

ON SYSTEMS WITH BULK ARRIVAL AND GROUP SERVICE. I

LÁSZLÓ LAKATOS

Computing Center of Eötvös Loránd University
1117 Budapest, Bogdánfy u. 10/b.

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1. In [1] for the investigation of characteristics of operating systems the tools of queueing theory are frequently used. Usually these models are connected only with the steady-state behaviour in terms of stationary probability distributions, thus the time-dependent case is not considered in these models. In the present paper we describe a new model which may be used for the investigation of nonstationary characteristics of such systems.

First we formulate the corresponding queueing problem. Let $\xi(t)$, $t \geq 0$ be an inhomogeneous Markov chain with state space $\{0, 1, \dots\}$ and transition probabilities for $[t, t + \Delta](\Delta \rightarrow 0)$:

$$\begin{aligned} (1) \quad \left\{ P \xrightarrow{[t, t+\Delta]} k - c + r \right\} &= \delta_{rc} + a_r(t)\Delta + o(\Delta), \\ r &\geq 0, k \geq c, \\ \left\{ P \xrightarrow{[t, t+\Delta]} r \right\} &= \delta_{kr} + b_{kr}(t)\Delta + o(\Delta), \\ r &\geq 0, 0 \leq k < c, \end{aligned}$$

where $b_{kk}(t) < 0$, $a_c(t) < 0$, $b_{kr}(t)(k \neq r)$ and $a_r(t)(r \neq c)$ nonnegative continuous functions, for which the relations

$$\sum_{r=0}^{\infty} b_{kr}(t) = 0 \quad \text{and} \quad \sum_{r=0}^{\infty} a_r(t) = 0$$

hold.

Consider a more general process. Associate with $\xi(t)$ a homogeneous second component $\eta(t) \in \mathbb{R}^d$, and create the process $\zeta(t) = \{\xi(t), \eta(t)\}$. The second component will be additive and so it can describe e.g. the costs connected with the functioning of the system. So we get a Markov process with homogeneous second component. These processes with property

$$\begin{aligned} P\{\xi(s) = r, \eta(s) \in A \mid \xi(t) = k, \eta(t) = x\} = \\ = P\{\xi(s) = r, \eta(s) \in A_{-x} \mid \xi(t) = k, \eta(t) = 0\}, \end{aligned}$$

where

$$A_{-x} = \{y \in \mathbb{R}^d: x+y \in A\},$$

were introduced and investigated by Ezhov and Skorohod in [4] and [5]. We determine $\eta(t)$ depending on the value of $\xi(t)$ as follows:

- (2)
1. if $\xi(t-0) = k, \xi(t+0) = k$ and $k \geq c$, then $\eta(t) = \varepsilon(t)$;
 2. if $\xi(t-0) = k, \xi(t+0) = r$ and $k \geq c$, then $\eta(t) = \varepsilon_{r-k}(t)$;
 3. if $\xi(t-0) = k, \xi(t+0) = k$ and $0 \leq k < c$, then $\eta(t) = \varepsilon_{kk}(t)$;
 4. if $\xi(t-0) = k, \xi(t+0) = r$ and $0 \leq k < c$, then $\eta(t) = \varepsilon_{kr}(t)$;

where $\varepsilon(t), \varepsilon_j(t), \varepsilon_{kr}(t) \in \mathbb{R}^d$ are homogeneous processes, $\eta(t)$ coincides with one of them, the choice depends on the actual jump of the Markov chain.

Moreover, let

$$\begin{aligned} M(e^{i(\lambda, \varepsilon(s) - \varepsilon(t))} | \xi(x) = \text{const}, x \in [t, s]) &= \exp \left\{ \int_t^s f(x, \lambda) dx \right\}, \\ M(e^{i(\lambda, \varepsilon_{r-k}(t+0) - \varepsilon_{r-k}(t-0))} | \xi(t+0) - \xi(t-0) = r - k) &= \varphi_{r-k}(t, \lambda), \\ M(e^{i(\lambda, \varepsilon_{kk}(s) - \varepsilon_{kk}(t))} | \xi(x) = \text{const}, x \in [t, s]) &= \exp \left\{ \int_t^s f_k(x, \lambda) dx \right\}, \\ M(e^{i(\lambda, \varepsilon_{kr}(t+0) - \varepsilon_{kr}(t-0))} | \xi(t-0) = k, \xi(t+0) = r) &= \varphi_{kr}(t, \lambda), \end{aligned}$$

