ON L_1-MEAN OSCILLATING RANDOM VARIABLES

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1. The $\mathcal{K}_p$-spaces are treated e.g. in the book by A. M. Garsia [1]. Let $X \in L^1(\Omega, \mathcal{A}, P)$ be a random variable defined on the probability space $(\Omega, \mathcal{A}, P)$ and consider the regular martingale

$$X_n = E(X|\mathcal{F}_n),\ n \geq 0,$$

where $\{\mathcal{F}_n\}$, $n \geq 0$, is an increasing sequence of σ-fields of events such that

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{n=0}^{\infty} \mathcal{F}_n\right) = \mathcal{A}.$$

We suppose that $X_0 = 0$ a.s. We denote by $d_0 = 0, d_1, d_2, \dots$ the difference sequence corresponding to the martingale $(X_n, \mathcal{F}_n)$.

For $1 \leq p \leq +\infty$ set

$$\Gamma_X^{(p)} = \{\gamma\colon \gamma \in L_p E,\ (|X - X_{n-1}|\,|\mathcal{F}_n) \leq E(\gamma|\mathcal{F}_n) \text{ a. s., } \forall\, n \geq 1\}.$$

We say that $X \in \mathcal{K}_p$ if the set $\Gamma_X^{(p)}$ is not empty and in this case we let

$$\|X\|_{\mathcal{K}_p} = \inf_{\gamma \in \Gamma_X^{(p)}} \|\gamma\|_p.$$

It easily can be seen that $\|\cdot\|_{\mathcal{K}_p}$ is a semi-norm on $\mathcal{K}_p$. The space $\mathcal{K}_\infty$ is the well-known BMO_1-space.

In [2] we generalized this notion in the following way. Consider a pair (Φ, Ψ) of conjugate Young functions and put

$$\Gamma_X^{(\Phi)} = \{\gamma : \gamma \in L^\Phi,\ E(|X - X_{n-1}|\,|\mathcal{F}_n) \leq E(\gamma|\mathcal{F}_n) \text{ a. s., } \forall\, n \geq 1\}.$$

We say that $X \in \mathcal{K}_\Phi$ if the set $\Gamma_x^{(\Phi)}$ is not empty. In this case we let

$$\|X\|_{\mathcal{K}_\Phi} = \inf_{\gamma \in \Gamma_X^{(\Phi)}} \|\gamma\|_\Phi,$$

where $\|\cdot\|_\Phi$ denotes the Luxemburg norm in the Orlicz space L^Φ. For the definition of the Young functions, Orlicz spaces and Luxemburg norms we refer to [3] and [4]. We easily prove that $\|\cdot\|_{\mathcal{K}_\Phi}$ is a semi-norm on $\mathcal{K}_\Phi$.

We say that the random variable X belongs to the Hardy space $\mathcal{H}_\Phi$ if

$$S = S(X) = \left(\sum_{i=1}^{\infty} d_i^2\right)^{1/2} \in L^\Phi,$$

or in other words $\|S\|_\Phi < +\infty$. In this case we write $\|X\|_{\mathcal{H}_\Phi} = \|S\|_\Phi$.

Since the Young functions Φ cannot be linear, the space L_1 is not contained amongst the Orlicz spaces. Therefore we define the Hardy space $\mathcal{H}_1$ as the set of all the random variables X for which $\|S\|_1 < +\infty$. In this case we let $\|X\|_{\mathcal{H}_1} = \|S\|_1$.

We recall the definition of the power of a Young function Φ. Let $\varphi(x)$ be the right hand side derivative of Φ. Then the quantity

$$p = \sup_{x>0} \frac{x\varphi(x)}{\varphi(x)}$$

is called the power of Φ. The finiteness of p is equivalent to say that Φ satisfies the so called Δ_2-condition. We define similarly the power q of the conjugate Young function $\Psi(x)$.

The inequality of Burkholder-Davis-Gundy says that if p is finite then $X \in \mathcal{H}_\Phi$ if and only if $X^* = \sup_{n \geq 0} |X_n| \in L^\Phi$ (cf. [5]). Also, Davis' inequality states that $X \in \mathcal{H}_1$ if and only if $X^* \in L^1$.

In paper [2] we proved if both Φ and Ψ have finite power then $X \in \mathcal{K}_\Phi$ is equivalent to the fact that $X \in \mathcal{H}_\Phi$, or in other words $X^* \in L^\Phi$. For the pair $\left(\frac{X^p}{p}, \frac{X^q}{q}\right)$ of conjugate Young functions, where $p > 1$ and $p^{-1} + q^{-1} = 1$ this fact has essentially been established by Garsia in [1] (Theorem III. 5.2.).

