FUNCTIONAL LIMIT THEOREMS FOR THE LIKELIHOOD RATIO PROCESSES

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Abstract. In this paper we give functional limit theorems for the likelihood ratio processes corresponding to given parametric statistical models. The theorems are proved for the simplest statistical models, i.e. with only two probability measures, as well as for the general ones. By applying these theorems we obtain „predictable" sufficient conditions for asymptotic normality of the maximum likelihood estimators.

Introduction

One of the basic objects in mathematical statistics is *the general statistical parametric model* $\mathcal{E} = \{(\Omega, F), P_\vartheta, \vartheta \in \Theta\}$ where (Ω, F) is measurable space, the $(P_\vartheta)_{\vartheta\in\Theta}$ are probability measures on (Ω, F) depending on $\vartheta \in \Theta \subseteq R^m$, and $m \geqq 1$. Suppose, the general statistical parametric model $\mathcal{E}$ is supplemented with right-continuous filtration $\mathbf{F} = (F_t)_{t\geqq 0}$ which is a family of non-decreasing right-continuous σ-algebras $(F_t)_{t\geqq 0}$, $\bigvee_{t\geqq 0} F_t = F$. This additional assumption allows us to associate with every $\mathcal{E}$ and fixed $\vartheta \in \Theta$ *the likelihood ratio process* z and to use for its investigation the general theory of stochastic processes. By definition $z = \big(z(\vartheta', \vartheta)\big)_{\vartheta'\in\Theta}$ where $z(\vartheta', \vartheta)$ is the likelihood ratio process of the simplest statistical model $\mathcal{E}_s = \{(\Omega, F, \mathbf{F}), P^n_{\vartheta'}, P^n_\vartheta\}$ defined in (1), $\vartheta \in \Theta$.

Among the various statistical parametric models an essential role is played by the *Gaussian* one, i.e. by the model with likelihood ratio process $Z = \big(Z_t(\vartheta)\big)_{t\geqq 0,\, \vartheta\in\Theta}$ given by

$$Z_t(\vartheta) = \exp\left(N_t(\vartheta) - \frac{1}{2}\langle N(\vartheta)\rangle_t\right) \tag{1}$$

where $N = \big(N_t(\vartheta)\big)_{t\geqq 0,\, \vartheta\in\Theta}$ is a continuous (a.s.) Gaussian field given on a probability space $(\bar\Omega, \bar F, \bar P)$ with filtration $\bar{\mathbf{F}} = (\bar F_t)_{t\geqq 0}$ and such that $N(\vartheta) = (N_t(\vartheta), \bar F_t)_{t\geqq 0}$ is a $\bar P$-martingale for every $\vartheta \in \Theta$, $\langle N(\vartheta)\rangle_t = DN_t(\vartheta)$.

Consider the sequence of general statistical parametric models $\mathcal{E}^n = \{(\Omega^n, F^n, \mathbf{F}^n), P^n_\vartheta, \vartheta\in\Theta\}$, $n\geq 1$, and the corresponding likelihood ratio processes $z^n = (z^n_t(\vartheta', \vartheta))_{t\geq 0, \vartheta'\in\Theta}$, $n\geq 1$. We are interested in weak convergence of distributions of the normalized likelihood ratio process

$$Z^n = (Z^n_t(u,(0))_{t\geq 0, u\in U^n{}_\vartheta},$$

where $Z^n_t(u, 0) = z^n_t(\vartheta + u/c_n, \vartheta)$, $(c_n)_{n\geq 1}$ is normalizing sequence, $c_n>0$, $U^n_\vartheta = \{u\colon \vartheta + c_n^{-1}u\in\Theta\}$, to the distributions of the likelihood ratio process Z of the Gaussian model. Since the likelihood ratio process Z^n is a function of (ω, t, u) we can consider functional limit theorems of three types:

(i) for fixed u with respect to t,
(ii) for fixed t with respect to u,
(iii) with respect to (t, u).

Functional limit theorems of type (i) in discrete time case have been obtained by Greenwood, Shiryayev [5] and by Kordzahia [12].

Functional limit theorems of type (ii) were proved in the papers of Prakasa Rao [20], [21], Inagaki [7], Weiss and Wolfowitz [26], Ibragimov and Hasminskij [6], Kutoyants [13].

The *aim of this paper* is to obtain functional limit theorems of type (i) and (ii) in „predictable" terms (i.e. with the conditions on predictable processes) for the continuous time case.

In **1** basic facts about the likelihood ratio processes, the formulas for the change of the measure in mathematical expectations, the inequality of Lenglart-Rebolledo and also the criteria of contiguity of probability measures are given.

In **2** we consider the simplest statistical models $\mathcal{E}_s = \{(\Omega, F, \mathbf{F}), \tilde{P}, P\}$, i.e. the models with two probability measures $\tilde{P}$ and P. We obtain here necessary and sufficient conditions for weak convergence of the distributions of likelihood ratio process Z^n to those of the likelihood ratio process $Z = (Z_t)_{t\geq 0}$ of the Gaussian model with

$$Z_t = \exp\left(N_t - \frac{1}{2}\langle N\rangle_t\right)$$

where $N = (N_t)_{t\geq 0}$ is a Gaussian martingale, i.e. a Gaussian process with independent increments, with mathematical expectation $EN_t = 0$ and variance $DN_t = \langle N\rangle_t$ (see Theorems 2, 3).

