DISCONTINUOUS VARIATIONAL PROBLEMS OF HIGHER ORDER

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In practice, in a number of higher order variational problems, discontinuity arises not only in the last derivative of the admissible functions, but also in their derivatives of lower order. An interesting special case, among others, has been motivated by investigations in economics [3]. The subject is generally treated in [6], [7] and [8]. In the present paper, a generalization of this topic is given in a direction similar to that of [9].

1. Notations and definitions

Let $n \in \mathbf{N}$ be fixed, and take arbitrary real numbers a, b $(a<b)$, α_i, β_i $(i \in \overline{0, n-1})$. We denote by A the class of all real functions defined and continuous in $[a, b]$ which are n-times differentiable except an at most countable set and each derivative of which is a function of first kind (see [5]). In other words, $x \in A$ implies that for every $i \in \overline{1, n}$, $t \in [a, b[$ resp. $s \in]a, b]$ the unilateral limits $x^{(i)}(t+0)$ resp. $x^{(i)}(s-0)$ exist. We also assume that each $x \in A$ satisfies the boundary condition

$$(1) \qquad x^{(i)}(a+0) = \alpha_i, \; x^{(i)}(b-0) = \beta_i \quad (i \in \overline{0, n-1}).$$

(Here and in the following the 0-th derivative of a function is the function itself.) In case of $\alpha_i = \beta_i = 0$ $(i \in \overline{0, n-1})$ the corresponding class of functions is denoted by A_0.

Put $j := \mathrm{id}_{[a, b]}$ and for any $\alpha \in \mathbf{R}$ define the function

$$\underset{\sim}{\alpha}:[a, b] \to \mathbf{R}, \; t \to \alpha.$$

Let $F:[a, b] \to \mathbf{R}$ be a function of first kind. Then for every $\tau \in [a, b]$ we shall use the following notation

$$\int_\tau F:[a, b] \to \mathbf{R}, \; t \to \int_\tau^t F.$$

For each function $x \in A$ define its n-th lifting by

$$\overset{n}{x} := (j, x, x', \ldots, x^{(n)}).$$

For any $\Phi : \mathbf{R}^{n+2} \to \mathbf{R}$ we denote

$$\Phi \overset{n}{x} := \Phi \circ \overset{n}{x} (x \in A).$$

Now, given a continuously differentiable function $f : \mathbf{R}^{n+2} \to \mathbf{R}$, we denote the partial derivative function of f with respect to its i-th variable ($i \in \overline{1, n+2}$) by $f_{.i}$.

Let φ and ψ be real functions defined in the interval $[a, b]$ except an at most countable set. By the equality

$$\varphi \doteq \psi$$

we mean the following: There exists an at most countable set $H \subset [a, b]$ such that

$$\varphi(t) = \psi(t) \quad (t \in [a, b] \setminus H).$$

Consider now the functional $I : A \to \mathbf{R}$, $x \to \int_a^b f \overset{n}{x}$. Let $x \in A$. Then the functional

$$\delta_x I : A_0 \to \mathbf{R}, \; h \to \int_a^b (f_{\cdot 2} \overset{n}{x} \cdot h + f_{\cdot 3} \overset{n}{x} \cdot h' + \ldots + f_{\cdot n+2} \overset{n}{x} h^{(n)})$$

is called the *first variation* of I with respect to x. It for some $x \in A$ the range of the first variation of I with respect to x is the set $\{0\}$, then x is called a *stationary function* of I. It is a well-known fact that I can attain an extremum only at a stationary function.

