S_{\nu \wedge n} - a(\nu \wedge n), \mathcal{F}_n\}_{n=0}^{\infty}$$

is the martingale $\{S_n - an, \mathcal{F}_n\}$ stopped at the moment ν. The a.s. limit of $\{S_{\nu \wedge n} - a(\nu \wedge n)\}$ is $S_\nu - a\nu$ as $n \to +\infty$, where

$$S_\nu - a\nu = \sum_{n=1}^{\infty} (S_n - na)\chi(\nu = n) = \sum_{i=1}^{\infty} (Y_i - a)\chi(+\infty > \nu \geq i)$$

and $\chi(A)$ stands for the indicator of the event A. Thus on the set $\{\nu = +\infty\}$ the random variable $S_\nu - a\nu$ is defined to be equal to 0. This is not an essential restriction since $P(\nu = +\infty) = 0$ and on the event $\{\nu = +\infty\}$ the symbol $S_\infty - a(+\infty)$ is not defined.

The following Wald identities are well-known:

a/ if $E(\nu) < +\infty$, then $E(S_\nu - a\nu) = 0$ $(E(S_\nu) = aE(\nu))$,

b/ if $E(\nu) < +\infty$ and $\sigma^2 = E((Y_i - a)^2) < +\infty$, then

$$E((S_\nu - a\nu)^2) = \sigma^2 E(\nu).$$

Let $X_n = S_{\nu \wedge n} - a(\nu \wedge n)$ and put $X^* = \sup_{n \geq 1} | X_n |$. The purpose of the present note is to give one- or two-sided inequalities for $E(X^{*p})$, where $1 \leq p < +\infty$. Since $| S_\nu - a\nu | \leq X^*$, in such a way we can obtain one- or two-sided inequalities for the moment $E(| S_\nu - a\nu |^p)$. These will be called Wald-type inequalities. E.g. one of them is the generalization of the two-sided inequality

$$c_p \sigma^p n^{p/2} \leq E(| \sum_{i=1}^{n} (Y_i - a) |^p) \leq C_p E(| Y_1 - a |^p) n^{p/2}$$

due to Marcinkiewicz and Zygmund , which is valid for $2 \le p < +\infty$. Here $c_p > 0$ and $C_p > 0$ are constants depending only on p. The proof of the assertions will be based on the use of the Burkholder–Davis–Gundy inequality and a Rosenthal-type inequality for martingales (see [1] and [3]).

It is also important to give estimates for the absolute p-th moment of the stopped sum S_ν of the random variables $Y_1, Y_2, \dots$. We do not suppose in this case that the Y_i's are centered at their expectation. Here

$$S_\nu = \sum_{n=1}^{\infty} S_n \chi(\nu = n) = \sum_{i=1}^{\infty} Y_i \chi(+\infty > \nu \ge i).$$

This will be done at the end of the present note in form of necessary and sufficient conditions. These results improve a part of the results obtained by A. Gut and S. Janson [9].

2. Wald-type inequalities

Given a martingale $(X_n, \mathcal{F}_n)_{n=0}^{\infty}$ let $d_0 = 0$ and $d_i = X_i - X_{i-1}$, $i \ge 1$, be its difference sequence. The quadratic variation of the martingale is given by the relation

$$S = S(X) = (\sum_{i=1}^{\infty} d_i{}^2)^{1/2}.$$

Let

$$X^* = \sup_{n \ge 1} | X_n |$$

be the maximal function of the martingale [2]. The Burkholder–Davis–Gundy inequality (see [1, Theorem 15.1]) says that for $1 \le p < +\infty$ we have

$$c_p E(S^p) \le E(X^{*p}) \le C_p E(S^p)$$

provided that any of the expectations $E(S^p)$ and $E(X^{*p})$ is finite. Here $c_p > 0$ and $C_p > 0$ are constants depending only on p. In this case $\{X_n\}_{n=1}^{\infty}$ is uniformly integrable and thus there exists a random variable $X \in L_p$ such that $X_n = E(X \mid \mathcal{F}_n)$ a.s. for every $n \geq 1$. Here X is the a.s. limit of the martingale $(X_n, \mathcal{F}_n)$. If $S \in L_p$ then we say that X belongs to the Hardy space $\mathcal{H}_p$. In this case we put

$$\| X \|_{\mathcal{H}_p} = \| S \|_p .$$

It is easy to see that $\| \cdot \|_{\mathcal{H}_p}$ is a norm on $\mathcal{H}_p$. By the above inequality $X \in \mathcal{H}_p$ if and only if $X^* \in L_p$.