The space $\mathcal{K}_1$ is less studied. We only know that $\mathcal{H}_1 \subset \mathcal{K}_1$. In fact, if $X^* \in L^1$, then

$$E(|X - X_{n-1}| \,|\mathcal{F}_n) \leq E(2X^* |\mathcal{F}_n),\ n \geq 1.$$

Consequently, $\Gamma_X^{(1)}$ is not empty, since $X^* \in L^1$ and so $2X^* \in \Gamma_X^{(1)}$. The reverse implication, i.e. $\mathcal{K}_1 \subset \mathcal{H}_1$, is false. Here is a counterexample. Consider a nonnegative random variable X belonging to L_1. Let $X_n = E(X|\mathcal{F}_n)$, $n \geq 0$ be the corresponding martingale. Also let $X'_n = X_n - X_0$, $n \geq 0$. Then $(X'_n, \mathcal{F}_n)$ is also a martingale. Suppose we have chosen such an X that the limit $X - X_0$ of $X_n - X_0$ does not belong to $\mathcal{H}_1$ but at the same time $|X'_{n+1} - X'_n| \leq 1$ a.s. We show that $X - X_0$ belongs to $\mathcal{K}_1$. In fact,

$$E(|X - X_0 - X'_{n-1}| \,|\mathcal{F}_n) = E(|X - X_{n-1}| \,|\mathcal{F}_n) \leq E(|X - X_n| \,|\mathcal{F}_n) +$$
$$+ |X_n - X_{n-1}| \leq E(X|\mathcal{F}_n) + E(X|\mathcal{F}_n) + |X'_n - X'_{n-1}| \leq E(2X + 1/\mathcal{F}_n),$$

which shows that $X - X_0 \notin \mathcal{H}_1$ and that $X - X_0 \in \mathcal{K}_1$ (cf. e.g. [1], p. 122.).

In what follows we shall use a maximal inequality which is proved in [2]. We state it in the form of the following

Theorem 1. *Let $(X_n, \mathcal{F}_n)$ be a martingale and let $\gamma \in L^1$ be a random variable such that for every $n \geq 1$ we have*

$$E(|X - X_{n-1}|\,|\mathcal{F}_n) \leq E(\gamma|\mathcal{F}_n) \text{ a.s.}$$

Then for arbitrary $\beta > \alpha > 0$ we have

$$(\beta - \alpha)E(\chi(X_n^* \geq \beta)) \leq E(\gamma\chi(X_n^* \geq \alpha)).$$

Here $\chi(A)$ denotes the indicator function of the event A.

2. About the behaviour of the random variables belonging to $\mathcal{X}_1$ we can prove the following

Theorem 2. *If $X \in \mathcal{X}_1$ then X^* is a.s. finite. Moreover, for arbitrary $\lambda > 0$ we have the inequality*

$$\lambda P(X^* \geq \lambda) \leq 2\|X\|_{\mathcal{X}_1}.$$

Proof. We use the inequality of Theorem 1. According to this if $\beta > \alpha > 0$ and if $X \in \mathcal{X}_1$ then with arbitrary $\gamma \in \Gamma_X^{(1)}$ we have

$$(\beta - \alpha)E(\chi(X^* \geq \beta)) \leq E(\gamma\chi(X^* \geq \alpha)).$$

Choose $\beta = 2\alpha$. Then

$$\alpha P(X^* \geq 2\alpha) \leq E(\gamma).$$

Since $\gamma \in \Gamma^{(1)}$ is arbitrary from this we get

$$\alpha P(X^* \geq 2\alpha) \leq \|X\|_{\mathcal{X}_1},$$

or, in other words

$$2\alpha P(X^* \geq 2\alpha) \leq 2\|X\|_{\mathcal{X}_1}.$$

Taking $\lambda = 2\alpha$ we obtain our inequality.

Further, since

$$P(X^* = +\infty) = \lim_{\lambda \to +\infty} P(X^* \geq \lambda),$$

the inequality just proved shows that

$$P(X^* = +\infty) \leq 2\|X\|_{\mathcal{X}_1} \lim_{\lambda \to +\infty} \frac{1}{\lambda} = 0.$$

This means that $P(X^* < +\infty) = 1$. □

3. When at least one of Φ and Ψ have no finite power then we cannot prove the equivalence of the norms $\|\cdot\|_{\mathcal{X}_\Phi}$ and $\|\cdot\|_{\mathcal{H}_\Phi}$. Assuming only the finiteness of the power p of Φ we are able to prove the validity of the following inequality: if $X \in \mathcal{X}_\Phi$, $P(X = 0) < 1$, then with arbitrary constants $c > 1$ and $\varrho > 1$ we have

$$(\varrho - 1)E\left(\Psi\left(\varphi\left(\frac{X_n^*}{\varrho A \dfrac{c}{c-1}\|X\|_{\mathcal{X}_\Phi}}\right)\right)\right) \leq 1.$$

Here $A = A(c)$ is the number for which

$$\varphi(ct) \leqq A\varphi(t)$$

is satisfied for every $t > 0$.

