SPLINE APPROXIMATIONS FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS. II

TH. FAWZY* and Z. RAMADAN**

* Dept. of Math., Fac. of Science, Suez Canal Univ., Ismailia, Egypt
** Dept. of Math., Fac. of Education, Ain Shams Univ., Cairo, Egypt

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Abstract. In this paper we present a method for approximating the solution of the system of nonlinear ordinary differential equations $y' = f_1(x, y, z)$, $z' = f_2(x, y, z)$ with $y(x_0) = y_0$ and $z(x_0) = z_0$ adopting spline functions which are not necessarily polynomial splines. It is a one-step method $O(h^{2+\alpha})$ in $y(x)$, $y'(x)$, $y''(x)$, $z(x)$, $z'(x)$ and $z''(x)$ where $0<\alpha\leq 1$, assuming that f_1, $f_2 \in C^1[a, b]$.

Description of the method

Consider the system of ordinary differential equations

$$y' = f_1(x, y, z), \quad y(x_0) = y_0, \tag{1}$$

$$z' = f_2(x, y, z), \quad z(x_0) = z_0, \tag{2}$$

where $f_1, f_2 \in C^1([0,1]\times \mathbf{R}^2)$.

Let Δ be the partition

$$\Delta: 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1,$$

where $\quad x_{k+1} - x_k = h$ and $k = 0,1. \ldots, n-1$.

Let L_1 and L_2 be the Lipschitz constants satisfied by the functions f_1, f_1' and f_2, f_2' respectively, i.e.,

$$|f_1^{(j)}(x, y_1, z_1) - f_1^{(j)}(x, y_2, z_2)| \leq L_1\{|y_1 - y_2| + |z_1 - z_2|\}, \quad j = 0,1 \tag{3}$$

and

$$|f_2^{(j)}(x, y_1, z_1) - f_2^{(j)}(x, y_2, z_2)| \leq L_2\{|y_1 - y_2| + |z_1 - z_2|\}, \quad j = 0,1 \tag{4}$$

for all (x, y_1, z_1) and (x, y_2, z_2) in the domain of definition of f_1, f_1', f_2 and f_2'.

Then we define the spline functions approximating $y(x)$ and $z(x)$ by $S_\Delta(x)$ and $\overline{S}_\Delta(x)$ by:

$$S_\Delta(x) = S_k(x), \; x_k \leq x \leq x_{k+1}, \; k = 0, 1, \ldots, n-1 \tag{5}$$

and

(6) $$\bar{S}_\Delta(x) = \bar{S}_k(x),\ x_k \le x \le x_{k+1},\ k = 0, 1, \ldots, n-1.$$

Both $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ are given by

(7) $$S_k(x) = S_{k-1}(x_k) + \int_{x_k}^{x} f_1[t, y_k^*(t), z_k^*(t)]\, dt,\ k = 0, 1, \ldots, n-1,$$

where

(8) $$y_k^*(t) = S_{k-1}(x_k) + f_1\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}(t-x_k) + \\ + \frac{1}{2} f_1'\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}(t-x_k)^2,$$

(9) $$z_k^*(t) = \bar{S}_{k-1}(x_k) + f_2\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}(t-x_k) + \\ + \frac{1}{2} f_2'\{x_k, S_{k-1}(x_k), S_{k-1}(x_k)\}(t-x_k)^2,\ x_k \le t \le x \le x_{k+1},$$

(10) $$S_{-1}(x_0) = y_0,\ \bar{S}_{-1}(x_0) = z_0$$

and

(11) $$\bar{S}_k(x) = \bar{S}_{k-1}(x_k) + \int_{x_k}^{x} f_2[t, y_k^*(t), z_k^*(t)]\, dt.$$

It is clear by the construction, that $S_\Delta(x)$ and $\bar{S}_\Delta(x) \in C[0, 1]$.

It should be noted that we use the Lipschitz conditions on f_1 and f_2 to guarantee the existence of a unique solution to the problem (1)–(2).

