L^p-APPROXIMATION BY SPLINES

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Abstract. In this paper a new spline interpolational method is presented in the L^p-space. For the construction of the approximating spline function integral meanvalues instead of function values have been used. The order of approximation – which is the best possible – is expressed by the L^p-modulus of continuity of the L^p-function to be approximated.

The approximation of differentiable functions and their derivatives by means of interpolating splines on equidistant subdivisions in the sense of L^p-metric $(1 \leq p < \infty)$ has been considered by several authors (see e.g. [2], [3], [4], [5], [6]). The common nature of these methods is that on one side, they can be used either on a finite interval, or for periodic functions only, on the other hand, the functions to be approximated cannot be arbitrary in L^p, namely, function values and even differentiability properties are used. Here we present a spline interpolational method in $L^p(a, b)$, where (a, b) can be either finite, or the whole real line. For the construction of the approximating spline function we use integral mean values instead of function values, and the method gives the same order of approximation expressed by the L^p-modulus of continuity for any L^p-function, which is the best possible one (see [8]). Concerning the main ideas see [7], [9].

In what follows p is a fixed real number with $1 \leq p < \infty$, and L^p denotes $L^p(a, b)$ for either (a, b) is a finite interval on the real line, or (a, b) is the whole real line. In the finite case all functions in L^p are supposed to be extended periodically to the whole line. For any integer $r \geq 0$ the space W_p consists of all r-times differentiable functions with an r-th derivative belonging to L^p. The L^p-modulus of continuity $\omega_r(f, h)_p$ for any f in L^p and $r \geq 0$ integer, is defined as usually (see e.g. [2], [3]).

For the estimations we shall make use of the following theorem, which is a modification of a theorem in [2] and can be proved similarly.

Theorem 1. *Let Γ be an arbitrary set and L_γ: $L^p \to L^p$ for any γ in Γ, uniformly bounded linear operators, for which there exists a nonnegative function*

$a: \Gamma \to \mathbf{R}$ such that $a(\gamma) \leq 1$, further there exists an integer $r \geq 1$ and a constant $c > 0$ for which we have whenever f is in W_p^r

$$\|L_\gamma(f) - f\|_p \leq c \cdot a(\gamma) \|f^{(r)}\|_p.$$

Then there exists a constant $d > 0$ such that for all f in L^p we have

$$\|L\gamma(f) - f\|_p \leq d \cdot \omega_r(f, a(\gamma)^{\frac{1}{r}})_p.$$

Proof. To prove this theorem we use the following result: if f belongs to L^p, $h > 0$ and $r \geq 1$ is an integer, then there exist constants c_r, d_r and a function $f_{r,h}$ in W_p^r such that

$$\|f - f_{r,h}\|_p \leq c_r \cdot \omega_r(f, h)_p$$

and

$$\|f_{r,h}^{(i)}\|_p \leq d_r \cdot h^{-i} \cdot \omega_r(fh)_p \quad (i = 1, 2, \ldots, r).$$

The proof of this theorem can be found in [4] under the assumption that f is bounded, but it is easy to see, that the construction in [4] – which depends on the use of the modified Steklov-transform – works well even if f is unbounded (see also [6]). We note that we shall use this result only in the case $r = 2$, $i = 2$ for which the corresponding statement can be found also in [1].

Let $h = a(\gamma)^{\frac{1}{r}}$, then $h \geq a(\gamma)$ and we have

$$\|L_\gamma(f) - f\|_p \leq \|L_\gamma(f) - L_\gamma(f_{r,h})\|_p + \|L_\gamma(f_{r,h}) - f_{r,h}\|_p + \|f_{r,h} - f\|_p \leq K\|f - f_{r,h}\|_p + c \cdot a(\gamma)\|f_{r,h}^{(r)}\|_p + \|f_{r,h} - f\|_p \leq (K+1)\|f - f_{r,h}\|_p + c \cdot a(\gamma) \cdot d_r \cdot h^{-r} \cdot \omega_r(f, h)_p \leq [(K+1)c_r + c \cdot d_r] \cdot \omega_r(f, a(\gamma)^{\frac{1}{r}})_p,$$

where K is a common bound for the operators L_γ. Hence the theorem is proved. □

Let $h > 0$ and $\{t_k\}$ be a subdivision of (a, b) with $t_{k+1} - t_k = h$. In the finite case we suppose that the number of t_k's is finite, and in all cases $(a, b) = \bigcup_k [t_k, t_{k+1}]$.