In these considerations the Young function Φ is „far" from the linear function. Namely, $\Phi(x)/x$ tends increasingly to $+\infty$ as $x \uparrow +\infty$. So, it is of interest to consider separately the case of the linear function, too. In analogy with the classical result of Doob stating that $X \in \mathcal{H}_1$ whenever $X \in L \log L$ we can deduce the following

Theorem 3. *Suppose that $X \in \mathcal{K}_\Phi$, where $\Phi(x) = x \log^+ x$. Then $X \in \mathcal{H}_1$ and we have*

$$E(X_n^*) \leq \frac{12e}{e-2}(e + \|X\|_{\mathcal{K}_{x\log^+ x}}) \log (e + \|X\|_{\mathcal{K}_{x\log^+ x}}).$$

Proof. For the proof we use the inequality of Theorem 1. Choosing $\beta = 2\alpha$ we get

$$\alpha E(\chi(x_n^* \geq 2\alpha)) \leq E(\gamma\chi(x_n^* \geq \alpha)),$$

where $\gamma \in \Gamma_X^{(\Phi)}$ is arbitrary. Multiply this inequality by $1/\alpha$ and integrate with respect to α on the interval $[1, +\infty)$. Then

$$E\left(\left(\frac{X_n^*}{2} - 1\right)^+\right) \leq E(\gamma \log^+ X_n^*),$$

or, in other words,

$$\frac{1}{\max(e, \|\gamma\|_\Phi)} E\left(\left(\frac{X_n^*}{2} - 1\right)^+\right) \leq E\left(\frac{\gamma}{\max(e, \|\gamma\|_\Phi)} \log^+ X_n^*\right).$$

Using the elementary inequality

$$a \log^+ b \leq a \log^+ a + \frac{b}{e}$$

which is valid for arbitrary $a \geq 0$ and $b \geq 0$ we obtain on the right-hand side

$$\frac{1}{\max(e, \|\gamma\|_\Phi)} E\left(\left(\frac{X_n^*}{2} - 1\right)^+\right) \leq E\left(\frac{\gamma}{\max(e, \|\gamma\|_\Phi)} \log^+ \frac{\gamma}{\max(e, \|\gamma\|_\Phi)} + \right.$$
$$\left. + \frac{X_n^*}{e \max(e, \|\gamma\|_\Phi)} + \frac{\gamma}{\max(e, \|\gamma\|_\Phi)} \log^+ \max(e, \|\gamma\|_\Phi)\right).$$

Note that

$$\frac{X_n^*}{2} \leq \left(\frac{X_n^*}{2} - 1\right)^+ + 1.$$

From this and from the above inequlity

$$\frac{1}{2\max(e, \|\gamma\|_\Phi)}E(X_n^*) \leq 2 + \frac{1}{e\max(e, \|\gamma\|_\Phi)}E(X_n^*) + $$
$$+\frac{E(\gamma)\log\max(e, \|\gamma\|_\Phi)}{\max(e, \|\gamma\|_\Phi)},$$

since

$$\log^+\max(e, \|\gamma\|_\Phi) = \log\max(e, \|\gamma\|_\Phi).$$

This implies that

$$\frac{e-2}{2e}E(X_n^*) \leq 2\max(e, \|\gamma\|_\Phi) + E(\gamma)\log\max(e, \|\gamma\|_\Phi).$$

Now we show that

$$E(\gamma) \leq 4\|\gamma\|_\Phi.$$

In fact, in case of any Young function Φ we have for every $x > 0$ the inequality

$$\Phi(x) \geq (x - x_0)\varphi(x_0),$$

where $x_0 > 0$ satisfies $\varphi(x_0) > 0$. Here, as usual, φ denotes the right-hand side derivative of Φ. Consequently, for arbitrary $Y \in L^\Phi$ such that $P(Y = 0) < 0$ we see that

$$1 \geq E\left(\Phi\left(\frac{|Y|}{\|Y\|_\Phi}\right)\right) \geq \varphi(\chi_0)E\left(\left(\frac{|Y|}{\|Y\|_\Phi} - x_0\right)^+\right).$$

Using the inequality $x \leq (x - x_0)^+ + x_0$ from the preceding inequality we get

$$E\left(\frac{|Y|}{\|Y\|_\Phi}\right) \leq \frac{1}{\varphi(x_0)} + x_0,$$

or, in other words

$$E(|Y|) \leq \left(\frac{1}{\varphi(x_0)} + x_0\right)\|Y\|_\Phi.$$

Now, turning to our case, we have $\varphi(x) = 1 + \log x$, if $x \geq 1$ and $\varphi(x) = 0$ if $x < 1$. Consequently, choosing $x_0 = e$ we get $\varphi(x_0) = 1 + \log e = 2$. So,

$$E(\gamma) \leq \left(\frac{1}{2} + e\right)\|\gamma\|_\Phi \leq 4\max(e, \|\gamma\|_\Phi).$$