We also make use of the following result of one of the present authors (see [3, Theorem 1 and Theorem 2]). Let

$$s = s(X) = (\sum_{i=1}^{\infty} E(d_i{}^2 \mid \mathcal{F}_{i-1}))^{1/2}$$

be the so called conditional quadratic variation of the square integrable martingale $(X_n, \mathcal{F}_n)$. Then for $p \geq 2$ we have

$$c_p\{E(s^p) + \sum_{i=1}^{\infty} E(| d_i |^p)\} \leq E(| X |^p) \leq E(X^{*p}) \leq$$

$$\leq C_p\{E(s^p) + \sum_{i=1}^{\infty} E(| d_i |^p)\},$$

where $c_p > 0$ and $C_p > 0$ are constants depending only on p. (See also [1].) Here X is the a.s. limit of $\{X_n, \mathcal{F}_n\}$.

In the case of the stopped random walk the martingale differences are $d_i = Y_i \chi(\nu \geq i)$, $i \geq 1$. Remark that Y_i and $\chi(\nu \geq i)$ are independent.

Theorem 1. a) *The a.s. limit $S_\nu - a\nu$ of the stopped martingale*

$$\{X_n = S_{\nu \wedge n} - a(\nu \wedge n), \mathcal{F}_n\}_{n=0}^{\infty}$$

belongs to the Hardy-space $\mathcal{H}_p$ for some p, where $1 \leq p < +\infty$, if and only if the condition

$$(*) \qquad E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right) < +\infty$$

holds. In this case for $p = 1$ we have

$$E(| S_\nu - a\nu |) \leq C_1 E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2 \ \chi(\nu \geq i)\right)^{1/2}\right),$$

whilst for $p > 1$

$$(\frac{p}{p-1})^{-p} c_p E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right) \leq$$

$$\leq E(| S_\nu - a\nu |^p) \leq E(X^{*p}) \leq$$

$$\leq C_p E\left(\left(\sum_{i=1}^{\infty}(Y_i - a))^2\chi(\nu \geq i)\right)^{p/2}\right),$$

where $c_p > 0$ and $C_p > 0$ are constants depending only on p.

b) *For $p \geq 2$ the random variable $S_\nu - a\nu$ belongs to $\mathcal{H}_p$ if and only if the expectations $E(| Y_1 |^p)$and $E(\nu^{p/2})$are finite. In this case we have the inequality*

$$c_p[\sigma^p E(\nu^{p/2}) + E(| Y_1 - a |^p)E(\nu)] \leq E(| S_\nu - a\nu |^p) \leq$$
$$\leq E(X^{*p}) \leq C_p[\sigma^p E(\nu^{p/2}) + E(| Y_1 - a |^p)E(\nu)].$$

Here $c_p > 0$ and $C_p > 0$ are constants depending only on p.

Proof. a) The differences of the stopped martingale $(X_n, \mathcal{F}_n)$ are $d_1 = (Y_1 - a)\chi(\nu \geq 1) = Y_1 - a$, and $d_i = (Y_i - a)\chi(\nu \geq i)$, $i = 2, 3, \ldots$. By the Burkholder–Davis–Gundy inequality (see [1, Theorem 15.1]) for arbitrary $p \geq 1$ we have

$$c_p E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right) \leq E(X^{*p}) \leq$$

$$\leq C_p E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right),$$

where $c_p > 0$ and $C_p > 0$ are constants depending only on p. Consequently, $X^* \in L_p$, i.e. $S_\nu - a\nu \in \mathcal{H}_p$, if and only if

$$(*) \qquad E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right) < +\infty$$

holds. In this case for $p = 1$

$$E(| S_\nu - a\nu |^p) \leq E(X^*) \leq C_1 E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right),$$

whilst for $p > 1$ by Doob's maximal lemma and by the above inequality

$$\left(\frac{p}{p-1}\right)^{-p} c_p E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right) \leq E(| S_\nu - a\nu |^p) \leq$$

$$\leq E(X^*) \leq C_p E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right).$$

b) Let us now turn to the case $p \geq 2$. Then by Theorem 1 and Theorem 2 of [3] the inequalities

