ON A UNIFIED THEORY OF ITERATION METHODS FOR SOLVING NONLINEAR OPERATOR EQUATIONS. II

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This paper represents a continuation of the works [6–9]. Here we have tried to present a unitary theory for certain classes of iteration methods, used for solving nonlinear operator equations, which were defined in the conditions of Banach spaces and linear semiordered spaces, respectively.

We have shown in the above mentioned papers that the concept of convergence's order implies a direct influence on the structure of iteration methods. In this way this concept permits to generate step by step and to classify as well the obtained iteration methods and enables in these circumstances the common treatment of these methods.

At the same time common conditions for convergence are given.

Based on a certain principle of construction in the condition of derivability our purpose is to generate systematically some new large classes of iteration methods. Obviously, the common treatment of all iteration methods of higher order represents a necessity in the development of this domain, so it's a fundamental question.

Of course, those iteration processes, algorithms are most important, which can be adapted and applied effectively at digital computers.

In the present work we resume the above problems and we shall generate also – from a common source – some interesting new classes of iteration methods, constructed partly by J. F. Traub and by P. Jarratt, in the case of real equations [22, 10].

1. Let us now consider the equation

$$P(x) = \Theta \tag{1}$$

where P is a nonlinear operator defined in a given domain D of a Banach space X, having – for simplicity – his values also in X, without restricting essentially the conditions.

Next we shall study only the case of simple solutions for the equation (1), i.e. the existence and boundedness in norm of $[P'(x^*)]^{-1}$ shall be assumed, x^* being the solution of (1).

In this part of the present work we shall not deal with the optimality of the generated methods [23].

Our basic problem is to substitute in a suitable manner the given operator equation $P(x) = \Theta$, by another equivalent operator equation $x-\psi(x) = \Theta$, in such a way that the order of convergence should $k \geqq 2$ be, for the iteration method

$$x_{n+1} = \psi(x_n). \tag{2}$$

For this purpose we use the following concept of convergence's order.

Definition. Let x^* be a solution of the operator equation (1). We say that the above considered iteration method (2) *posesses the order of convergence k*, iff

(i) the norm $\|x-x_n\|$ tends to zero for $n \to \infty$;

(ii) the Fréchet derivatives of the iteration operator $\psi(x)$ exist and satisfy the equalities

$$\psi'(x^*) = O_1,\ \psi''(x^*) = O_2,\ \ldots,\ \psi^{(k-1)}(x^*) = O_{k-1},\ \psi^{(k)}(x^*) \neq O_k, \tag{3}$$

where O_i, $(i = 1, 2, \ldots, k)$ are i-linear null-operators.

In our works [6—9] we have considered at the first step the following iteration operator

$$\psi(x) := x - \{P'(x) + \mu_1(x)P(x)\}^{-1}P(x) + \lambda_2(x)[P(x)]^2, \tag{4}$$

where $\psi(x)$ represents a nonlinear operator with domain $D \subset X$ and range, in X. Moreover $\mu_1(x)$ and $\lambda_2(x)$ are bilinear operators for fixed x, being defined in the domain $D \times D \subset X \times X$ and having his values also in X.

Using the above iteration operator $\psi(x)$, we have contructed corresponding iteration method (2), and we have applied it for solving equation (1). In this way we have generated two essential classes of iteration methods of second and third order, respectively.

So we have obtained the well-known Newton-Kantorovich's method, the Chebyshev's method and the method of tangent hyperbolas as particular cases. Moreover, we have shown that besides there exists a class of transfinit number of methods of second and third order, respectively [7].

In the case in which the iteration operator is chosen in another form, i.e.

$$\tilde{\psi}(x) := x - [P'(x) + \mu_1(x)P(x)]^{-1}(P(x) + \lambda_2(x)[P(x)]^2), \tag{4'}$$

we recover the L. K. Vohandu's method and the method of Ü. Kaasik, which are included as particular cases in the above mentioned methods [12, 21, 9, 24].

Some general iteration operators were constructed by us in the following forms

$$\psi(x) := x - [P'(x) + R(x)]^{-1}P(x) + Q(x) \tag{5}$$

and

$$\psi(x) := x - [P'(x) + R(x)]^{-1}(P(x) + Q(x)) \tag{5'}$$

respectively, where

$$R(x): = \mu_1(x)P(x) + \ldots + \mu_{i+1}(x)[P(x)]^{i+1}$$

and

$$Q(x): = \lambda_2(x)[P(x)]^2 + \ldots + \lambda_{j+1}(x)[P(x)]^{j+1}, \ (i+j = k-1).$$

Let us mention here that the operators $Q(x)$ and $R(x)$ possess so certain "multilinear" or "polynomial" character, being constructed by the above multilinear operators λ and μ. On the other hand we notice that the iteration method $x_{n+1}=\psi(x_n)$ constructed by (5) contains among others the Chebyshev-type methods indicated by a formal development of the inverse of the non-linear operator $P(x)$ [17, p. 72].

Now we are going to give some more general iteration operators and we shall generate certain interesting iteration methods, without to limit, of course, the above mentioned multilinear character.

