

ON AN INFINITE-PERSON DYNAMIC COALITION GAME

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In this paper cooperative solutions of infinite-person games with time-depending fuzzy coalitions are studied. Infinite-person games were introduced by Aumann in [1]. For their applications in economics we refer to Hildebrandt [2]. The idea of using fuzzy coalitions (i. e. coalitions with “rates of participation”) is due to Aubin (see [3], [4] and [5]). Dynamic coalition models were considered in [6]. Cooperative solutions of N -person dynamic games were studied e. g. in [7].

In the present paper the community of players is modelled by a compact metric space. A finite subset of players is supposed to control the environment. The dynamics of the latter is described by a linear differential equation, and its final state determines the pay-off of each player. Under continuity assumptions on the pay-off functions, the existence of a cooperative solution for the whole infinite set of players is proved.

1. Let Ω be a compact metric space. The elements of Ω are interpreted as *players*. We suppose that the environment of the players is described by the differential equations

$$\dot{x} = Ax$$

with an appropriate matrix $A \in \mathbf{R}^{n \times n}$. Then we consider a finite subset

$$\Omega_N = \{\omega_1, \dots, \omega_N\} \subset \Omega$$

of distinguished players which have the option to influence the environment. The influence of player ω_i is given by the equation

$$\dot{x} = Ax + b_i$$

where $b_i \in \mathbf{R}^n$ is fixed ($i \in \overline{1, N}$).

For a given $c \subset \mathcal{N} := \{1, \dots, N\}$ the set of players

$$(1) \quad \Omega_c := \{\omega_i : i \in c\}$$

is interpreted as a *coalition*. The influence of a coalition is supposed to be the superposition of the single influences:

$$(2) \quad \dot{x} = Ax + \sum_{t \in c} b_t.$$

For a more flexible model, we suppose that, instead of forming a “strict” coalition, each player in Ω_N may have a “rate of participation” $v_i \in [0, 1]$ ($i \in \overline{1, N}$).

Definition 1. Any element $v \in U := [0, 1]^N$ is called a *fuzzy coalition*. Any $v \in U_0 := \{0, 1\}^N$ is called a *pure coalition*.

Clearly, there is a natural one-to-one correspondence between the pure coalitions and the sets (1). Moreover, let’s consider both U and U_0 as subsets of \mathbf{R}^N . Then the set of fuzzy coalitions is nothing else than the convex hull of the set of pure coalitions.

Now fix $T \in \mathbf{R}_+$ and put

$$\mathcal{U} := \{u \in L_2^N[0, T], u(t) \in U \text{ for a.e. } t \in [0, T]\}.$$

The elements of U are interpreted as *time-dependent fuzzy coalitions*.

We fix an initial state $x_0 \in \mathbf{R}^n$ and consider the mapping $L: L_2^N[0, T] \rightarrow \mathbf{R}^n$ which associates with each $u \in L_2^N[0, T]$ the point $x(T)$ of the solution of the following initial value problem

$$(3) \quad \begin{aligned} \dot{x} &= Ax + \sum_{i=1}^N b_i u_i, \\ x(0) &= x_0. \end{aligned}$$

For a $u \in U$, (3) is interpreted as the dynamics of the environment under the influence of the time-dependent fuzzy coalition u . Clearly, (3) is an extension of (2) in some sense.

2. Having set up the dynamics of the game, we turn to the formalization of the pay-off. Let

$$g: \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}$$

be an upper semicontinuous function such that for every $y \in \mathbf{R}^n$ the function $g(\cdot, y)$ is lower semicontinuous. The function g is interpreted as follows. For every $(\omega, y) \in \Omega \times \mathbf{R}^n$, $g(\omega, y)$ is the pay-off received by the player ω , provided that the process (3) has ended at y .

Let’s define

$$G: \mathbf{R}^n \rightarrow C(\Omega), G(y) := g(\cdot, y);$$

$$F := G \circ L|_U.$$

According to the above interpretations, the function $F: U \rightarrow C(\Omega)$ associates the resulting pay-off as a continuous function of the player with every time-dependent fuzzy coalition $u \in U$.

Definition 2. The pair $\Gamma := (U, F)$ is called an *infinite-person dynamic coalition game*.

Definition 3. A $u_* \in \mathcal{U}$ is called a *cooperative solution* of the game Γ if for every $u \in U$ the inequalities

$$F(u)(\omega) \geq F(u_*)(\omega) \quad (\omega \in \Omega)$$

imply that

$$F(u)(\omega) = F(u_*)(\omega) \quad (\omega \in \Omega).$$

Remark. In terms of vector optimization, a cooperative solution u_* provides a maximal value of the function F with respect to the partial ordering of $C(\Omega)$ according to the closed convex cone K of nonnegative functions in $C(\Omega)$. (See e. g. [8].)

