

ON DIFFERENTIABLE ADDITIVE FUNCTIONS

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1. Let \mathcal{A} denote the set of all those real sequences $\{\lambda_n\}$ for which $\lambda_n > \lambda_{n+1} > 0$ ($n \in \mathbb{N}$) and $L := \sum_{n=1}^{\infty} \lambda_n < \infty$.

Definition 1.1. A sequence $\{\lambda_n\} \in \mathcal{A}$ is said to be *interval filling*, if for any number $x \in [0, L]$ there exists a sequence $\varepsilon_n \in \{0, 1\}$ ($n \in \mathbb{N}$), such that

$$(1.1) \quad x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n. \quad \square$$

As is known ([1]), we have the following

Theorem 1.2. A sequence $\{\lambda_n\} \in \mathcal{A}$ is interval filling if and only if for any $n \in \mathbb{N}$

$$(1.2) \quad \lambda_n \leq \sum_{i=n+1}^{\infty} \lambda_i$$

holds.

Let $\{\lambda_n\} \in \mathcal{A}$ be an interval filling sequence and $x \in [0, L]$. Moreover, let us define by induction on n

$$(1.3) \quad \varepsilon_n(x) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x) \lambda_i + \lambda_n \leq x \\ 0 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x) \lambda_i + \lambda_n > x. \end{cases}$$

Then ([1], [2])

$$(1.4) \quad x = \sum_{n=1}^{\infty} \varepsilon_n(x) \lambda_n$$

and the representation (1.4) will be called the *regular expansion* of x with respect to the sequence $\{\lambda_n\}$.

Definition 1.3. Let $\{\lambda_n\} \in \mathcal{A}$ be an interval filling sequence. We call a bounded function $F: [0, L] \rightarrow \mathbf{R}$ *additive*, if for any $x \in [0, L]$ the equality

$$(1.5) \quad F(x) = \sum_{n=1}^{\infty} \varepsilon_n(x) F(\lambda_n)$$

holds, where $\varepsilon_n(x)$ equals 0 or 1 according to the algorithm (1.3). \square

It is known that if we put

$$a_n := F(\lambda_n) \quad (n \in \mathbf{N}), \text{ then } \sum_{n=1}^{\infty} |a_n| < \infty,$$

and conversely any sequence $a_n \in \mathbf{R}$ satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$ determines an additive function.

In this paper we investigate the following question: What can be said about the structure of a differentiable additive function defined on the interval $[0, L]$?

2. Let $\{\lambda_n\} \in \mathcal{A}$ be a fixed interval filling sequence.

Definition 2.1. We call a number $x \in [0, L]$ *terminating*, if there exists a number N , such that $\varepsilon_n(x) = 0$ for $n > N$. If x is terminating and $\varepsilon_m(x) = 1$, moreover $\varepsilon_n(x) = 0$ for $n > m$ then we say that x has *length* m , written $h(x) = m$. \square

Lemma 2.1. Let $\{\lambda_n\} \in \mathcal{A}$ be an interval filling sequence and $F: [0, L] \rightarrow \mathbf{R}$ an additive function (with respect to $\{\lambda_n\}$), differentiable at the terminating point $x \in [0, L]$. If $a_n := F(\lambda_n)$ ($n \in \mathbf{N}$) then

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{a_n}{\lambda_n} = F'(x).$$

Proof. Let $h(x) = m$, i. e.

$$x = \sum_{i=1}^m \varepsilon_i(x) \lambda_i.$$

Then there exists an $N > m$, such that for $n > N$ the relation

$$\varepsilon_k(x + \lambda_n) = \begin{cases} \varepsilon_k(x) & \text{for } k = 1, 2, \dots, m \\ 1 & \text{for } k = n \\ 0 & \text{for } k \in \mathbf{N} \setminus \{1, 2, \dots, m\} \cup \{n\}. \end{cases}$$

Thus

$$\frac{a_n}{\lambda_n} = \frac{F(x + \lambda_n) - F(x)}{\lambda_n} \quad (n > N)$$

and, by differentiability at x , this implies (2.1). \square

3. Let us now ask the following more difficult question. What will the consequences be in case the additive function F is differentiable at a *non-terminating* point x ? For arbitrary interval filling sequences this question still remains open, nevertheless we are able to give an answer for quite a large class of sequences.