2. Results

Theorem 1. *Suppose that $x \in A$ is a stationary function of the functional I. Then for each $t \in]a, b[$ we have*

(2) $$f_{\cdot 3}(t, x(t), x'(t-0), \ldots, x^{(n)}(t-0)) = \\ = f_{\cdot 3}(t, x(t), x'(t+0), \ldots, x^{(n)}(t+0)),$$

and

(3) $$f_i(t, x(t), x'(t \pm 0), \ldots, x^{(n)}(t \pm 0)) = 0 \; (i \in \overline{4, n}).$$

Proof. To simplify the setting, denote

$$u_i := f_{\cdot i+2} \overset{n}{x} \quad (i \in \overline{0, n}).$$

Since f is continuously differentiable, instead of (2) and (3), it is enough to prove that

$$u_1(t-0) = u_1(t+0) \quad (t \in]a, b[) \tag{4}$$

and

$$u_i(t \pm 0) = 0 \quad (t \in]a, b[;\ i \in \overline{2, n}). \tag{5}$$

The function x is stationary so for all $h \in A_0$ we have

$$\delta_x I(h) = \int_a^b (u_0 h + u_1 h' + \ldots + u_n h^{(n)}) = 0. \tag{6}$$

Now fix an arbitrary number $c \in]a, b[$ and $k_0 \in \mathbf{N}$ such that $[c-1/k_0, c+1/k_0] \subset [a, b]$. For $p \in \overline{1, n}$ and $k \in \mathbf{N} \setminus \overline{1, k_0}$ define

$$\varphi_{p,k} : [a, b] \to \mathbf{R},$$

$$t \to \begin{cases} \dfrac{k}{p!}\left(x - c + \dfrac{1}{k}\right)^p & , \text{ if } t \in \left[c - \dfrac{1}{k}, c\right], \\ \dfrac{1}{p!}\left(\dfrac{1}{k}\right)^{p-2}\left(c + \dfrac{1}{k} - x\right), & \text{if } t \in \left[c, c + \dfrac{1}{k}\right], \\ 0 & , \text{ if } t \in [a, b] \setminus \left[c - \dfrac{1}{k}, c + \dfrac{1}{k}\right]. \end{cases}$$

Clearly, $\varphi_{p,k} \in A_0$ and, except three points, it is differentiable any times in $[a, b]$. A simple calculation shows the following:

$$\varphi'_{1,k}(t) = \begin{cases} k, & \text{if } t \in]c - \dfrac{1}{k}, c[, \\ -k, & \text{if } t \in]c, c + \dfrac{1}{k}[\end{cases} \tag{7}$$

$$(k \in \mathbf{N} \setminus \overline{1, k_0});$$

$$\varphi^{(i)}_{p,k}\left(c - \frac{1}{k}\right) = 0 \quad (p \in \overline{2, n};\ i \in \overline{0, p-1}); \tag{8}$$

$$\varphi^{(k)}_{p,k}(t) = \begin{cases} k, & \text{if } t \in]c - \dfrac{1}{k}, c[, \\ 0, & \text{if } t \in]c, c + \dfrac{1}{k}[\end{cases} \tag{9}$$

$$(p \in \overline{2, n};\ k \in \mathbf{N} \setminus \overline{1, k_0});$$

$$\varphi^{(\alpha)}_{p,k}(t\pm 0) = 0 \tag{10}$$

$$(t\in\,]a,b[;\ p\in\overline{1,n};\ \alpha\in\mathbf{N}\setminus\overline{1,p});$$

$$|\varphi^{(i)}_{p,k}(t\pm 0)| \leqq 1 \tag{11}$$

$$(t\in\,]a,b[;\ p\in\overline{1,n};\ i\in\overline{1,p-1}).$$

Introduce now the notations

$$V_{i,\alpha} := \overset{\alpha}{\overbrace{\int_c \dots \int_c}}\, u_i \quad (i\in\overline{0,n};\ \alpha\in\overline{1,n}). \tag{12}$$

It is obvious that $V_{i,\alpha}$ is continuous and $V_{i,\alpha}(c) = 0$.