We now discuss the convergence of these approximants.

For all $x \in [x_k, x_{k+1}]$, $k = 0,1. \ldots. n-1$ the exact solutions of (1) and (2) can be written – by using Taylor's expansion – in the forms:

(12) $$y(x) = y_k + \int_{x_k}^{x} f_1[t, y_k(t), z_k(t)]\, dt,$$

where

(13) $$y_k(t) = y_k + y_k'(t-x_k) + \frac{1}{2} y''(\xi_k)(t-x_k)^2,$$

(14) $$z_k(t) = z_k + z_k'(t-x_k) + \frac{1}{2} z''(\eta_k)(t-x_k)^2,$$

$$\xi_k, \eta_k \in (x_k, x_{k+1})$$

and

$$z(x) = z_k + \int_{x_k}^{x} f_2[t, y_k(t), z_k(t)]\, dt. \tag{15}$$

We now estimate $|y(x)-s_0(x)|$ where $x \in [x_0, x_1]$.
Using (7–10), (12–14) and the Lipschitz condition (3), we get

$$|y(x) - s_0(x)| \leq \int_{x_0}^{x} |f_1[t, y_0(t), z_0(t)] - f_1[t, y_0^*(t), z_0^*(t)]|\, dt \leq$$

$$\leq L_1 \int_{x_0}^{x} \{|y_0(t) - y_0^*(t)| + |z_0(t) - z_0^*(t)|\}\, dt. \tag{16}$$

Now let

$$U = |y_0(t) - y_0^*(t)| \text{ and } v = |z_0(t) - z_0^*(t)|.$$

Then

$$U = \frac{1}{2}|y''(\xi_0) - y_0''|\,|t - x_0|^2 \tag{17}$$

and

$$V = \frac{1}{2}|z''(\eta_0) - z_0''|\,|t - x_0|^2. \tag{18}$$

Thus using (16), (17) and (18) we get:

$$|y(x) - s_0(x)| \leq \frac{h^3}{6} L_1\{\omega(y'', h) + \omega(z'', h)\} \leq \frac{1}{3} L_1 h^3 \omega(h) = O(h^{3+\alpha}), \tag{19}$$

where $\omega(y'', h)$ and $\omega(z'', h)$ are the moduli of continuity of the functions y'' and z'' respectively, and

$$\omega(h) = \max\{\omega(y'', h), \omega(z'', h)\}. \tag{20}$$

We now estimate $|y'(x) - z'(x)|$.
Using (7–10), (12–14) and the Lipschitz condition (3) we get

$$|y'(x) - s_0'(x)| \leq \frac{1}{2} L_1\{|y''(\xi_0) - y_0''| + |z''(\eta_0) - z_0''|\}|x - x_0|^2 \leq$$

$$\leq \frac{1}{2} L_1\{\omega(y'', h) + \omega(z'', h)\}h^2 \leq \tag{21}$$

$$\leq L_1 h^2 \omega(h) = O(h^{2+\alpha}).$$

We also estimate $|y''(x)-s_0''(x)|$. Thus using (7–10), (12–14) and the Lipschitz condition (3) we get

$$|y''(x)-s_0''(x)| \leq \frac{1}{2}L_1\{|y''(\xi_0)-y_0''|+|z''(\eta_0)-z_0''|\}|x-x_0|^2 \leq \tag{22}$$

$$\leq L_1h^2\omega(h) = O(h^{2+\alpha}).$$

By the same way, using (8–11), (13–15) and employing the Lipschitz condition (4), it can be shown that

$$|z(x)-\bar{s}_0(x)| \leq \frac{1}{3}L_2h^3\omega(h) = O(h^{3+\alpha}), \tag{23}$$

$$|z'(x)_0'-\bar{s}_0(x)| \leq L_2h^2\omega(h) = O(h^{2+\alpha}), \tag{24}$$

and

$$|z''(x)-\bar{s}_0''(x)| \leq L_2h^2\omega(h) = O(h^{2+\alpha}). \tag{25}$$

Now, we are going to consider the general subinterval $[x_k, x_{k+1}]$, $k = 1, 2, \ldots, n-1$.