where $f(t, \lambda)$ and $f_k(t, \lambda)$ are cumulants of processes with independent increments in $\mathbb{R}^d$, $\varphi_r(t, \lambda)$ and $\varphi_{kr}(t, \lambda)$ are characteristic functions for certain distributions in $\mathbb{R}^d$. It can be easily seen that we get our original system with $\lambda = 0$.

2. The described Markov chain with homogeneous second component described above is regular. We derive the direct system of Kolmogorov differential equations for the transition probabilities

$$P_{kr}(t, s, A) = P\{\xi(s) = r, \eta(s) - \eta(t) \in A | \xi(t) = k\}.$$

Here A is the set of possible values of the second component. Let

$$P_{kr}(t, s, \lambda) = \int_{\mathbb{R}^d} e^{i(\lambda, z)} P\{\xi(s) = r, \eta(s) - \eta(t) \in dz | \xi(t) = k\}$$

and determine $P_{kr}(t, t+\Delta, \lambda)$:

1. $k \geq c, r \geq 0$. 1. $r \neq c$

$$\begin{aligned} P_{k, k-c+r}(t, t+\Delta, \lambda) &= [a_r(t)\Delta + o(\Delta)] M e^{i(\lambda, \varepsilon_{r-c}(t+\Delta) - \varepsilon_{r-c}(t))} = \\ &= [a_r(t)\Delta + o(\Delta)] \varphi_{r-c}(t, \lambda) = a_r(t, \lambda)\Delta + o(\Delta); \end{aligned}$$

2. $r = c$

$$\begin{aligned} P_{k,k}(t, t + \Delta, \lambda) &= [1 + a_c(t)\Delta + o(\Delta)] Me^{l(\lambda, e(t+\Delta) - e(t))} = \\ &= [1 + a_c(t)\Delta + o(\Delta)] \exp \left\{ \int_t^{t+\Delta} f(x, \lambda) dx \right\} = [1 + a_c(t)\Delta + o(\Delta)] \cdot \\ &\cdot [1 + f(t, \lambda)\Delta + o(\Delta)] = 1 + [a_c(t) + f(t, \lambda)]\Delta + o(\Delta) = \\ &= 1 + a_c(t, \lambda)\Delta + o(\Delta); \end{aligned}$$

II. $0 \leq k < c$. 1. $k \neq r$

$$\begin{aligned} P_{k,r}(t, t + \Delta, \lambda) &= [b_{kr}(t)\Delta + o(\Delta)] Me^{l(\lambda, e_{kr}(t+\Delta) - e_{kr}(t))} = \\ &= [b_{kr}(t)\Delta + o(\Delta)] \varphi_{kr}(t, \lambda) = b_{kr}(t, \lambda)\Delta + o(\Delta); \end{aligned}$$

2. $k = r$

$$\begin{aligned} P_{k,k}(t, t + \Delta, \lambda) &= [1 + b_{kk}(t)\Delta + o(\Delta)] M_e^{i(\lambda, e_{kk}(t+\Delta) - e_{kk}(t))} = \\ &= [1 + b_{kk}(t)\Delta + o(\Delta)] \exp \left\{ \int_t^{t+\Delta} f_k(x, \lambda) dx \right\} = [1 + b_{kk}(t)\Delta + o(\Delta)] \cdot \\ &\cdot [1 + f_k(t, \lambda)\Delta + o(\Delta)] = 1 + [b_{kk}(t) + f_k(t, \lambda)]\Delta + o(\Delta) = \\ &= 1 + b_{kk}(t, \lambda)\Delta + o(\Delta). \end{aligned}$$