$$c_p\{E\left(\left(\sum_{i=1}^{\infty} E((Y_i - a)^2 \mid \mathcal{F}_{i-1})\chi(\nu \geq i)\right)^{p/2}\right) +$$

$$+\sum_{i=1}^{\infty} E(\mid Y_i - a \mid^p \chi(\nu \geq i))\} \leq E(\mid S_\nu - a\nu \mid^p) \leq E(X^{*p}) \leq$$

$$\leq C_p\{E\left(\left(\sum_{i=1}^{\infty} E((Y_i - a)^2 \mid \mathcal{F}_{i-1})\chi(\nu \geq i)\right)^{p/2}\right) +$$

$$+\sum_{i=1}^{\infty} E(\mid Y_i - a \mid^p)\chi(\nu \geq i))\}$$

are satisfied, where c_p and C_p are positive constants depending only on p. From this we get

$$c_p\{E(\mid Y_1 - a \mid^p)E(\nu) + \sigma^p E(\nu^{p/2})\} \leq E(\mid S_\nu - a\nu \mid^p) \leq E(X^{*p}) \leq$$

$$\leq C_p\{E(\mid Y_1 - a \mid^p)E(\nu) + \sigma^p E(\nu^{p/2})\}.$$

Here we have used that $(Y_i - a)^2$ and $\mathcal{F}_{i-1}$ as well as $\mid Y_i - a \mid^p$ and $\chi(\nu \geq i)$ are independent. Since $E(\nu) \leq E(\nu^{p/2})$ and $\sigma^p \leq E(\mid Y_1 - a \mid^p)$, the above inequality means that $S_\nu - a\nu$ belongs to $\mathcal{H}_p$ and

$$c_p\{E(\mid Y_1 - a \mid^p)E(\nu) + \sigma^p E(\nu^{p/2})\} \leq E(\mid S_\nu - a\nu \mid^p) \leq$$

$$\leq C_p\{E(\mid Y_1 - a \mid^p)E(\nu) + \sigma^p E(\nu^{p/2})$$

if and only if $E(\nu^{p/2}) < +\infty$ and $E(\mid Y_1 \mid^p) < +\infty$. □

REMARKS. Part b) of Theorem 1 gives an exact estimate for $E(\mid S_\nu - a\nu \mid^p)$, where $2 \leq p < +\infty$.

The following remarks and observations help us to use the necessary and sufficient condition (*) in case when $1 \leq p < 2$.

1/ If the necessary and sufficient condition (*) is satisfied for some p, $1 \leq p < +\infty$, then necessarily $E(|Y_1|^p) < +\infty$. In fact, under this condition

$$E(|Y_1 - a|^p) = E\left(((Y_1 - a)^2\chi(\nu \geq 1))^{p/2}\right) \leq$$
$$\leq E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right) < +\infty.$$

This generalizes a result of [9] where the condition $E(\nu) < +\infty$ is also assumed.

2/ The necessary and sufficient condition (*) is satisfied if the expectations $E(|Y_1|^p)$ and $E(\nu)$ are finite. In fact, in this case

$$E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \geq i)\right)^{p/2}\right) \leq$$
$$\leq E\left(\sum_{i=1}^{\infty}|Y_i - a|^p\,\chi(\nu \geq i)\right) =$$
$$= E\left(\sum_{i=1}^{\nu}|Y_i - a|^p\right) = E(|Y_1 - a|^p)E(\nu),$$

where we have used the fact $p/2 < 1$ and the equation a/ of Wald. Since $|S_\nu - a\nu| \leq X^*$, we deduce from part a) of Theorem 1 that

$$E(|S_\nu - a\nu|^p) \leq E(X^{*p}) \leq C_pE(|Y_1 - a|^p)E(\nu)$$

with the same constant C_p as in part a) of Theorem 1.

3/ If $\sigma^2 = E((Y_1 - a)^2)$ is finite then by the concavity inequality (see [6, Theorem 1])

$$(\frac{p}{2})^{p/2}E\left(\left(\sum_{i=1}^{\infty}(Y_i-a)^2\chi(\nu\geq i)\right)^{p/2}\right)\leq$$

$$\leq E\left(\left(\sum_{i=1}^{\infty}E((Y_i-a)^2\mid F_{i-1})\chi(\nu\geq i)\right)^{p/2}\right)\leq\sigma^pE(\nu^{p/2}).$$

Consequently, (*) is satisfied if $E(\nu^{p/2})<+\infty$ *and* $\sigma^2<+\infty$. In this case we have

$$E(\mid S_\nu-a\nu\mid^p)\leq(\frac{p}{2})^{p/2}E(\nu^{p/2})\sigma^pC_p,$$

where C_p is a constant depending only on p. A generalization of this remark will be given in Theorem 4.