Using (7–9), (12–14) and the Lipschitz condition (3), we get

$$|y(x)-s_k(x)| \leq |y_k-s_{k-1}(x_k)|+L_1\int_{x_k}^{x}\{|y_k(t)-y_k^*(t)|+|z_k(t)-z_k^*(t)|\}\,dt. \tag{26}$$

Now let

$$U_1 = |y_k(t)-y_k^*(t)|.$$

Then

$$U_1 \leq |y_k-s_{k-1}(x_k)|+|y_k'-f_1\{x_k, s_{k-1}(x_k), \bar{s}_{k-1}(x_k)\}|\,|t-x_k|+ \tag{27}$$

$$+\frac{1}{2}|y''(\xi_k)-f_1'\{x_k, s_{k-1}(x_k), \bar{s}_{k-1}(x_k)\}|\,|t-x_k|^2.$$

Using the fact that $s_\Delta(x)\in C[0, 1]$, $\bar{s}_\Delta(x)\in C[0, 1]$ and the notations

$$e(x) = |y(x)-s_k(x)|,$$

$$e_k(x) = |y_k-s_k(x_k)|,$$

$$\bar{e}(x) = |z(x)-\bar{s}_k(x)|,$$

$$\bar{e}_k(x) = |z_k-\bar{s}_k(x_k)|,$$

and if we let

$$T = |y_k'-f_1\{x_k, s_{k-1}(x_k), \bar{s}_{k-1}(x_k)\}|,$$

then

$$T \leq L_1\{|y_k-s_{k-1}(x_k)|+|z_k-\bar{s}_{k-1}(x_k)|\},$$

i. e.

$$T \le L_1(e_k + \bar{e}_k), \tag{29}$$

and if we let

$$T_1 = |y''(\xi_k) - f_1'\{x_k, s_{k-1}(x_k), \bar{s}_{k-1}(x_k)\}|,$$

then

$$T_1 \le |y''(\xi_k) - y_k''| + |f_1'(x_k, y_k, z_k) - f_1'\{x_k, s_{k-1}(x_k), \bar{s}_{k-1}(x_k)\}|,$$

i. e.

$$T_1 \le \omega(y'', h) + L_1(e_k + \bar{e}_k). \tag{30}$$

Now, using (28 – 30) in the inequality (27), we get

$$U_1 \le e_k + L_1(e_k + \bar{e}_k)|t - x_k| + \frac{1}{2}\{\omega(y'', h) + L_1(e_k + \bar{e}_k)\}|t - x_k|^2. \tag{31}$$

Similarly, let

$$V_1 = |z_k(t) - z_k^*(t)|.$$

Then, using (28) and the Lipschitz condition (4), we get

$$V_1 \le \bar{e}_k + L_2(e_k + \bar{e}_k)|t - x_k| + \frac{1}{2}\{\omega(z'', h) + L_2(e_k + \bar{e}_k)\}|t - x_k|^2. \tag{32}$$

Using (26), (28), (31) and (32) we can easily get

$$e(x) \le e_k(1 + c_0 h) + c_0 h \bar{e}_k + \frac{1}{3} L_1 h^3 \omega(h), \tag{33}$$

where $c_0 = L_1 + \frac{2}{3}L_1^2 + \frac{2}{3}L_1 L_2$ is a constant independent of h and $h < 1$.

Similarly using (13 – 15), (8, 9, 11) and the Lipschitz conditions (3 – 4) we can easily see that

$$\bar{e}(x) \le \bar{e}_k(1 + c_1 h) + c_1 h e_k + \frac{1}{3} L_2 h^3 \omega(h), \tag{34}$$

where $c_1 = L_2 + \frac{2}{3}L_2^2 \quad \frac{2}{3}L_1 L_2$ is a constant independent of h and $h < 1$.

