(0,2,4) LACUNARY INTERPOLATION WITH G-SPLINES

THARWAT FAWZY

Math. Dept., Fac. of Science, Suez-Canal Univ.
Ismailia, Egypt

(Received August 5, 1985)

1. Introduction. Recently, R. S. Mishra and K. K. Mathur [1] introduced a global method for solving the (0, 2, 4) interpolation problem with splines. They constructed spline polynomial of degree 6 and deficiency 2 for functions $f \in C^5$. They introduced the following convergence theorem.

Theorem 1.1. (Mishra and Mathur). *Let* $f \in C^5[0, 1]$ *and* $S_n(x)$ *be the unique spline satisfying the conditions of Theorem 3 of* [1]. *Then* $(n = 2m+1)$

$$\|S_n^{(q)}(x) - f^{(q)}(x)\|_\infty \leq 2245\, m^{q-5} \omega_5\left(\frac{1}{m}\right) + 4m^{q-5}\|f^{(5)}\|_\infty$$

where $q = 0, 1, 2, 3, 4$.

In this paper we study the following interpolation problem.

Problem. Given $\Delta: \{x_i = ih\}_{i=0}^n$ and arbitrary real numbers $\{f_i, f_i'', f_i^{(iv)}\}_{i=0}^n$. Find S such that

$$S^{(q)}(x_i) = f^{(q)}(x_i) = f_i^{(q)},\ q = 0, 2, 4,\ i = 0, 1, \ldots, n. \ \square \tag{1.1}$$

The purpose of this paper is to construct a local spline method for solving this problem using piece-wise polynomials of degree 6, such that for all functions $f \in C^6$, the order of approcimation is the same as the best approximation using splines of degree 6.

2. Construction of the spline interpolant. We shall construct a solution S of the given problem in the form:

$$S_\Delta(x) = S_k(x) = \sum_{j=0}^{6} \frac{S_k^{(j)}}{j!}(x - x_k)^j,\ x_k \leq x \leq x_{k+1}, \tag{2.1}$$

where $k = 0, 1, \ldots, n-1$.

We shall define each of the $S_k^{(j)}$ explicitly in terms of the data. In particular we choose

$$S_k^{(q)} = f_k^{(q)},\ q = 0, 2, 4 \text{ and } k = 0, 1, \ldots, n-1. \tag{2.2}$$

For k = 1, 2, . . . , $n-2$ we take

(2.3) $$S_k^{(6)} = \frac{1}{h^2}\left\{f_{k+1}^{(iv)} - 2f_k^{(iv)} + f_{k-1}^{(iv)}\right\},$$

(2.4) $$S_k^{(5)} = \frac{1}{h}\left\{f_{k+1}^{(iv)} - f_k^{(iv)} - \frac{h^2}{2}S_k^{(6)}\right\},$$

(2.5) $$S_k^{(3)} = \frac{1}{h}\left\{f''_{k+1} - f''_k - \frac{h^2}{2}f_k^{(iv)} - \frac{h^3}{3!}S_k^{(5)} - \frac{h^4}{4!}S_k^{(6)}\right\} \text{ and}$$

(2.6) $$S_k^{(1)} = \frac{1}{h}\left\{f_{k+1} - f_k - \frac{h^2}{2}f''_k - \frac{h^3}{3!}S_k^{(3)} - \frac{h^4}{4!}f_k^{(iv)} - \frac{h^5}{5!}S_k^{(5)} - \frac{h^6}{6!}S_k^{(6)}\right\}.$$

For $k = 0$ we take

(2.7) $$S_0^{(6)} = S_1^{(6)},$$

(2.8) $$S_0^{(5)} = \frac{1}{h}\left\{f_1^{(iv)} - f_0^{(iv)} - \frac{h^2}{2}S_0^{(6)}\right\},$$

(2.9) $$S_0^{(3)} = \frac{1}{h}\left\{f''_1 - f''_0 - \frac{h^2}{2}f_0^{(iv)} - \frac{h^3}{3!}S_0^{(5)} - \frac{h^4}{4!}S_0^{(6)}\right\} \text{ and}$$