4/ Suppose that $P(\mid Y_1-a\mid\geq K)=1$, where $K>0$ is a constant. Then

$$E\left(\left(\sum_{i=1}^{\infty}(Y_i-a)^2\chi(\nu\geq i)\right)^{p/2}\right)\geq K^pE(\nu^{p/2}).$$

Consequently, if (*) is satisfied and $P(\mid Y_1-a\mid\geq K)=1$, $K>0$, then necessarily $E(\nu^{\frac{p}{2}})<+\infty$.

5/ From 3/ and 4/ it follows that if $P(K_2\geq\mid Y_1-a\mid\geq K_1)=1$ with some constants $K_2>K_1>0$, then

$$K_1^pE(\nu^{p/2})\leq E\left(\left(\sum_{i=1}^{\infty}(Y_i-a)^2\chi(\nu\geq i)\right)^{p/2}\right)\leq K_2^pE(\nu^{p/2}).$$

Therefore in this special case the necessary and sufficient condition (*) is equivalent to the condition $E(\nu^{p/2})<+\infty$.

6/ Burkholder and Gundy [8] established the following two-sided maximal inequality: let $0 < p < 2$ and suppose that $E(Y_1) = 0$, Var $Y_1 = 1$, $E(| Y_1 |) = d > 0$. Let ν be a stopping time with respect to the increasing sequence of σ -fields $\{\mathcal{F}_n\}_{n=1}^{\infty}$ satisfying $P(\nu < +\infty) = 1$. Then

$$c_{p,d}E(\nu^{p/2}) \leq E(X^{\star p}) \leq C_p E(\nu^{p/2}),$$

where $C_p > 0$ and $c_{p,d} > 0$ are constants. Remark that the constant $c_{p,d}$ on the left-hand side depends also on $d = E(| Y_1 |)$, whilst C_p is universal, depending only on p. Especially, it follows that for $0 < p < 2$

$$E(| S_\nu - a\nu |^p) \leq C_p E(\nu^{p/2})$$

and for $1 < p < 2$, by the maximal inequality of Doob ,

$$\left(\frac{p}{p-1}\right)^{-p} c_{p,d}E(\nu^{p/2}) \leq E(| S_\nu - a\nu |)^p) \leq C_p E(\nu^{p/2}).$$

Note that Example 8.2 of [8] shows that for $0 < p < 2$ there is no two-sided maximal inequality with an absolute constant c_p on the left-hand side.

7/ The a.s. limit $S_\nu - a\nu$ of the stopped martingale $\{S_{\nu\wedge n} - a(\nu \wedge n)\}$ can belong to L_p without the conditions of Theorem 1. Let us consider e.g. the first recurrence time ν to the origin in a symmetric random walk, where $P(Y_i = \pm 1) = 2^{-1}$. We then trivially have $P(\nu < +\infty) = 1$ and $S_\nu \equiv 0$. Consequently, $E(| S_\nu |^p) = 0$ for every $p \geq 1$. However,

$$X^\star = \sup_{n \geq 1} | S_{\nu\wedge n} |$$

does not belong to L_p which would ensure the uniform integrability of $\{S_{\nu\wedge n}\}$. $\{S_{\nu\wedge n}\}$ does not converge in L_p to its a.s. limit

$S_\nu = 0$. This follows from Remark 5/ according to which the condition (*) is not satisfied since for every $p \geq 1$ we have $E(\nu^{p/2}) = +\infty$.

Theorem 1 and the assertions below ensure that the stopped walk centered at the mean converges to its a.s. limit in L_p by stating that under the conditions of the assertion $X^\star \in L_p$. This fact, by means of the Doob maximal lemma, implies that $E(| S_\nu - a\nu |^p) < +\infty$.