If we use the matrix notations

$$E(x) = (e(x)\,\bar{e}(x))^T,$$

$$E_k = (e_k\,\bar{e}_k)^T, \quad k = 0, 1, \ldots, n-1,$$

then the estimations (33) and (34) can be written in the form

$$E(x) \le \begin{pmatrix} 1 + c_0 h & c_0 h \\ c_1 h & 1 + c_1 h \end{pmatrix} E_k + \frac{1}{3} h^3 \omega(h) \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

i. e.

(35) $$E(x) \le (I + hA)E_k + \frac{1}{3}h^3\omega(h)\, B,$$

where

$$A = \begin{pmatrix} c_0 & c_0 \\ c_1 & c_1 \end{pmatrix}, \quad B = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

and I is the identity matrix of order 2.

At this point, we use the following definition of the matrix norm. Let $\tau = [t_{ij}]$ be an $m \times n$ matrix, then we define

$$\|\tau\| = \max_i \sum_{j=1}^{n} |t_{ij}| .$$

Using this definition, we get

(36) $$\|E_k\| = \max(e_k, \bar{e}_k), \; k = 0, 1, \dots, n-1.$$

Now, since (35) is valid for all $x \in [x_k, x_{k+1}]$, $k = 0, 1, \dots, n-1$, the following inequalities hold:

$$\|E(x)\| \le (1 + h\|A\|)\|E_k\| + \frac{1}{3}h^3\omega(h)\, \|B\|,$$

$$(1 + h\|A\|)\|E_k\| \le (1 + h\|A\|)^2\|E_{k-1}\| + \frac{1}{3}h^3\omega(h)(1 + h\|A\|)\|B\|,$$

$$(1 + h\|A\|)^2\|E_{k-1}\| \le (1 + h\|A\|)^3\|E_{k-2}\| + \frac{1}{3}h^3\omega(h)(1 + h\|A\|)^2\|B\|,$$

$$\dots$$

$$(1 + h\|A\|)^k\|E_1\| \le (1 + h\|A\|)^{k+1}\|E_0\| + \frac{1}{3}h^3\omega(h)(1 + h\|A\|)^k\|B\|.$$

Adding L.H.S. and R.H.S. in these inequalities and noting that $e_0 = 0$ we get

(37) $$E(x) \le c_2 h^2\omega(h),$$

where $c_2 = \dfrac{e^{\|A\|}}{3} \dfrac{\|B\|}{\|A\|}$ is a constant independent of h and $h < 1$.

By the definition (36), it follows that

(38) $$e(x) \le c_2 h^2\omega(h) = O(h^{2+\alpha})$$

and

(39) $$\bar{e}(x) \le c_2 h^2\omega(h) = O(h^{2+\alpha}).$$

We now estimate $|y'(x) - s_k'(x)|$. For this purpose we use equations (7—9), (12—14) and the Lipschitz conditions (3—4) and get

(40) $$e'(x) = |y'(x) - s_k'(x)| \le c_3(e_k + \bar{e}_k) + L_1 h^2\omega(h),$$

where $c_3 = \frac{3}{2}L_1^2 + \frac{3}{2} L_1L_2 + L_1$ is a constant independent of h and $h<1$.

Using (38) and (39) inequality (40) becomes

(41) $$e'(x) \leq c_4 h^2 \omega(h) = O(h^{2+\alpha}),$$

where $c_4 = 2c_2c_3 + L_1$ is a constant independent of n and $h<1$.

In a similar manner we estimate $|z'(x) - \bar{s}_k'(x)|$.

From equations (8–11), (13–15) and using the Lipschitz conditions (3–4) it follows that

(42) $$\bar{e}'(x) = |z'(x) - \bar{s}_k'(x)| \leq c_5(e_k + e_k) + L_2 h^2 \omega(h),$$

where $c_5 = \frac{3}{2}L_2^2 + \frac{3}{2} L_1L_2 + L_2$ is a constant independent of h and $h<1$.