For all x in L^p we define

$$y_k = \frac{1}{h} \int_{t_k - \frac{h}{2}}^{t_k + \frac{h}{2}} x(t)\, dt$$

and

$$S_h(x)(t) = y_k + \frac{1}{h} \Delta y_k \cdot (t - t_k) - \frac{1}{h^2} \Delta^2 y_k [(t - t_k)^2 - \frac{1}{h}(t - t_k)^3],$$

whenever t is in $[t_k, t_{k+1}]$, where the operator Δ has the usual meaning

$$\Delta y_k = y_{k+1} - y_k, \quad \Delta^2 y_k = y_{k+2} - 2y_{k+1} + y_k.$$

Obviously $S_h(x)$ is continuously differentiable and $S_h(x) = x$ whenever x is linear.

Theorem 2. *For all x in L^p we have*

$$\|S_h(x)-x\|_p \leq \text{const} \cdot \omega_2(x, h)_p.$$

Proof. The proof of this theorem depends on Theorem 1. Obviously, the operator $x \to S_h(x)$ is linear. On the other hand,

$$\|S_h(x)\|_p \leq \|A_h(x)\|_p + \|B_h(x)\|_p + \|C_h(x)\|_p + \|D_h(x)\|_p,$$

where

$$A_h(x)(t) = y_k,$$

$$B_h(x)(t) = \frac{1}{h}\Delta y_k \cdot (t-t_k),$$

$$C_h(x)(t) = -\frac{1}{h^2}\Delta^2 y_k \cdot (t-t_k)^2,$$

$$D_h(x)(t) = \frac{1}{h^3}\Delta^2 y_k (t-t_k)^3$$

for all t in $[t_k, t_{k+1}]$.

In the estimations which follow we shall use the Hölder-, and Minkowski-inequalities several times, as stated in [1]. We have

$$\|A_h(x)\|_p = \left[\sum_k \int_{t_k}^{t_{k+1}} |y_k|^p \, dt\right]^{1/p} = \left[\sum_k h|y_k|^p\right]^{1/p} =$$

$$= \left[h \sum_k \left|\frac{1}{h} \int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}} x(t)\, dt\right|^p\right]^{1/p} = \left[\frac{1}{h^{p-1}} \sum_k \left|\int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}} x(t)\, dt\right|^p\right]^{1/p} \leq$$

$$\leq \left[\frac{1}{h^{p-1}} \sum_k \int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}} |x(t)|^p \, dt \cdot h^{p-1}\right]^{1/p} = \|x\|_p;$$

$$\|B_h(x)\|_p = \left[\sum_k \int_{t_k}^{t_{k+1}} \left|\frac{1}{h}\Delta y_k (t-t_k)\right|^p dt\right]^{1/p} =$$

$$= \left[\frac{h^{p+1}}{h^p(p+1)} \sum_k |\Delta y_k|^p\right]^{1/p} = \left[\frac{h}{p+1} \sum_k \left|\frac{1}{h} \int_{t_{k+1}-\frac{h}{2}}^{t_{k+1}+\frac{h}{2}} x(t)\, dt - \right.\right.$$

$$-\frac{1}{h}\int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}} x(t)\,dt\Big|^p\Big]^{1/2} = \Big[\frac{1}{(p+1)h^{p-1}}\sum_k\Big|\int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}}[x(t+h)-x(t)]\,dt\Big|^p\Big]^{1/p}\le$$

$$\le\Big[\frac{1}{(p+1)h^{p-1}}\sum_k\int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}}|x(t+h)-x(t)|^r\,dt\cdot h^{p-1}\Big]^{1/p}\le 2\cdot\|x\|_p;$$

$$\|C_h(x)\|_p = \Big[\sum_k\int_{t_k}^{t_{k+1}}\Big|\frac{1}{h^2}\Delta^2y_k(t-t_k)^2\Big|^p dt\Big]^{1/p} =$$

$$=\Big[\frac{h^{2p+1}}{h^{2p}(2p+1)}\sum_k|\Delta^2y_k|^p\Big]^{1/p} = \Big[\frac{h}{2p+1}\sum_k|\Delta y_{k+1}-\Delta y_k|^p\Big]^{1/p}\le$$

$$\le\Big[\frac{2^{p-1}h}{2p+1}\sum_k(|\Delta y_{k+1}|^p+|\Delta y_k|^p)\Big]^{1/p}\le\Big(\frac{2^{p-1}}{2p+1}\Big)^{1/p}\cdot 2^{\frac{p+1}{p}}\cdot\|x\|_p\le 4\cdot\|x\|_p;$$