3. The concave case

For $0 < p < 2$ it is not possible in general to obtain a two-sided inequality for $E(| S_\nu - a\nu |^p)$. In this connection we refer to [8, Example 8.2] . See also Remark 6/ after Theorem 1. However, an upper estimate can easily be given by using a result of Y. S. Chow and H. Teicher (see [5, Chapter 11, Section 3, Theorem 3]), which is a modification of a theorem of Burkholder (see [1, Theorem 20.2]) .

Let $\Phi(x)$ be a concave Young function, i.e. of the form

$$\Phi(x) = \int_0^x \varphi(t)dt, \ x \geq 0,$$

where $\varphi(t)$ is right-continuous, nonincreasing and nonnegative so that its integral is finite for every $x > 0$ with $\Phi(0) = 0$. Let further $(X_n, \mathcal{F}_n)$ be a martingale with the difference sequence $d_0 = 0, d_1, d_2, \ldots$. For $\alpha \in [1,2]$ let $s_\alpha(X)$ be the so called conditional variation of order α of the martingale, i.e.

$$s_\alpha(X) = \left(\sum_{i=1}^{\infty} E(| d_i |^\alpha \mid \mathcal{F}_{i-1})\right)^{1/\alpha}.$$

The result of Chow and Teicher is now the following:

Let Φ be a concave Young function and $\alpha \in [1,2]$ a fixed number. If $(X_n, \mathcal{F}_n)$ is a martingale with the difference sequence $d_0 = 0, d_1, d_2, \ldots$ such that $E(| d_i |^\alpha) < +\infty$ for $i = 1, 2, \ldots,$ then there exists a constant $A = A_\alpha$ such that

$$E(\Phi(X^{\star\alpha})) \leq A_\alpha E(\Phi(s_\alpha^\alpha(X))),$$

where

$$s_\alpha^\alpha(X) = \sum_{i=1}^{\infty} E(| d_i |^\alpha \mid \mathcal{F}_{i-1}).$$

We are now able to formulate our

Theorem 3. *Let Φ be a concave Young function and suppose that for some $\alpha \in [1,2]$ the expectation $M_\alpha = E(| Y_1 - a |^\alpha)$ is finite. Consider the stopped martingale $(S_{\nu\wedge n} - a(\nu \wedge n), \mathcal{F}_n)$ and its a.s. limit $S_\nu - a\nu$. Then there exists a constant A_α such that*

$$E(\Phi(X^{\star\alpha})) \leq A_\alpha \max(1, M_\alpha) E(\Phi(\nu))$$

and the same inequality holds for $E(\Phi(| S_\nu - a\nu |^\alpha))$.

Proof. We only have to use the above theorem of Chow and Teicher and to note that in our case

$$s_\alpha^\alpha(X) =$$

$$= \sum_{i=1}^{\infty} E(| Y_i - a |^\alpha \mid \mathcal{F}_{i-1}) \chi(\nu \geq i) = M_\alpha \sum_{i=1}^{\infty} \chi(\nu \geq i) = M_\alpha \nu.$$

It follows that

$$E(\Phi(| S_\nu - a_\nu |^\alpha)) \leq E(\Phi(X^{\star\alpha})) \leq A_\alpha E(\Phi(M_\alpha \nu)).$$

Note that for arbitrary $a > 0$ we have $\Phi(ax) \leq \max(1, a)\Phi(x)$. □

Theorem 4. *Let $0 < p \leq \alpha \leq 2$. Then there exists a constant $A = A_\alpha$ such that*

$$E(| S_\nu - a\nu |^p) \leq E(X^{\star p}) \leq A_\alpha M_\alpha^{p/\alpha} E(\nu^{p/\alpha}), \alpha \in [1,2].$$

Proof. Consider the concave function $\Phi(x) = x^{p/\alpha}$ and apply the preceding theorem. □

4. The moments of S_ν

By means of the results of Section 2 we now improve some results of paper [9].

Theorem 5. *Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables and suppose that $P(Y_1 > 0) > 0$. Let ν be a stopping time with respect to the increasing sequence $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$, $n \geq 1$, of σ-fields which satisfies $P(\nu < +\infty) = 1$. Consider the corresponding stopped random walk, i.e. let $S_0 = 0$ and*

$$S_{\nu\wedge n} = \sum_{i=1}^{n} Y_i \chi(\nu \geq i), \; n = 1, 2, \ldots \; .$$

Let S_ν denote the a.s. limit of the random walk. Then for $p \geq 1$ the moment $E({S_\nu}^p)$ is finite iff $E(Y_1^p) < +\infty$ and $E(\nu^p) < +\infty$.