(2.10) $$S_0^{(1)} = \frac{1}{h}\left\{f_1 - f_0 - \frac{h^2}{2}f''_0 - \frac{h^3}{3!}S_0^{(3)} - \frac{h^4}{4!}f_0^{(iv)} - \frac{h^5}{5!}S_0^{(5)} - \frac{h^6}{6!}S_0^{(6)}\right\}.$$

Finally for $k = n-1$ we take

(2.11) $$S_{n-1}^{(6)} = S_{n-2}^{(6)},$$

(2.12) $$S_{n-1}^{(5)} = \frac{1}{h}\left\{f_n^{(4)} - f_{n-1}^{(4)} - \frac{h^2}{2}S_{n-1}^{(6)}\right\},$$

(2.13) $$S_{n-1}^{(3)} = \frac{1}{h}\left\{f''_n - f''_{n-1} - \frac{h^2}{2}f_{n-1}^{(iv)} - \frac{h^3}{3!}S_{n-1}^{(5)} - \frac{h^4}{4!}S_{n-1}^{(6)}\right\} \text{ and}$$

(2.14) $$S_{n-1}^{(1)} = \frac{1}{h}\left\{f_n - f_{n-1} - \frac{h^2}{2}f''_{n-1} - \frac{h^3}{3!}S_{n-1}^{(3)} - \frac{h^4}{4!}f_{n-1}^{(iv)} - \frac{h^5}{5!}S_{n-1}^{(5)} - \frac{h^6}{6!}S_{n-1}^{(6)}\right\}.$$

The relation (2.3) is an assumed formula and it has been chosen to give a good approximation to $f^{(6)}(x)$, $x_k \leq x \leq _{k+1}$.

The relations (2.4), (2.5) and (2.6) have been chosen to make S_Δ satisfying the following condition for $k = 1, 2, \dots, n-2$:

$$S_k^{(q)}(x_{k+1}) = S_{k+1}^{(q)}(x_{k+1}) = f_{k+1}^{(q)}, q = 0, 2, 4. \tag{2.15}$$

Similarly, the relations (2.8), (2.9) and (2.10) have been chosen to make S_Δ satisfying the condition (2.15) for $k = 0$, while the relation (2.7) has been chosen to make

$$D_L^6 S_0(x_1) = D_R^6 S_1(x). \tag{2.16}$$

where D_R is the right derivative.

Finally, the relation (2.11) has been chosen to make

$$D_L^6 S_{n-2}(x_{n-1}) = D_R^6 S_{n-1}(x_{n-1}), \tag{2.17}$$

while the relations (2.12), (2.13) and (2.14) have been chosen to make S_Δ satisfying the condition

$$S_\Delta^{(q)}(x_n) = f_n^{(q)}, q = 0, 2, 4. \tag{2.18}$$

Clearly, the function S defined in (2.1)–(2.14) solves the interpolation problem. Moreover, by construction it is clear that S is a piece-wise polynomial of degree 6.

Indeed S is the unique piece-wise polynomial of degree 6 in

$$C^{(0,2,4,6)}[x_0, x_2] \cap C^{(0,2,4)}[x_0, x_n] \cap C^{(0,2,4,6)}[x_{n-2}, x_n] \tag{2.19}$$

and satisfying the interpolation condition (1.1), where

$$C^{(0,2,4,6)}[a, b] = \{f: f, D_R^2 f, D_R^4 f, D_R^6 f \in C[a, b]\} \text{ and} \tag{2.20}$$

$$C^{(0,2,4)}[a, b] = \{f: f, D_R^2 f, D_R^4 f \in C[a, b]\}. \tag{2.21}$$

S is a special kind of g-splines, we refer to it as lacunary g-spline.

3. Error bounds for the spline interpolant. Suppose $f \in C^6[x_0, x_n]$. Then using Taylor and dual Taylor expansions it is easy to establish the following lemma estimating how well the $S_k^{(j)}$ approximate $f^{(j)}(x_k)$ in term of the modulus of continuity $\omega(D^6 f; h)$ of $f^{(6)}(x)$.