Let us consider the following large class of iteration operators

$$\psi(x): = x - \{\alpha U(x) + \sum_i a_i U(x + R_i(x)) + r(x)\} \cdot \tag{6}$$

$$\cdot \{\beta P(x) + \sum_{i \geq 2} b_i P(x + Q_i(x)) + q(x)\},$$

where

$$R_i(x): = \sum_j \mu_j^{(i)}(x)[P(x)]^i; \quad r(x): = \sum_k \nu_k(x)[P(x)]^k;$$

$$Q_i(x): = \sum_j \lambda_j^{(i)}(x)[P(x)]^j; \quad q(x): = \sum_k \chi_k(x)[P(x)]^k.$$

Then α, β, a_i, b_i denote real numbers and $\mu_j^{(i)}(x)$, $\nu_k(x)$, $\lambda_j^{(i)}(x)$, $\chi_k(x)$ are also multilinear operators for fixed x. In the next we shall give the operator $U(x)$ certain concrete expressions.

Let us now consider the construction of a few interesting methods:

A) As a first particular case we shall consider the above iteration operator in the following particular form

$$\psi(x): = x - P(x + \lambda_1(x)P(x)),$$

where $\lambda_i(x)$ is a linear operator for fixed x. Imposing the condition

$$\psi'(x^*): = I - P'(x^*)[I + \lambda_1(x^*)P'(x^*)] = O_1,$$

we obtain

$$\lambda_1(x^*): = (\Gamma(x^*) - I)\Gamma(x^*)$$

where

$$\Gamma(x): = [P'(x)]^{-1}.$$

We can see that, for the moment, the operator $\lambda_1(x)$ is determined only for $x = x^*$. For any $x \in D$ it may be chosen in the following form

$$\lambda_1(x) := \{\Gamma(x) - I\}\Gamma(x),$$

and more generally

$$\lambda_1(x) := \{\Gamma(x) - I\}\Gamma(x) + \sum_k \nu_k(x)[P(x)]^k.$$

Of course, instead of the variable x we can replace also an expression of $P(x)$, so

$$\sum_l \varrho_l(x)[P(x)]^l,$$

where $\varrho_l(x)$ are multilinear operators in the above sense. By this composition we have constructed a certain large set of iteration methods of second order.

We mention for the next, if we choose the iteration operator $\psi(x)$ as follows

$$\psi(x) := x - U(x)P(x + \lambda_1(x)P(x))$$

and impose $\psi'(x^*) = O_1$, $\psi''(x^*) = O_2$, then for the determination of $U(x)$ and $\lambda_1(x)$ we shall obtain certain system of differential-operator equations.

B) In the next step we consider

$$\psi(x) := x - U(x)P(x + \lambda_2(x)[P(x)]^2).$$

In this case the condition $\psi'(x^*) = O_1$ leads to the relation

$$\psi'(x^*) := I - U(x^*)P'(x^*) = O_1,$$

which implies $U(x^*) := \Gamma(x^*)$. This means that the operator $U(x)$ may be chosen more generally in the form

$$U(x) := \Gamma(x + \sum_j \mu_j(x)[P(x)]^j) + \sum_k \nu_k(x)[P(x)]^k.$$

In order to choose $\lambda_2(x)$ we use the condition $\psi''(x^*) = O_2$, i.e. $\psi''(x^*)(\Delta x)^2 = \Theta$ for any $\Delta x \in D$, where Θ is the null-element of X; so we have

$$(7) \qquad \psi''(x^*)(\Delta x)^2 := -2U'(x^*)P'(x^*)(\Delta x)^2 - U(x^*)P''(x^*))(\Delta x)^2 - \\ - U(x^*)P'(x^*)\{2\lambda_2(x^*)[P'(x^*)(\Delta x)]^2 = 0_2(\Delta x)^2.$$

Based on the above expression of $U(x)$ and on his Fréchet's derivative of the form

$$U'(x^*)\Delta x := -\Gamma(x^*)P''(x^*)(\Delta x)\Gamma(x^*)(\Delta x)$$

we get from (7) the relation

$$\Gamma(x^*)P''(x^*)(\Delta x)^2 - 2\lambda_2(x^*)[P'(x)\Delta x]^2 = \Theta.$$

Thus we have

$$\lambda_2(x^*)(\Delta x)^2 = \frac{1}{2}\Gamma(x^*)P''(x^*)[\Gamma(x^*)\Delta x]^2$$

and for any x we can choose

$$\lambda_2(x)(\Delta x)^2 = \frac{1}{2}\Gamma(x)P''(x)[\Gamma(x)\Delta x]^2 + \sum_l \varrho_l(x)P(x)]^l.$$

For the generality we can also pose here a multilinear form of $P(x)$ instead of the variable x.