Substituting inequalities (38) and (39) into inequality (42) we get:

(43) $$\bar{s}'(x) \leq c_6 h^2 \omega(h) = O(h^{2+\alpha}),$$

where $c_6 = 2c_2c_5 + L_2$ is a constant independent of h and $h<1$.

We are going to estimate $|y''(x) - s_k''(x)|$ and $|z''(x) - \bar{s}_k''(x)|$, where we are using the following definitions for $s_k''(x)$ and $\bar{s}_k''(x)$:

(44) $$s_k''(x) = f_1'\{x, s_k(x), \bar{s}_k(x)\}$$

and

(45) $$\bar{s}_k''(x) = f_2'\{x, s_k(x), \bar{s}_k(x)\}.$$

Now, using (1) and (44) we get

$$e''(x) \equiv |y''(x) - s_k''(x)| = |f_1'(x, y, z) - f_1'\{x, s_k(x); \bar{s}_k(x)\}.$$

Using (38) and (39) we get

(46) $$e''(x) \leq c_7 h^2 \omega(h) = O(h^{2+\alpha}),$$

where $c_7 = 2L_1c_2$ is a constant independent of h and $h<1$.

Similarly, it can be shown that

(47) $$\bar{e}''(x) \equiv |z''(x) - \bar{s}_k''(x)| \leq c_8 h^2 \omega(h) = O(h^{2+\alpha}),$$

where $c_8 = 2L_2c_2$ is a constant independent of h and $h<1$.

Thus, we have proved the following

Theorem. *Let $s_\Delta(x)$ and $\bar{s}_\Delta(x)$ be the approximate solutions to problem (1)–(2) given by equations (5–11), and let $f_1, f_2 \in c^1([x_0, x_n] \times \mathbf{R}^2)$.*
Then, for all $x \in [x_0, x_1]$ we have

$$|y(x) - s_0(x)| \leq \frac{1}{3} L_1 h^3 \omega(h),$$

$$|y^{(j)}(x) - s_0^{(j)}(x)| \le L_1 h^2 \omega(h),\ j = 1, 2,$$

$$|z(x) - s^*(x)| \le \frac{1}{3} L_2 h^2 \omega(h)$$

and

$$|z^{(j)}(x) - \bar{s}_0^{(j)}(x)| \le L_2 h^2 \omega(h),\ j = 1, 2,$$

and for all $x \in [x_k, x_{k+1}]$, $k = 1(1)n-1$ *we have*

$$|y^{(j)}(x) - \bar{s}_k^{(j)}(x)| \le c_9 h^2 \omega(h)$$

and

$$|z^{(j)}(x) - \bar{\bar{s}}_k^{(j)}(x)| \le c_{10} h^2 \omega(h),$$

where $j = 0, 1$ *and* 2, $c_9 = \max(c_2, c_4, c_7)$ *and* $c_{10} = \max(c_2, c_6, c_8)$.

Numerical example

Consider the following system of differential equations

$$y' = y + z - x - x^2 - e^{2x},$$

$$z' = 2y + 2z - 2\,e^x - 2\,x^2 - 2,\ y(0) = 1,\ z(0) = 2.$$

The method is tested using this example, in the interval [0, 1] with step size $h = 0.1$.

The analytical solution is

$$y(x) = e^x + x$$

and

$$z(x) = e^{2x} + x^2 + 1.$$

The tabuleted results, appearing in the following table, are evaluated at the point $x = 0.25$.

	exact value	approximate value	absolute error
y	1.5340254	1.533906117	0.000119283
y'	2.284025329	2.283397416	0.000627913
y''	1.284025155	1.282951722	0.001073433
z	2.7112212	2.710982672	0.000238528
z'	3.797442367	3.796186541	0.001255826
z''	8.594884559	8.59273769	0.002146869

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