Proof. Suppose $E({S_\nu}^p) < +\infty$. Since $S_\nu \geq Y_1 \geq 0$ we have $E(Y_1^p) < +\infty$. From this it also follows that the sequence $(S_{\nu\wedge n}, \mathcal{F}_n)$, $n \geq 1$ is an integrable nonnegative submartingale whose a.s. limit is S_ν . $S_{\nu\wedge n}$ converges to S_ν increasingly as $n \to +\infty$. Consequently, by the monotone convergence theorem,

$$\lim_{n\to+\infty} E(S^p_{\nu\wedge n}) = \sup_{n\geq 1} E(S^p_{\nu\wedge n}) = E(S^p_\nu).$$

By Theorem 1 of [10] this is equvivalent to the conjunction of the following two conditions

$$\sup_{n\geq 1} E(| \, S_{\nu\wedge n} - a(\nu \wedge n) \, |^p) < +\infty$$

$$\text{and } E(A^p_\infty) = a^p E(\nu^p) < +\infty,$$

where $a = E(Y_1) > 0$ and

$$S_{\nu\wedge n} = [S_{\nu\wedge n} - a(\nu \wedge n)] + a(\nu \wedge n) = M_n + A_n$$

is the Doob decomposition of the submartingale $(S_{\nu\wedge n}, \mathcal{F}_n)$, further

$$A_\infty = \lim_{n\to+\infty} A_n = a\nu.$$

It follows that $E(\nu^p) < +\infty$ since $0 < a < +\infty$. This, together with $E(Y_1^p) < +\infty$ proves the necessity part of the assertion.

Let us now turn to the sufficiency part. First consider the case $1 \le p < 2$. By part a) of Theorem 1 we have

$$E(|S_\nu - a\nu|^p) \le$$

$$\le E(X^{*p}) \le C_p E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2\chi(\nu \ge i)\right)^{p/2}\right) \le$$

$$\le C_p E\left(\sum_{i=1}^{\infty} |Y_i - a|^p \chi(\nu \ge i)\right) = C_p E(|Y_1 - a|^p)E(\nu),$$

since $p/2 < 1$ and the random variables $Y_i - a$ and $\chi(\nu \ge i)$ are independent. Here $C_p > 0$ is a constant depending only on p. By our assumption the right-hand side of this inequality is finite. Consequently, by the C_p-inequality,

$$E(S_\nu^p) \le 2^{p-1}[E(|S_\nu - a\nu|^p) + a^p E(\nu^p)] < +\infty.$$

The case $p \ge 2$ can be treated similarly by using part b) of Theorem 1. The moment

$$E(|S_\nu - a\nu|^p)$$

is finite if and only if so are $E(Y_1^p)$ and $E(\nu^{p/2})$. The later is implied by the assumption that $E(\nu^p) < +\infty$. Therefore, again, by the C_p-inequality,

$$E(S_\nu^p) \le 2^{p-1}\left[E(|S_\nu - a\nu|^p) + E(\nu^p)\right]$$

and now the right-hand side is finite by assumption. □

REMARK. In Theorem 2.1 of [9] the necessity part of this assertion is only proved.

We can also improve the assertion of Theorem 2.4 of [9].

Theorem 6. *Let $p > 1$ and suppose that $E(Y_1) = 0$. Then* $S_\nu \in \mathcal{H}_p$ *implies that $E(| Y_1 |^p) < +\infty$. Further, for $p \geq 2$ the condition* $S\nu \in \mathcal{H}_p$ *is equivalent to the conjunction of the two conditions*

$$E(| Y_1 |^p) < +\infty \text{ and } E(\nu^{p/2}) < +\infty.$$

Proof. Generalizing the assumption on $E(Y_1)$ we suppose that $E(Y_1) = a$ is finite. We prove that for $p > 1$ the condition

$$E(| S_\nu - a\nu |^p) < +\infty$$

implies that $E(| Y_1 |^p) < +\infty$. In fact, by the Burkholder–Davis–Gundy inequality (Theorem 1. a) and the maximal inequality of Doob we have

$$c_p(\frac{p}{p-1})^{-p} E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2 \chi(\nu \geq i)\right)^{p/2}\right) \leq$$
$$\leq E(| S_\nu - a\nu |^p) < +\infty.$$

Now by the nonnegativity of the summands

$$c_p(\frac{p}{p-1})^{-p} E(| Y_1 - a |^p) \leq$$
$$\leq c_p(\frac{p}{p-1})^{-p} E\left(\left(\sum_{i=1}^{\infty}(Y_i - a)^2 \chi(\nu \geq i)\right)^{p/2}\right) < +\infty,$$

which proves that $E(| Y_1 |^p) < +\infty$, since c_p is a positive constant.