In the next section we shall prove the existence of a cooperative solution to the game Γ . To this end we shall need the following.

Lemma. For any convex compact set $V \in \mathbf{R}^n$ the set

$$\mathcal{O} := \{u \in L_2^N[0, T] : u(t) \in V \text{ for a. e. } t \in [0, T]\}$$

is weakly compact in $L_2^N[0, T]$.

Proof. It is obvious that \mathcal{O} is convex and bounded in norm. First we prove that \mathcal{O} is closed in the norm topology.

Assume that $u_n \in \mathcal{O}$ ($n \in \mathbf{N}$) and $\lim (u_n) = u_0 \in \mathcal{O}$ in the norm topology. We prove that $u_0 \in \mathcal{O}$. $V \subset \mathbf{R}^n$ being convex, closed and $V \neq \mathbf{R}^n$, V can be represented as the intersection of a countable set of closed support half-spaces. Thus, it is sufficient to show that for any $a \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$

$$V \subset H := \{z \in \mathbf{R}^n : \langle a, z \rangle \leq \alpha\}$$

implies that

$$(4) \quad u_0(t) \in H \text{ for a.e. } t \in [0, T].$$

For the characteristic function $\chi_M : [0, T] \rightarrow \mathbf{R}$ of the set

$$M := \{t \in [0, T] : u_0(t) \notin H\}$$

we have

$$\int_0^T \chi_M \langle a, u_n \rangle d\lambda \leq \alpha \lambda(M) \quad (n \in \mathbf{N}),$$

where λ is the Lebesgue measure in $[0, T]$. Since the sequence (u_n) also weakly converges u_0 , we get

$$\begin{aligned} \lim \left(\int_M \langle a, u_n \rangle d\lambda \right) &= \lim \left(\int_0^T \chi_M \langle a, u_n \rangle d\lambda \right) = \int_0^T \chi_M \langle a, u_0 \rangle d\lambda = \\ &= \int_M \langle a, u_0 \rangle d\lambda \leq \alpha \lambda(M). \end{aligned}$$

To prove (4) suppose the contrary: $\lambda(M) > 0$. Then from

$$\langle a, u_0(t) \rangle > \alpha \quad (t \in M)$$

we obtain that

$$\int_M \langle a, u_0 \rangle d\lambda > \alpha \lambda(M),$$

which contradicts to (5).

Consequently, \mathcal{O} is closed in the norm topology. By its convexity \mathcal{O} is also weakly closed. On the other hand, any weakly closed convex and norm-bounded set in a reflexive Banach space is weakly compact (see e. g. [9]). \square

3. In this section we prove the following existence theorem.

Theorem. *The infinite-person dynamic coalition game Γ has a cooperative solution.*

Proof. According to Krein's theorem (see [10]), for any separable Banach space Z ordered by a closed convex cone $K \subset Z$, there exists a functional $p \in Z^*$ which is strictly positive in the sense that

$$\langle p, z \rangle > 0 \quad (z \in K \setminus \{0\}).$$

Since Ω is a compact metric space, $C(\Omega)$ is separable. Applying Krein's theorem with $Z := C(\Omega)$ and

$$K := \{\varphi \in C(\Omega) : \varphi(\omega) \geq 0 \quad (\omega \in \Omega)\},$$

also using Riesz's theorem, we obtain that there exists a Borel measure μ in Ω with the following properties: For any $\varphi \in C(\Omega)$

$$(6) \quad \begin{aligned} \varphi(\omega) \geq 0 (\omega \in \Omega) &\Rightarrow \int_{\Omega} \varphi d\mu \geq 0, \\ \forall \omega \in \Omega: \varphi(\omega) \geq 0 \\ \exists \bar{\omega} \in \Omega: \varphi(\bar{\omega}) > 0 &\Rightarrow \int_{\Omega} \varphi d\mu > 0. \end{aligned}$$

Now we define

$$f: \mathcal{U} \rightarrow \mathbf{R}, \quad f(u) := \int_{\Omega} F(u) d\mu.$$

Now we shall prove that the continuity assumptions on g in Section 2 imply the upper semicontinuity of f in the weak topology of \mathcal{U} . Hence, by the weak compactness of \mathcal{U} , the classical Weierstrass theorem guarantees the existence of a minimum point u_* of f . We shall see that u_* is the required solution to Γ .