Definition 3.1. We call the interval filling sequence $\{\lambda_n\} \in \mathcal{A}$ *smooth*, if there exists a constant $K > 1$, such that

$$(3.1) \quad \sum_{i=n+1}^{\infty} \lambda_i < K\lambda_n$$

for any $n \in \mathbb{N}$. \square

The inequality (3.1) expressing smoothness can be regarded as being complementary to the inequality (1.2).

Theorem 3.2. Let $\{\lambda_n\} \in \mathcal{A}$ be a smooth interval filling sequence and $F: [0, L] \rightarrow \mathbb{R}$ an additive function (with respect to $\{\lambda_n\}$). If F is differentiable at the nonterminating point $x \in [0, L]$, then with the notation $a_n := F(\lambda_n)$ ($n \in \mathbb{N}$) we have

$$(3.2) \quad \lim_{\substack{N \rightarrow \infty \\ N \in \mathbf{N}_1}} \frac{a_N}{\lambda_N} = F'(x),$$

where $\mathbf{N}_1 := \{n | n \in \mathbb{N}, \varepsilon_n(x) = 1\}$.

Proof. Let

$$x = \sum_{n=1}^{\infty} \varepsilon_n(x) \lambda_n,$$

where $\mathbf{N}_1 := \{n | n \in \mathbb{N}, \varepsilon_n(x) = 1\}$ is an infinite index set. Let moreover $S_N(x) := \sum_{n=1}^N \varepsilon_n(x) \lambda_n$, where the right-hand side is the regular expansion of $S_N(x)$. Since F is differentiable at x , there exists a function $E_x:]0, L[\rightarrow \mathbb{R}$, such that

$$\lim_{y \rightarrow x} E_x(y) = 0$$

and

$$(3.3) \quad F(x) - F(y) = F'(x)(x - y) + E_x(y)(x - y)$$

is satisfied for every $y \in]0, L[$.

If $N \in \mathbf{N}_1$ (i.e. $\varepsilon_N(x) = 1$), then (3.3) implies

$$F(x) - F(s_N(x)) = F'(x)(x - s_N(x)) + E_x(s_N(x))(x - s_N(x))$$

and

$$F(x) - F(s_{N-1}(x)) = F'(x)(x - s_{N-1}(x)) + E_x(s_{N-1}(x))(x - s_{N-1}(x)).$$

Hence

$$\begin{aligned} a_N = F(s_N(x)) - F(s_{N-1}(x)) &= F'(x)\lambda_N + E_x(s_{N-1}(x)) \left[\lambda_N + \right. \\ &\quad \left. + \sum_{i=N+1}^{\infty} \varepsilon_i(x)\lambda_i \right] - E_x(s_N(x)) \left[\sum_{i=N+1}^{\infty} \varepsilon_i(x)\lambda_i \right]. \end{aligned}$$

From the previous equality we get for $N \in \mathbb{N}_1$ that

$$\frac{a_N}{\lambda_N} = F'(x) + E_x(s_{N-1}(x)) \left[1 + \eta_N(x) \right] - E_x(s_N(x))\eta_N(x),$$

where

$$0 < \eta_N(x) := \frac{\sum_{i=N+1}^{\infty} \varepsilon_i(x)\lambda_i}{\lambda_N} \leq \frac{\sum_{i=N+1}^{\infty} \lambda_i}{\lambda_N} < K.$$

Thus for $N \rightarrow \infty$ ($N \in \mathbb{N}_1$) we obtain (3.2). \square

4. On the basis of the foregoing we are able now to formulate the following

Theorem 4.1. *Let $\{\lambda_n\} \in \mathcal{A}$ be a smooth interval filling sequence. Let moreover $F: [0, L] \rightarrow \mathbb{R}$ be an additive function (with respect to $\{\lambda_n\}$), differentiable at every point. Then there exists a $c \in \mathbb{R}$ such that $F(x) = cx$ for every $x \in [0, L]$.*

Proof. By Lemma 2.1. there exists $c := \lim_{n \rightarrow \infty} \frac{a_n}{\lambda_n}$ with $a_n := F(\lambda_n)$. On the basis of Lemma 2.1. and Theorem 3.2. $F'(x) = c$ for any $x \in [0, L]$. Hence $F(x) = cx$. \square

It is known that for $1 < q < 2$ the sequence

$$\lambda_n := \frac{1}{q^n} \quad (n \in \mathbb{N}) \text{ is interval filling } \left(L = \frac{1}{q-1} \right).$$

On the other hand, in view of

$$\sum_{i=n+1}^{\infty} \frac{1}{q^i} = \frac{L}{q^n},$$

this sequence is smooth, hence Theorem 4.1. can be applied.

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