The second part of the assertion is proved in Theorem 1. b). □

REMARK. Note that the first part of the assertion improves Theorem 2.4 of [9], where the authors suppose in addition that $E(\nu) < +\infty$. Taking especially $E(Y_1) = 0$, we get the assertion of these authors without supposing $E(\nu) < +\infty$.

The following theorem is also an improved form of Theorem 2.2 of Gut and Janson [9].

Theorem 7. *Suppose $E(Y_1)$ is finite and $E(Y_1) \neq 0$. Then for $p \geq 1$ the condition*

$$S_\nu - a\nu \in \mathcal{H}_p$$

is equivalent to the conjunction of the two conditions

$$E(| Y_1 |^p) < +\infty \text{ and } E(\nu^p) < +\infty.$$

Proof. a/ We first consider the case $p = 1$. The condition $E(| Y_1 |) < +\infty$ and $E(\nu) < +\infty$ imply by Wald's identity that $E(S_\nu)$ is finite (which, of course, implies $E(| S_\nu |) < +\infty$). Conversely, an argumentation of Blackwell, used also by Gut and Janson in their paper, shows that $E(Y_1) \neq 0$ and $E(| S_\nu |) < +\infty$ imply the finiteness of $E(\nu)$. There is another proof to this fact in the book [11] by A. A. Borovkov.

b/ Consider now the case $1 < p < 2$ and suppose that $E(| Y_1 |^p)$ and $E(\nu^p)$ are finite. Then by the C_p-inequality

$$E(| S_\nu |^p) \leq 2^{p-1}[E(| S_\nu - a\nu |^p) + E(\nu^p)]$$

and we only have to prove that $E(| S_\nu - a\nu |^p)$ is finite. By Theorem 1 a).

$$E(| S_\nu - a\nu |^p) \leq E(X^{*p}) \leq C_p E((\sum_{i=1}^{\infty}(Y_i - a)^2 \chi(\nu \geq i))^{p/2}) \leq$$

$$\leq C_p E(\sum_{i=1}^{\infty} | Y_i - a |^p \chi(\nu \geq i)) = C_p E(| Y_1 - a |^p) E(\nu) \leq$$

$$\leq C_p E(| Y_1 - a |^p) E(\nu^p) < +\infty.$$

Here we used the fact that $\frac{p}{2} < 1$ and that $Y_i - a$ and $\chi(\nu \geq i)$ are independent. $C_p > 0$ is a constant depending only on p. Conversely, suppose that $E(| S_\nu |^p)$ is finite. Then $E(S_\nu)$ is finite and we proved in the case $p = 1$ that this fact together with $E(Y_1) = a \neq 0$ implies that $E(\nu)$ is finite. It follows as above that

$$E(| S_\nu - a\nu |^p) \leq C_p E(| Y_1 - a |^p) E(\nu) < +\infty,$$

since by the preceding theorem $E(| Y_1 - a |^p) < +\infty$. Consequently, by the C_p-inequality

$$| a |^p E(\nu^p) \leq 2^{p-1}[E(| S_\nu - a\nu |^p) + E(| S_\nu |^p)]$$

and the right-hand side is finite.

c/ Let us now turn to the case $p \geq 2$. Suppose that $E(| Y_1 |^p)$ and $E(\nu^p)$ are finite. Then, again by the C_p-inequality

$$E(| S_\nu |^p) \leq 2^{p-1}[E(| S_\nu - a\nu |^p) + | a |^p E(\nu^p)]$$

and the finiteness of the first term on the right-hand side follows from Theorem 1. b) by the fact that $E(\nu^{\frac{p}{2}}) \leq E(\nu^p) < +\infty$ and that $E(| Y_1 |^p) < +\infty$.