We have

$$\begin{aligned} (3) \quad P_{lk}(t, s + \Delta, \lambda) &= \sum_{i=0}^{c-1} P_{li}(t, s, \lambda) [\delta_{ik} + b_{ik}(s, \lambda)\Delta + o(\Delta)] + \\ &+ \sum_{i=c}^{k+c} P_{li}(t, s, \lambda) [\delta_{ik} + a_{k-i+c}(s, \lambda)\Delta + o(\Delta)], \quad l \geq 0. \end{aligned}$$

To establish (3) it is enough to present it in case $0 \leq k < c$ in the form

$$\begin{aligned} P_{lk}(t, s + \Delta, A) &= \\ &= \sum_{i=0}^{c-1} \int_{\mathbb{R}^d} P_{li}(t, s, dx) [b_{ik}(s)\Delta + o(\Delta)] P\{\varepsilon_{ik}(s + \Delta) - \varepsilon_{ik}(s) + x \in A\} + \\ &+ \int_{\mathbb{R}^d} P_{lk}(t, s, dx) [1 + b_{kk}(s)\Delta + o(\Delta)] P\{\varepsilon_{kk}(s + \Delta) - \varepsilon_{kk}(s) + x \in A\} + \\ &+ \sum_{i=c}^{k+c} \int_{\mathbb{R}^d} P_{li}(t, s, dx) [a_{k-i+c}(s)\Delta + o(\Delta)] P\{\varepsilon_{k-i}(s + \Delta) - \varepsilon_{k-i}(s) + x \in A\}, \end{aligned}$$

and in case $k \geq c$

$$P_{lk}(t, s + \Delta, A) =$$

$$\begin{aligned}
&= \sum_{i=0}^{c-1} \int_{\mathbb{R}^d} P_{li}(t, s, dx) [b_{ik}(s)\Delta + o(\Delta)] P\{\varepsilon_{ik}(s+\Delta) - \varepsilon_{ik}(s) + x \in A\} + \\
&+ \sum_{\substack{i=c \\ i \neq k}}^{k+c} \int_{\mathbb{R}^d} P_{li}(t, s, dx) [a_{k-i+c}(s)\Delta + o(\Delta)] P\{\varepsilon_{k-i}(s+\Delta) - \varepsilon_{k-i}(s) + x \in A\} + \\
&+ \int_{\mathbb{R}^d} P_{lk}(t, s, dx) [1 + a_c(s)\Delta + o(\Delta)] P\{\varepsilon(s+\Delta) - \varepsilon(s) + x \in A\},
\end{aligned}$$

and to apply the Fourier transform. From (3)

$$\begin{aligned}
(4) \quad \frac{\partial P_{lk}(t, s, \lambda)}{\partial s} &= \sum_{i=0}^{c-1} P_{li}(t, s, \lambda) b_{ik}(s, \lambda) + \sum_{i=c}^{k+c} P_{li}(t, s, \lambda) a_{k-i+c}(s, \lambda) = \\
&= \sum_{i=0}^{c-1} P_{li}(t, s, \lambda) b_{ik}(s, \lambda) + \sum_{i=0}^k P_{l, i+c}(t, s, \lambda) a_{k-i}(s, \lambda).
\end{aligned}$$

Introduce the notations

$$\begin{aligned}
(5) \quad b_i(t, \lambda, \Theta) &= \frac{1}{\Theta^i} \sum_{k=0}^{\infty} b_{ik}(t, \lambda) \Theta^k, \quad i = 0, 1, \dots, c-1; \\
a(t, \lambda, \Theta) &= \frac{1}{\Theta^c} \sum_{k=0}^{\infty} a_k(t, \lambda) \Theta^k, \\
P_l(t, s, \lambda, \Theta) &= \sum_{k=0}^{\infty} P_{lk}(t, s, \lambda) \Theta^k, \quad l \geq 0.
\end{aligned}$$