First, let $p := 1$ and for every $k\in\mathbf{N}\setminus\overline{1,k_0}$ consider the value of the functional $\delta_x I$ at the function $\varphi_{1,k}$.
Using the equalities (6) and (10) we get

$$\delta_x I(\varphi_{p,k}) = \int_{c-\frac{1}{k}}^{c+\frac{1}{k}} (u_0\varphi_{1,k} + u_1\varphi'_{1,k}) = 0.$$

Hence, by a simple transformation (cf. [6]), taking into account (7), (8), (9) and (12), we obtain that

$$\int_{c-\frac{1}{k}}^{c} (-V_{0,1}+u_1)k - \int_{c}^{c+\frac{1}{k}} (-V_{0,1}+u_1)k = 0.$$

From this it follows that

$$k\left(\int_{c-\frac{1}{k}}^{c} u_1 - \int_{c}^{c+\frac{1}{k}} u_1\right) = k\left(\int_{c-\frac{1}{k}}^{c} V_{0,1} - \int_{c}^{c+\frac{1}{k}} V_{0,1}\right).$$

In each side of the last equality the k-th member of a sequence appears. The continuity of $V_{0,1}$ and the equality $V_{0,1}(c) = 0$ imply that the limit of the sequence corresponding to the right-hand side is equal to zero. The same holds for the sequence corresponding to the left-hand side. Thus, since u_1 is a function of first kind, we obtain that

$$0 = \lim\left(k\left(\int_{c-\frac{1}{k}}^{c} u_1 - \int_{c}^{c+\frac{1}{k}} u_1\right)\right) = u_1(c-0) - u_1(c+0)$$

and the equality (4) is proved.

Now let us turn to the proof of equality (5) for $n \geq 2$. We shall confine ourselves to the left limit, the right limit can be treated similarly (see [6]).

We fix an arbitrary number $p \in \overline{2, n}$ and for every $k \in \mathbf{N} \setminus \overline{1, k_0}$ consider the value of the functional $\delta_x I$ at the function $\varphi_{p,k}$. Equalities (6) and (10) imply that

$$\delta_x I(\varphi_{p,k}) = \int_{c-\frac{1}{k}}^{c+\frac{1}{k}} (u_0 \varphi_{p,k} + u_1 \varphi'_{p,k} + \ldots + u_p \varphi_{p,k}^{(p)}) = 0.$$

Hence, by repeated integration by parts, applying also the equalities (8), (9), (10) and (12) (cf. [6]) we obtain that

$$\int_{c-\frac{1}{k}}^{c} \left\{ \sum_{i=0}^{p-1} (-1)^{p-i} V_{i,\, p-i} + u_p \right\} k + \int_{c}^{c+\frac{1}{k}} (u_1 - V_{0,1}) \varphi' \left(c + \frac{1}{k} - 0 \right) = 0.$$

Hence

$$k \int_{c-\frac{1}{k}}^{c} u_p = k \int_{c-\frac{1}{k}}^{c} \sum_{i=0}^{p-1} (-1)^{p-i+1} V_{i,\, p-i} + \int_{c}^{c+\frac{1}{k}} (V_{0,1} - u_1) \varphi' \left(c + \frac{1}{k} - 0 \right).$$

On each side of the above equation the k-th member of a sequence appears. Taking into account the continuity of the functions $V_{i,\alpha}$, the equality $V_{i,\alpha}(c) = 0$, the boundedness of u and the inequality (11), we obtain that the sequence corresponding to the right hand side tends to zero. The same holds for the sequence corresponding to the left hand side. Thus, being u_p a function of first kind,

$$0 = \lim \left(k \int_{c-\frac{1}{k}}^{c} \right) = u_p(c-0).$$

Theorem 1 is proved. □

In the following we shall use the following generalization of the Du Bois Reymond lemma.

Lemma. *Let $m\colon [a, b] \to \mathbf{R}$ be a function of first kind and suppose that for every $h \in A_0$*

$$\int_a^b m h' = 0. \tag{13}$$

Then there exists an $\alpha \in \mathbf{R}$ such that

$$m \doteq \underset{\sim}{\alpha}.$$

Proof. We shall show that m takes the same value at any of its continuity points (that is in the whole interval $[a, b]$ except an at most countable set). [The usual elegant proof of the Du Bois Reymond lemma does not apply (see [2]). It is expedient to go back to Hilbert's original idea (see [1], p. 28.) and use an appropriate variant of the latter (see [4], p. 16. and [8], p. 111.).]