Conversely, if $E(| S_\nu |^p)$ is finite then $E(S_\nu)$ is finite and by part a/ of this proof it follows that $E(\nu) < +\infty$. From the preceding theorem it also follows that $E(| Y_1 |^p) < +\infty$. In order to show that $E(\nu^p)$ is also finite we proceed as follows. For $r \geq 2$ such that $p \geq r$ the finiteness of $E(\nu^{r/2})$ and of $E(| Y_1 |^r)$ by Theorem 1. b) ensures that

$$E(| S_\nu - a\nu |^r) < +\infty.$$

From this by the C_p-inequality we deduce that

$$| a |^r E(\nu^r) \leq 2^{r-1}[E(| S_\nu - a\nu |^r) + E(| S_\nu |^r)] < +\infty.$$

This idea helps us to prove that $E(\nu^p)$ is finite. For this purpose let k be the smallest positive integer for which $1 \leq p \cdot 2^{-k} < 2$. Then

$$E(|\, S_\nu - a\nu \,|^{p\cdot 2^{-k}})$$

is finite since $1 \leq p \cdot 2^{-k} < 2$ and by part b/ of the present proof

$$E(|\, S_\nu - a\nu \,|^{p\cdot 2^{-k}}) \leq C_{p,k} E(|\, Y_1 - a \,|^{p\cdot 2^{-k}}) E(\nu)$$

and the right-hand side is finite. Consequently, if $E(|\, S_\nu \,|^p)$ is finite then so is

$$E(|\, S_\nu - a\nu \,|^{p\cdot 2^{-k}}).$$

From this it follows that

$$|\, a \,|^{p\cdot 2^{-k}} \; E(\nu^{p\cdot 2^{-k}}) < +\infty$$

and since $a \neq 0$, we deduce that

$$E(\nu^{p\cdot 2^{-k}}) < +\infty.$$

Since k is the smallest integer for which $1 \leq p \cdot 2^{-k} < 2$, we see that $p \cdot 2^{-k+1} \geq 2$. Consider now the moment

$$E(|\, S_\nu - a\nu \,|^{p\cdot 2^{-k+1}}).$$

This is finite, since by Theorem 1. b)

$$E(|\, S_\nu - a\nu \,|^{p\cdot 2^{-k+1}}) \leq$$
$$\leq C_{p,k-1} \left[(\sigma^{p\cdot 2^{-k+1}}) E(\nu^{p\cdot 2^{-k}}) + E(|\, Y_1 - a \,|^{p\cdot 2^{-k+1}}) E(\nu) \right]$$

and the right-hand side is finite since we proved that $E(\nu^{p\cdot 2^{-k}}) < +\infty$ and $E(|\, Y_1 - a \,|^{p\cdot 2^{-k+1}}) \leq E(|\, Y_1 - a \,|^p) < +\infty$. From the finiteness of

$$E(|\, S_\nu - a\nu \,|^{p\cdot 2^{-k+1}}),$$

by the C_p-inequality we deduce again

$$E(\nu^{p\cdot 2^{-k+1}}),$$

since $a \neq 0$. If $k-1>1$, we proceed similarly as above. We prove thus that

$$E(|\, S_\nu - a\nu \,|^{p\cdot 2^{-k+2}})$$

is finite. This follows from Theorem 1. b) since

$$E(|\, S_\nu - a\nu \,|^{p\cdot 2^{-k+2}}) \leq$$
$$\leq C_{p,k-2}[\sigma^{p\cdot 2^{-k+2}} E(\nu^{p\cdot 2^{-k+1}}) + E(|\, Y_1 - a \,|^{p\cdot 2^{-k+2}})E(\nu)]$$

and the right-hand side is finite because we have just proved that $E(\nu^{p\cdot 2^{-k+1}}) < +\infty$ and $E(|\, Y_1 - a \,|^{p\cdot 2^{-k+2}}) < +\infty$, since $k-2 \geq 1$. From this we deduce by the C_p-inequality as above that

$$E(\nu^{p\cdot 2^{-k+2}}) < +\infty.$$

If $k-2>1$, then we continue this procedure to show that

$$E(\nu^{p\cdot 2^{-k+3}}) < +\infty.$$

In such a way we arrive at the finiteness of $E(\nu^{p\cdot 2^{-1}})$. By the same procedure as before we finally arrive at the conclusion $E(\nu^p) < +\infty$. □

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