Multiplying both sides of (4) by Θ^k and summing from 0 till ∞ we get

$$\begin{aligned}
(6) \quad \frac{\partial P_l(t, s, \lambda, \Theta)}{\partial s} &= P_l(t, s, \lambda, \Theta) a(s, \lambda, \Theta) + \\
&+ \sum_{i=0}^{c-1} P_{li}(t, s, \lambda) \Theta^i [b_i(s, \lambda, \Theta) - a(s, \lambda, \Theta)]
\end{aligned}$$

with initial condition

$$(7) \quad P_l(t, t, \lambda, \Theta) = \Theta^l.$$

So we proved

Theorem 1. *If*

$$P_{lk}(t, s, \lambda) = \int_{\mathbb{R}^d} e^{t(\lambda, z)} P\{\xi(s) = k, \eta(s) - \eta(t) \in dz \mid \xi(t) = l\},$$

where $P\{\xi(s) = k \mid \xi(t) = l\}$ are the transition probabilities for inhomogeneous Markov chain $\xi(t)$ with local characteristics (1), the second component is deter-

mined by (2), then the generating function of transition probabilities $P_l(t, s, \lambda, \Theta)$ is the solution of equation (6) with initial condition (7) for any fixed $l \geq 0$, where $a(t, \lambda, \Theta)$ and $b_i(t, \lambda, \Theta)$ ($i = 0, 1, \dots, c-1$) are determined by (5).

3. (6) is a linear differential equation for $P_l(t, s, \lambda, \Theta)$, so its solution according to [6] is

$$(8) \quad P_l(t, s, \lambda, \Theta) = \exp \left\{ \int_t^s a(x, \lambda, \Theta) dx \right\} \left[\Theta^l + \sum_{i=0}^{c-1} \int_t^s P_{li}(t, y, \lambda) \Theta^i [b_i(y, \lambda, \Theta) - a(y, \lambda, \Theta)] \exp \left\{ - \int_t^y a(x, \lambda, \Theta) dx \right\} dy \right] = \exp \left\{ \int_t^s a(x, \lambda, \Theta) dx \right\} \Theta^l + \sum_{i=0}^{c-1} \int_t^s P_{li}(t, y, \lambda) \Theta^i [b_i(y, \lambda, \Theta) - a(y, \lambda, \Theta)] \exp \left\{ \int_t^s a(x, \lambda, \Theta) dx \right\} dy.$$

We introduce the auxiliary process with independent increments $\xi^*(t)$ with state space $\{0, \pm 1, \pm 2, \dots\}$ and transition probabilities for $[t, t + \Delta]$

$$P\{k \xrightarrow{[t, t+\Delta]} k - c + r\} = \delta_{cr} + a_r(t)\Delta + o(\Delta).$$

Then

$$M\Theta^{\xi^*(s) - \xi^*(t)} e^{i(\lambda, \eta(s) - \eta(t))} = \exp \left\{ \int_t^s a(x, \lambda, \Theta) dx \right\}$$

Let further

$$(9) \quad \varrho_k(t, s, \lambda) = \frac{1}{2\pi i} \oint_{|\Theta|=1} \exp \left\{ \int_t^s a(x, \lambda, \Theta) dx \right\} \frac{d\Theta}{\Theta^{k+1}}.$$

Now comparing the coefficients of Θ^k in (8) we obtain

$$P_{lk}(t, s, \lambda) = \varrho_{k-l}(t, s, \lambda) + \sum_{i=0}^{c-1} \int_t^s P_{li}(t, y, \lambda) \sum_r [b_{ir}(y, \lambda) - a_r(y, \lambda)] \varrho_{k-r-i}(y, s, \lambda) dy.$$

Since

$$\exp \left\{ \int_t^s a(x, \lambda, \Theta) dx \right\} = \sum_{j=-\infty}^{\infty} \varrho_j(t, s, \lambda) \Theta^j,$$

we get

$$-a(y, \lambda, \Theta) \exp \left\{ \int_t^s a(x, \lambda, \Theta) dx \right\} = \sum_{j=-\infty}^{\infty} \frac{\partial \varrho_j(y, s, \lambda)}{\partial y} \Theta^j,$$

and

$$\begin{aligned} & \sum_r [b_{ir}(y, \lambda) - a_r(y, \lambda)] \varrho_{k-r-i}(y, s, \lambda) = \\ & = \sum_r b_{ir}(y, \lambda) \varrho_{k-r-i}(y, s, \lambda) + \frac{\partial \varrho_{k-i}(y, s, \lambda)}{\partial y}. \end{aligned}$$