Let's fix a $u_0 \in \mathcal{U}$. In order to prove the upper semicontinuity of f at u_0 , we pick an arbitrary positive real number ε . We define

$$Q := L(\mathcal{U}), \quad y_0 := L(u_0).$$

First we show that there exists a neighbourhood Q_{y_0} of y_0 in Q such that

$$(7) \quad g(\omega, y) < g(\omega, y_0) + \varepsilon/\mu(\Omega) \quad (y \in Q_{y_0}, \omega \in \Omega).$$

We fix an arbitrary $\omega_0 \in \Omega$. Then, by the upper semicontinuity of g and the lower semicontinuity of $g((\cdot, y_0))$ there exist an open neighbourhood Q_{ω_0} of y_0 in Q and an open neighbourhood Ω_{ω_0} of ω_0 such that for every $\omega \in \Omega_{\omega_0}$, $y \in Q_{\omega_0}$ we have

$$\begin{aligned} g(\omega, y) &< g(\omega_0, y_0) + \varepsilon/2\mu(\Omega) \\ g(\omega, y_0) &> g(\omega_0, y_0) - \varepsilon/2\mu(\Omega). \end{aligned}$$

The family

$$P_{\omega_0} := \Omega_{\omega_0} \times Q_{\omega_0} \subset \Omega \times Q \quad (\omega_0 \in \Omega)$$

is an open covering of the compact subset $\Omega \times \{y_0\} \subset \Omega \times Q$. Hence there exist a $k \in \mathbb{N}$ and elements

$$\omega^1, \dots, \omega^k \in \Omega$$

such that

$$\Omega \times \{y_0\} \subset \bigcup_{j=1}^k P_{\omega^j}.$$

The set $Q_{y_0} := \bigcap_{j=1}^k Q_{\omega^j}$ is a neighbourhood of y_0 in Q such that for every $\omega \in \Omega$

and $y \in Q_{y_0}$ there exists a $j \in \overline{1, k}$ with $(\omega, y) \in P_{\omega^j}$ and

$$\begin{aligned} g(\omega, y) - g(\omega, y_0) &= g(\omega, y) - g(\omega^j, y_0) + \\ &+ g(\omega^j, y_0) - g(\omega, y_0) < \varepsilon/2\mu(\Omega) + \varepsilon/2\mu(\Omega) = \varepsilon/\mu(\Omega). \end{aligned}$$

Now, from the variation of parameters formula, it follows easily that the operator L is continuous in the weak topology of $L_2^N[0, T]$. Therefore there exists a neighbourhood \mathcal{U}_{u_0} of u_0 in \mathcal{U} such that

$$L(u) \in Q_{y_0} \quad (u \in \mathcal{U}_{u_0}).$$

Hence, by (7), we get that for any $u \in \mathcal{U}_{u_0}$ and $\omega \in \Omega$

$$\begin{aligned} F(u)(\omega) &= G(L(u))(\omega) = g(\omega, L(u)) < \\ &< g(\omega, L(u_0)) + \varepsilon/\mu(\Omega) = F(u_0)(\omega) + \varepsilon/\mu(\Omega). \end{aligned}$$

By integration with respect to the measure μ , we obtain that for every $u \in \mathcal{U}_{u_0}$

$$f(u) = \int_{\Omega} F(u) d\mu < \int_{\Omega} F(u_0) d\mu + \varepsilon = f(u_0) + \varepsilon$$

that is f is upper semicontinuous in the weak topology of $L_2^N[0, T]$.

Since \mathcal{U} is compact in the latter topology, by the Weierstrass theorem it follows that there exists a $u_* \in \mathcal{U}$ such that

$$(8) \quad f(u) \leq f(u_*) \quad (u \in \mathcal{U}).$$

Finally, we show that u_* is a cooperative solution of Γ . Indeed, let's suppose the contrary. Then, by Definition 3, there exists a $u \in \mathcal{U}$ such that

$$F(u)(\omega) \geq F(u_*)(\omega) \quad (\omega \in \Omega)$$

and for some $\bar{\omega} \in \Omega$

$$F(u)(\bar{\omega}) > F(u_*)(\bar{\omega}).$$

Since μ has the property (6), taking $\varphi := F(u) - F(u_*)$ we get

$$f(u) - f(u_*) = \int_{\Omega} [F(u) - F(u_*)] d\mu > 0$$

in contradiction with (8). \square

Remark. It is easy to see that the above theorem also holds for games with time-depending dynamics, i. e. in case in (3) A and b_i ($i \in \overline{1, N}$) are, say, continuous functions.

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