C) Let us mention here that in the particular case when we have

$$\psi(x):= x - U(x+\mu(x))P(x)$$

and $U = P'$ then we obtain for $\psi'(x^*) = O_1$ and for $\psi''(x^*) = O_2$

$$U(x^*):= \Gamma(x^*),\ \mu(x^*):= \frac{1}{2}\Gamma(x^*)P(x^*).$$

This iteration method $x_{n+1} = \psi(x_n)$ of third order constructed in this way needs two inverses but only the Fréchet's derivative of first order, (sometimes the Fréchet's derivative of second order can be very complicated). This particular method was treated in [13] and in [8 p. 171].

D) At last let us show that the generalized Traub's method

$$x_{n+1} = x_n - \Gamma(x_n)\{P(x_n) + P(x_n - \Gamma(x_n)P(x_n))\}$$

treated in [15, p. 187] belongs also to our class of iteration methods given by (6). For this purpose we consider $\alpha = 1$, $a_i = 0$, $r(x) = \Theta$, $\beta = 1$, $b_1 = 1$, $b_2 = O$, $(i = 2, \ldots)$, $q(x) = \Theta$. For simplicity we can put $Q_1(x) \equiv U(x)P(x)$. Thus we can use the following special iteration operator

$$\psi(x):= x + U(x)\{P(x) + P[x + U(x)P(x)]\}.$$

For $\psi'(x^*)\,\Delta x = \Theta$ we get $U(x^*) = -\Gamma(x^*)$ and if we choose $U(x) := -\Gamma(x)$ for any $x \in D$, then the condition $\psi''(x^*)(\Delta x)^2 = \Theta$ is satisfied as well.

Refering to the construction of classes of iteration methods we observe that in the definition of the convergence's order instead of (3) we could have posed the consitions

$$\text{(3')} \qquad \psi^{(\nu)}(x_n) = O_\nu,\ (\nu = 1, 2, \ldots, k-1),$$
$$\psi^{(k)}(x_n) \neq O_k,$$

Of course. these definitions are equivalent [5]. However, using this second definition for the construction of methods we shall obtain certain systems of differential equations [6].

2. In this second section of our work we are going to establish common conditions of convergence of order k for the general iteration method $x_{n+1} = \psi(x_n)$, where $\psi(x)$ is defined by (6).

For this purpose we consider again the nonlinear operator equation $P(x) = \Theta$, where $x \in D \subset X$ and $P(x) \in X$.

We assume the following conditions:

1°. $P(x) = \Theta$ is equivalent to the equation $x = \psi(x)$;

2°. there exists an initial approximate solution $x_0 \in S$, where $S: \|x - x_0\| \leqq \leqq r$, and for the first derivative of the iteration operator $\psi(x)$ we have

$$\|\psi'(x)\| \leqq p < 1, \quad \forall x \in S, \tag{8}$$

where

$$r: = \varrho \|x_1 - x_0\| \text{ and } \varrho \geqq \frac{1}{1-q},$$

which ensure the following properties:

$$\psi(\tilde{x}) \in S \text{ if } \tilde{x} \in S;$$

3°. for the Fréchet's derivatives of $\psi(x)$ we impose the conditions (3);

4°. $$\|\psi^{(k-1)}(\xi_1) - \psi^{(k-1)}(\xi_2)\| \leqq M_k \|\xi_1 - \xi_2\|.$$

for any $\xi_1, \xi_2 \in S$, where M_k is a fixed real constant and $O < M_k < +\infty$;

5°. $$r^{k-1} M_k < k!.$$

Theorem. *Let us assume that the conditions* 1°–5°. *are fulfilled. Then the operator equation $P(x) = \Theta$ possesses a single solution $x^* \in S$ and the general iteration method $x_{n+1} = \psi(x_n)$ is convergent of order k, having the following estimation*

$$\|x^* - x_n\| \leqq r^{k^n} \left(\frac{M_k}{k!}\right)^{\frac{k^n - 1}{k-1}}. \tag{9}$$

Proof. Based on a Kantorowich's theorem [11, p. 536] we observe that the conditions 1°–2°. ensure the existence and the uniqueness for the solution of the equation $P(x) = \Theta$, moreover the iterations $x_n \in S$ for any natural n.

Further, for the convergence of order k we can consider the generalized Taylor's formula for $\psi(x)$ in the following form

$$\|\psi(x_{n-1}) - \psi(x^*) - \psi'(x^*)(x_{n-1} - x^*) - \dots -$$
$$- \frac{1}{(k-1)!} \psi^{(k-1)}(x^*)(x_{n-1} - x^*)^{k-1}\| \leqq \frac{1}{k!} M_k \|x_{n-1} - x^*\|^k.$$

This implies the delimitation

$$\|x_n - x^*\| \leqq \frac{M_k}{k!} \|x_{n-1} - x^*\|^k$$

or

$$\|x^* - x_n\| \leqq \|x^* - x_0\|^{k^n} \left(\frac{M_k}{k!}\right)^{\frac{k^n - 1}{k-1}},$$

which still leads to the estimation (9). □

Observation. It's worth mentioning here that — having in view the continuity of $\psi'(x)$ — in this way the condition (8) is also necessary for the linear convergence of our class of iteration methods $x_{n+1} = \psi(x_n)$ and for the existence of the solution $x^* \in S$ [11, p. 536].

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