Thus we have proved

Theorem 2. $P_{ik}(t, s, \lambda)$ is equal to

$$(10) \quad \begin{aligned} P_{ik}(t, s, \lambda) &= \varrho_{k-i}(t, s, \lambda) + \\ &+ \sum_{i=0}^{c-1} \int_t^s P_{ii}(t, y, \lambda) \left[\sum_r b_{ir}(y, \lambda) \varrho_{k-r-i}(y, s, \lambda) + \frac{\partial \varrho_{k-i}(y, s, \lambda)}{\partial y} \right] dy. \end{aligned}$$

where $\varrho_k(t, s, \lambda)$ are determined by (9).

4. In order to compute $P_{ik}(t, s, \lambda)$ ($k = 0, 1, \dots$) it is necessary to determine the unknown $P_{ii}(t, s, \lambda)$ ($k = 0, 1, \dots$) it is necessary to determine the unknown $P_{ii}(t, s, \lambda)$ ($i = 0, 1, \dots, c-1$). We introduce the notations

$$(11) \quad \begin{aligned} \pi_{ik}(t, s, \lambda) &= \sum_r b_{ir}(t, \lambda) \varrho_{k-r-i}(t, s, \lambda) + \frac{\partial \varrho_{k-i}(t, s, \lambda)}{\partial t}, \\ F(t, s, \lambda) &= \|\pi_{ik}(t, s, \lambda)\|, i, k = 0, 1, \dots, c-1, \\ \vec{P}_i(t, s, \lambda) &= \{P_{i0}(t, s, \lambda), P_{i1}(t, s, \lambda), \dots, P_{i, c-1}(t, s, \lambda)\}, \\ \vec{\varrho}_i(t, s, \lambda) &= \{\varrho_{-i}(t, s, \lambda), \varrho_{1-i}(t, s, \lambda), \dots, \varrho_{c-1-i}(t, s, \lambda)\}. \end{aligned}$$

Now from (10) we obtain the system of equations

$$\vec{P}_i(t, s, \lambda) = \vec{\varrho}_i(t, s, \lambda) + \int_t^s \vec{P}_i(t, y, \lambda) F(y, s, \lambda) dy.$$

According to the theory of the Volterra integral equations (see e.g. [10]) for any fixed T we get

$$\vec{P}_i(t, s, \lambda) = \vec{\varrho}_i(t, s, \lambda) + \int_t^s \vec{\varrho}_i(t, y, \lambda) G(y, s, \lambda) dy \quad (0 \leq s \leq t \leq T),$$

where $G(t, s, \lambda)$ is the resolvent of the matrix $F(t, s, \lambda)$. Let

$$(12) \quad \vec{f}_k(t, s, \lambda) = \begin{bmatrix} \pi_{0k}(t, s, \lambda) \\ \pi_{1k}(t, s, \lambda) \\ \dots\dots\dots \\ \pi_{c-1, k}(t, s, \lambda) \end{bmatrix}, k \geq c,$$

then

$$(13) \quad \begin{aligned} P_{ik}(t, s, \lambda) &= \varrho_{k-i}(t, s, \lambda) + \int_t^s \vec{P}_i(t, y, \lambda) \vec{f}_k(y, s, \lambda) dy = \varrho_{k-i}(t, s, \lambda) + \\ &+ \int_t^s \vec{\varrho}_i(t, y, \lambda) \vec{f}_k(y, s, \lambda) dy + \int_t^s \int_{x \leq y \leq s} \vec{\varrho}_i(t, x, \lambda) G(x, y, \lambda) \vec{f}_k(y, s, \lambda) dx dy, \end{aligned}$$

$k \geq c$.

Thus we have proved

Theorem 3. *The transition probabilities $P_{ik}(t, s, \lambda)$ of $\zeta(t)$ are determined by (13), where $\varrho_k(t, s, \lambda)$ are given by (9), $\vec{f}_k(t, s, \lambda)$ by (11) and (12), and $G(t, s, \lambda)$ is the resolvent of matrix $F(t, s, \lambda)$.*

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