Comparing this with the inequality above we get

$$\frac{e-2}{2e}E(X_n^*) \leq 6\max(e, \|\gamma\|_\Phi)\log\max(e, \|\gamma\|_\Phi),$$

which implies the inequality

$$E(X_n^*) \le \frac{12e}{e-2}\left((e+\|X\|_{\mathcal{X}_\Phi}) \log (e+\|X\|_{\mathcal{X}_\Phi}\right).$$

This proves the assertion. □

Remark. $\Phi(x) = x \log^+ x$ will be a Young function only in the case when we define its right hand side derivative $\varphi(x)$ to be right continuous at $x = 1$. This means that $\varphi(+1)$ must be equal to 1.
As usual, we say that $X \in L \log L$ if $E(|X| \log^+ |X|) < +\infty$. Consider the set

$$\Gamma_X'^{(x \log^+ x)} = \{\gamma : \gamma \in L \log L, E(|X - X_{n-1}| \,|\, \mathcal{F}_n) \leqq E(\gamma) | \mathcal{F}_n) \text{ a. s. } \forall n \geqq 1\}.$$

Then $\Gamma_X'^{(x \log^+ x)}$ is a subset of $\Gamma_X^{(x \log^+ x)}$ and we have

$$\|\gamma\|_{x \log^+ x} \le \max \left(1, E(\gamma \log^+ \gamma)\right).$$

In fact, if $\gamma \in \Gamma_X'^{(X \log^+ X)}$, then $E(\gamma \log^+ \gamma) < +\infty$. Consequently, if $E(\gamma \log^+ \gamma) > 1$ then by the convexity of $\Phi(x) = x \log^+ x$,

$$E\left(\frac{\gamma}{E(\gamma \log^+ \gamma)} \log^+ \frac{\gamma}{E(\gamma \log^+ \gamma)}\right) \le \frac{1}{E(\gamma \log^+ \gamma)} E(\gamma \log^+ \gamma) = 1.$$

If, conversely, $E(\gamma \log^+ \gamma) \le 1$, then trivially $\|\gamma\|_{x \log^+ x} \le 1$. Therefore,

$$\|\gamma\|_{x \log^+ x} \leqq \max \left(1, \mathrm{E}(\gamma \log^+ \gamma)\right). \quad \square$$

It seems to be interesting to deduce an inequality, like the preceding one for $E(X_n^*)$ in case of the class $\Gamma_X'^{(x \log^+ x)}$. It can be shown that in this case the inequality to be proved is simpler than that of the preceding assertion.

Theorem 4. *Let X be a random variable and suppose that the set $\Gamma_X'^{(x \log^+ x)}$ defined by the formula*

$$\Gamma_X'^{(x \log^+ x)} = \{\gamma : \gamma \in L \log L, E(|X - X_{n-1}| \,|\, \mathcal{F}_n) \le E(\gamma | \mathcal{F}_n) \text{ a. s. } \forall n \ge 1\}.$$

is not empty. Then $X \in \mathcal{H}_1$ and we have

$$E(X_n^*) \le \frac{e-2}{2e}\left(1 + \inf_{\gamma \in \Gamma_X'^{(x \log^+ x)}} E(\gamma \log^+ \gamma)\right).$$

Proof. Again, we shall use the inequality

$$(\beta - \alpha) E\left(\chi(X_n^* \ge \beta)\right) \le E\left(\gamma \chi(X_n^* \ge \alpha)\right)$$

and we choose $\beta = 2\alpha$. Here $\gamma \in \Gamma_X'^{(x \log^+ x)}$ is arbitrary. Then

$$\alpha E\left(\chi\left(\frac{X_n^*}{2} \ge \alpha\right)\right) \le E(\gamma \chi(X_n^* \ge \alpha)).$$

Integrate this with respect to the measure $d\alpha/\alpha$ on the interval $[1, +\infty)$. We then get

$$E\left(\left(\frac{X_n^*}{2} - 1\right)^+\right) \le E(\gamma \log^+ X_n^*) \le E\left(\gamma \log^+ \gamma + \frac{X_n^*}{e}\right).$$

From this

$$\frac{e-2}{2e} E(X_n^*) \leq 1 + E(\gamma \log^+ \gamma)$$

and finally

$$E(X_n^*) \leq \frac{2e}{e-2} [1 + \inf_{\gamma \in \Gamma'^{(x \log^+ x)}_X} E(\gamma \log^+ \gamma)].$$

This proves the assertion. □

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