and finally

$$\|D_h(x)\|_p = \Big[\sum_k\int_{t_k}^{t_{k+1}}\Big|\frac{1}{h^3}\Delta^2y_k(t-t_k)^3\Big|^p dt\Big]^{1/p} =$$

$$=\Big[\frac{h^{3p+1}}{h^{3p}(3p+1)}\sum_k|\Delta y_{k+1}-\Delta y_k|^p\Big]^{1/p}\le$$

$$\le\Big[\frac{2^{p-1}}{3p+1}\Big]^{1/p}[h\sum_k|\Delta y_{k+1}|^p+|\Delta y_k|^p]^{1/p}\le 4\cdot\|x\|_p;$$

and hence $\|S_h(x)\|_p\le 11\cdot\|x\|_p$, that is, the operators S_h are uniformly bounded. Now let $x\in W_p^2$. By the Taylor-formula we have

$$x(t) = y(t)+z(t)+w(t),$$

where

$$y(t) = x(t_k)$$

$$z(t) = x'(t_k)(t-t_k)$$

$$w(t) = \int_{t_k}^{t}(t-\xi)\cdot x''(\xi)d(\xi),$$

whenever t is in $[t_k, t_{k+1}]$. Hence

$$\|S_h(x)-x\|_p\le\|A_h(x)-y\|_p+\|B_h(x)-z\|_p+\|w\|_p+\|C_h(x)\|_p+\|D_h(x)\|_p.$$

Further, we obtain

$$\|w\|_p = \left[\sum_k \int_{t_k}^{t_{k+1}} \left| \int_{t_k}^{t} (t-\xi)\cdot x''(\xi)d\xi \right|^p dt\right]^{1/p} \leq$$

$$\leq \left[\sum_k \int_{t_k}^{t_{k+1}} \left[\int_{t_k}^{t_{k+1}} |t-\xi|\,|x''(\xi)|\,d\xi\right]^p dt\right]^{1/p} \leq$$

$$\leq h\left[\sum_k h\left[\int_{t_k}^{t_{k+1}} |x''(\xi)|^p d\xi\cdot h^{p-1}\right]\right]^{1/p} = h^2\|x''\|_p.$$

On the other hand,

$$\|C_h(x)\|_p = \left[\sum_k \int_{t_k}^{t_{k+1}} \left|\frac{1}{h^2}\Delta^2 y_k\right|^p (t-t_k)^{2p}\,dt\right]^{1/p} =$$

$$= \left(\frac{h^{2p+1}}{2p+1}\right)^{1/p}\left[\sum_k \left|\frac{1}{h^2}\Delta^2 y_k\right|^p\right]^{1/p}.$$

But we have

$$\sum_k \left|\frac{1}{h^2}\Delta^2 y_k\right|^p = \sum_k \left|\frac{1}{h}\left(\frac{x_h(t_{k+2})-x_h(t_{k+1})}{h} - \frac{x_h(t_{k+1})-x_h(t_k)}{h}\right)\right|^p =$$

$$= \sum_k \left|\frac{x_h'(\xi_{k+1})-x_h'(\xi_k)}{h}\right|^p = \sum_k \left|\frac{1}{h}\int_{\xi_k}^{\xi_{k+1}} x_h''(\xi)d(\xi)\right|^p \leq$$

$$\leq \frac{1}{h^p}\sum_k \left[\int_{t_k}^{t_{k+2}} |x_h''(\xi)|\,d\xi\right]^p \leq \frac{1}{h^p}\sum_k \int_{t_k}^{t_{k+2}} |x_h''(\xi)|^p\,d\xi\cdot 2^{p-1}\cdot h^{p-1} =$$

$$= \frac{2^p}{h}\|x_h''\|_p^p,$$

where we used the notation

$$x_h(t) = \frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(s)\,ds,$$

and the fact, that

$$x_h(t) = \frac{1}{h}\left[x\left(t+\frac{h}{2}\right) - x\left(t-\frac{h}{2}\right)\right].$$