Lemma 3.1. *For $1 \le k \le n-2$ and $j = 1, 3, 5$ and 6, we have*

$$|S_k^{(j)} - f^{(j)}(x_k)| \le C_{jk} h^{6-j} \omega(D^6 f; h), \tag{3.1}$$

where the constants c_{jk} are the following: $c_{1k} = \frac{7}{240}$, $c_{3k} = \frac{5}{48}$, $c_{5k} = \frac{3}{4}$ *and* $c_{6k} = \frac{3}{2}$.

Theorem 3.1. *Let $f \in C^6[x_0, x_n]$ and let S_Δ be the lacunary g-spline constructed in* (2.1)–(2.14). *Then for all $0 \le j \le 6$ and all $1 \le k \le n-2$, we have*

$$\|D^j(f - S_\Delta)\|_{L_\infty[x_k, x_{k+1}]} \le c_{jk}^* h^{6-j} \omega(D^6 f; h),$$

where the constants $\overset{*}{c}_{jk}$ are: $\overset{*}{c}_{0k} = \frac{79}{1440}$, $\overset{*}{c}_{1k} = \frac{1}{8}$, $\overset{*}{c}_{2k} = \frac{7}{24}$, $\overset{*}{c}_{3k} = = \frac{35}{48}$, $\overset{*}{c}_{4k} = \frac{3}{2}$, $\overset{*}{c}_{5k} = \frac{9}{4}$ and $\overset{*}{c}_{6k} = \frac{3}{2}$.

Proof. Usint the Taylor expansion of $f(x)$, $x_k \le x \le_{k+1}$, and (2.1) we get

$$|f(x)-S_k(x)| \le \sum_{j=0}^{5} \frac{|f^{(j)}(x_k) - S_k^{(j)}|}{j!} h^j +$$

$$+\frac{h^6}{6!}|f^{(6)}(\xi) - S_k^{(6)}| \le \sum_{j=0}^{6} \frac{|f^{(j)}(x_k) - S_k^{(j)}|}{j!} h^j +$$

$$+\frac{h^6}{6!}|f^{(6)}(\xi) - f^{(6)}(x_k)|,$$

where $x_k < \xi < x_{k+1}$. Using Lemma 3.1, (2.2) and the definition of the modulus of continuity of $f^{(6)}(x)$ we get

$$|f(x)-S_k(x)| \le \frac{79}{1440} h^6 \omega(D^6 f; h).$$

Similar procedures for the derivatives will easily complete the proof. □

Finally, we can use the above theorem to prove the following lemma and theorem.

Lemma 3.1. *For $k = 0, n-1$ and $j = 1, 2, 3, 5$ and 6 we have*

$$|S_k^{(j)} - f^{(j)}(x_k)| \le c_{j,k} h^{6-j} \omega(D^6 f; h),$$

where the constants $c_{j,k}$ are given by $c_{1,k} = \frac{7}{144}$, $c_{3,k} = \frac{5}{12}$, $c_{5,k} = \frac{5}{4}$ and $c_{6,k} = \frac{5}{2}$.

Theorem 3.2. *Let $f \in C^6[x_0, x_n]$ and let S_Δ be the lacunary g-spline given in (2.1)–(2.14). Then for all $0 \le j \le 6$ and $k = 0, n-1$ we have*

$$\|D^j(f - S_\Delta)\|_{L_\infty[x_k, x_{k+1}]} \le \overset{*}{c}_{j,k} h^{6-j} \omega(D^6 f; h),$$

where the constants $\overset{*}{c}_{j,k}$ are given by $\overset{*}{c}_{0,k} = \frac{19}{144}$, $\overset{*}{c}_{1,k} = \frac{95}{288}$, $\overset{*}{c}_{2,k} = \frac{35}{48}$, $\overset{*}{c}_{3,k} = \frac{35}{24}$, $\overset{*}{c}_{4,k} = \frac{5}{2}$, $\overset{*}{c}_{5,k} = \frac{15}{4}$ and $\overset{*}{c}_{6,k} = \frac{5}{2}$.

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