Suppose the contrary to our statement: There exist points $\alpha, \beta \in]a, b[$, $\alpha < \beta$ such that m is continuous at both α and β, furthermore, say $m(\alpha) > m(\beta)$. Hence it follows that there exist positive numbers d and δ $(2\delta < \beta - \alpha)$ such that

$$\text{(14)} \qquad \begin{cases} m(t) > d, & \text{if} \quad t \in [\alpha, \alpha + \delta], \\ m(t) < d, & \text{if} \quad t \in [\beta - \delta, \beta]. \end{cases}$$

We fix an arbitrary positive number k and define a function $\omega\colon [a, b] \to \mathbf{R}$ in such a way that its value is equal to k at the points of $[\alpha, \alpha + \delta]$, $-k$ at the points of $[\beta - \delta, \beta]$ and 0 elsewhere in $[a, b]$. Let us define, finally, $\bar{h} := \int_a \omega$. Then, by (14) we get that

$$\int_a^b m h' = k \int_\alpha^{\alpha+\beta} m - k \int_{\beta-\delta}^{\beta} m > k\delta - k\delta = 0$$

which contradicts to (13). Lemma is proved. □

Making use of our Lemma, we obtain the corresponding Du Bois Reymond equation in the usual simple way (see e. g. [9]):

Theorem 2. *Suppose that $x \in A$ is a stationary function of the functional I. Then there exists an $\alpha \in \mathbf{R}$ such that*

$$f_{\cdot 3}\overset{n}{x} \int_a f_{\cdot 2}\overset{n}{x} \doteq \underset{\sim}{\alpha}$$

3. Remarks

1. For $n = 1$ a detailed treatment of the problem can be found in [9]. The first part of the proof of Theorem 1 provides another proof of the so called Weierstrass-Erdmann corner condition.

2. The corner conditions (2)–(3) clearly make sense only if the derivatives of the admissible functions are of first kind or „better" than that. This implies that the results of Theorem 1 can not be generalized by extending the class of admissible functions.

3. In case $n > 1$ the stationary functions of the considered problem formally satisfy the same necessary conditions as for $n = 1$. This also refers to the rest of the necessary conditions not considered here (such as the Lagrange and the Weierstrass conditions etc., cf. [6], [7], [8]). Therefore, the necessary conditions related to first order problems of calculus of variations are not characteristic to the first order problem itself, but they also hold for any higher order problem in which only the continuity of the admissible functions is assumed and not that of their derivatives.

REFERENCES

[1] *Bolza O.:* Vorlesungen über Variationsrechnung, Kochler-Amelang, Leipzig, 1909.
[2] *Bliss G. E.:* Lectures on the Calculus of Variations, University of Chicago, Illinois, 1946.
[3] *Pixley H. H.:* A problem in the calculus of variations suggested by a problem in economics. Contributions to the Calculus of Vaiiations, 1931 – 1932, 131 – 189.
[4] *Kósa A.:* Variációszámítás (Calculus of Variations), Felsőoktatási Jegyzetellátó Vállalat, Budapest, 1955 (in Hungarian).
[5] *Dieudonné J.:* Foundations of Modern Analysis. Academic Press, New York, 1960.
[6] *Kósa A.:* Notwendige Bedingungen für die diskontinuierlichen Lösungen von den Variationsproblemen *n*-ter Ordnung. *Acta Math. Acad. Sci. Hung.* **11** (1960), 23 – 49.
[7] *Kósa A.:* Über ein reguläres Problem der Variationsrechnung. *Acta Math. Acad. Sci. Hung.* **11** (1960), 423 – 428.
[8] *Kósa A.:* Über die diskontinuierlichen Lösungen von räumlichen Variationsprobleme höherer Ordnung. *Annales Univ. Sci. Budapest., Sectio Matematica* **3 – 4** (1960 – 61), 109 – 121.
[9] *Kósa A.* and *Shamandy A.:* On the smoothness properties of stationary functions arising in calculus of variations. *Annales Univ. Sci. Budapest., Sectio Computatorica* **5** (1984), 29 – 35.