Now observe, that

$$\|x_h''\|_p = \left[\int |x_h''(t)|^p\,dt\right]^{1/p} = \left[\int \left|\frac{1}{h}x'\left(t+\frac{h}{2}\right)-\frac{1}{h}x'\left(t-\frac{h}{2}\right)\right|^p dt\right]^{1/p} =$$

$$= \frac{1}{h}\left[\int |x'(t+h)-x'(t)|\,dt\right]^{1/p} = \frac{1}{h}\left[\int \left|\int_t^{t+h} x''(\xi)\,d\xi\right|^p dt\right]^{1/p} \le$$

$$\le \frac{1}{h}\left[\sum_k \int_{t_k}^{t_{k+1}} \left[\int_t^{t+h} |x''(\xi)|\,d\xi\right]^p dt\right]^{1/p} \le \frac{1}{h}\left[\sum_k \int_{t_k}^{t_{k+1}} \left[\int_{t_k}^{t_{k+2}} |x''(\xi)|\,d\xi\right]^p dt\right]^{1/p} \le$$

$$\le \frac{1}{h}\left[h\sum \int_{t_k}^{t_{k+2}} |x''(\xi)|^p\,d\xi\cdot 2^{p-1}\cdot h^{p-1}\right]^{1/p} = 2\|x''\|_p.$$

Hence we have

$$\|C_h(x)\|_p \le \left(\frac{h^{2p+1}}{2p+1}\right)^{1/p}\cdot\frac{2}{h^{1/p}}\cdot 2\cdot\|x''\|_p = \frac{4}{(2p+1)^{1/p}}\cdot h^2\cdot\|x''\|_p.$$

Similarly we get

$$\|D_h(x)\|_p \le \frac{4}{(3p+1)^{1/p}}\cdot h^2\cdot\|x''\|_p.$$

To estimate $\|A_h(x)-y\|_p$ we notice, that by the Taylor-formula we have

$$x(s) = x(t_k)+x'(t_k)(s-t_k)+\int_{t_k}^{s}(s-\xi)x''(\xi)\,d\xi$$

for all s in $[t_k, t_{k+1}]$. It follows

$$y_k = \frac{1}{h}\int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}} x(s)\,ds = \frac{1}{h}\int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}}\int_{t_k}^{s}(s-\xi)x''(\xi)\,d\xi\,ds + x(t_k),$$

and then we obtain

$$\|A_h(x)-y\|_p = \left[\sum_k \int_{t_k}^{t_{k+1}} |y_k - x(t_k)|^p\,dt\right]^{1/p} =$$

$$= \left[\sum_k \int_{t_k}^{t_{k+1}} \left|\frac{1}{h}\int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}}\int_{t_k}^{s}(s-\xi)x''(\xi)\,d\xi\,ds\right|^p dt\right]^{1/p} \le$$

$$\leq \left[h \sum_k \left[\int_{t_k}^{t_{k+1}} \int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}} |x''(\xi)|\,d\xi\,ds\right]^p\right]^{1/p} =$$

$$= \left[h \sum_k h^p \left[\int_{t_k}^{t_{k+1}} |x''(\xi)|\,d\xi\right]^p\right]^{1/p} \leq \left[h \sum_k h^p \int_{t_k}^{t_{k+1}} |x''(\xi)|^p\,d\xi \cdot h^{p-1}\right]^{1/p} =$$

$$= h^2\|x''\|_p.$$

Finally, to estimate $\|B_h(x)-z\|_p$ we use the identity

$$\frac{1}{h}\Delta y_k - x'(t_k) = \frac{x_h(t_k+h)-x_h(t_k)}{h} - x'(t_k) =$$

$$= x_h'(\xi_k) - x'(t_k) = \frac{1}{h}\left[x\left(\xi_k+\frac{h}{2}\right) - x\left(\xi_k-\frac{h}{2}\right)\right] - x'(t_k) =$$

$$= x'(\eta_k) - x'(t_k) = \int_{t_k}^{\eta_k} x''(\xi)\,d\xi,$$

where ξ_k is in $[t_k, t_k+h]$ and η_k is in $\left[\xi_k-\frac{h}{2}, \xi_k+\frac{h}{2}\right]$, hence η_k is in $\left[t_k-\frac{h}{2}, t_k+\frac{3h}{2}\right]$. This yields

$$\|B_h(x)-z\|_p = \left[\sum_k \int_{t_k}^{t_{k+1}} \left|\frac{1}{h}\Delta y_k - x'(t_k)\right|^p (t-t_k)^p\,dt\right]^{1/p} \leq$$

$$\leq \left[\frac{h^{p+1}}{p+1} \sum_k \int_{t_k}^{t_{k+2}} |x''(\xi)|^p\,d\xi \cdot 2^{p-1} \cdot h^{p-1}\right]^{1/p} = \frac{2}{(p+1)^{1/p}} \cdot h^2 \cdot \|x''\|_p.$$

Summarizing our results, we have

$$\|S_h(x)-x\|_p \leq C \cdot h^2 \cdot \|x''\|_p,$$

where the constant C depends only on p.
Now, our theorem is a consequence of Theorem 1. □

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