In **3** we consider the general statistical parametric models $\mathcal{E}^n$ and give "predictable" conditions for the convergence of finite dimensional distributions of the process $Z^n_T = (Z^n_T(u, 0))_{u\in U^n{}_\vartheta}$, with fixed $T>0$ to the ones of the process $Z_T = (Z_T(u))_{u\in R^m}$ defined by (1), as $n\to\infty$ (see Theorem 4).

In **4** we prove a functional limit theorem (see Theorem 5) stating that under some sufficient conditions the distributions of (with respect to u) continuous modifications $\hat{Z}^n_T$ of the processes Z^n_T weakly converge in $\mathbf{C}(R^m, P^\vartheta_n)$ to the distributions of the processes Z_T as $n\to\infty$.

Theorem 5 of 4 can be easily applied to the investigation of properties of statistical estimators. As an example, we obtain sufficient "predictable" conditions for the asymptotic normality of maximum likelihood estimators (see Theorem 6). In particular, this theorem imply the classical results of Ibragimov and Hasminskij [6] for discrete time and the "schemes of independent random variables".

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1. Basic facts

Suppose $(\Omega^n, F^n)_{n\geq 1}$ is a sequence of measurable spaces with two probability measures P^n and $\tilde{P}^n$ and right-continuous filtration $\mathbf{F}^n = (F_t^n)_{t\geq 0}$, $\bigvee_{t\geq 0} F_t^n = F^n$, on each of them. Denote by $\tilde{P}_t^n = P^n|F_t^n$, $P_t^n = \tilde{P}^n|F_t^n$ the restrictions of the measures P^n and $\tilde{P}^n$ to the σ-algebra F_t^n and set $Q^n = (P^n + P^n)/2$, $Q_t^n = Q^n|F_t^n$, Assume that the filtration $\mathbf{F}^n$ is completed by Q^n-null sets belonging to F^n.

By $\zeta^n = (\zeta_t^n, F_t^n)_{t\geq 0}$ and $\tilde{\zeta}^n = (\tilde{\zeta}_t^n, F_t^n)_{t\geq 0}$ we denoted the likelihood processes with paths in the space $\mathbf{D}$ of right-continuous functions with left-hand limits and in a way that for every stopping time τ (Q^n – a.s.)

$$\zeta_\tau^n = \frac{dP_\tau^n}{dQ_\tau^n} \quad \text{and} \quad \tilde{\zeta}_\tau^n = \frac{d\tilde{P}_\tau^n}{dQ_\tau^n}. \tag{2}$$

We also introduce the likelihood ratio processes $Z^n = (Z_t^n, F_t^n)_{t\geq 0}$, $\tilde{Z}^n = (\tilde{Z}_t^n, F_t^n)_{t\geq 0}$ with $Z_t^n = \tilde{\zeta}_t^n/\zeta_t^n$, $\tilde{Z}_t^n = \zeta_t^n/\tilde{\zeta}_t^n$ where we put $0/0 = 0$, $a/0 = \infty$ for every $a > 0$.

In the following lemma the properties of the processes ζ^n, $\tilde{\zeta}^n$ and Z^n are given.

Lemma 1. (see [16]). (a) *The processes ζ^n and $\tilde{\zeta}^n$ are non-negative uniformly integrable Q^n-martingales; for every stopping time τ $\zeta_\tau^n + \tilde{\zeta}_\tau^n = 2$* ($Q^n$ – a.s.);

$$P^n(\inf_{t\geq 0} \zeta_t^n > 0) = 1, \; \tilde{P}^n(\inf_{t\geq 0} \tilde{\zeta}_t^n > 0) = 1.$$

(b) *The likelihood ratio process Z^n is a non-negative P^n-supermartingale and for every stopping time τ $E^n Z^n_\tau \leqq 1$, where E^n is the expectation with respect to P^n; the points $\{0\}$ and $\{\infty\}$ are absorbing states (Q^n – a.s.) for Z^n and*

$$P^n(\sup_{t\geqq 0} Z^n_t \geqq a) \leqq \frac{1}{a}, \quad \tilde{P}^n(\inf_{t\geqq 0} Z^n_t \leqq a) \leqq a$$

for every $a > 0$.

The following lemma contains the formulas for the change of the measure in expectations. We set $\alpha^n_t = Z^n_t / Z^n_{t-}$ putting $0/0 = 0$, $\infty/\infty = \infty$, and denote by $\tilde{E}^n$ and E^n the expectations with respect to $\tilde{P}^n$ and P^n respectively.

Lemma 2. (see [16]). *For every stopping time τ and every nonnegative F^n_τ measurable random variable η we have*

$$\tilde{E}^n \eta = E^n \eta Z^n_\tau + \tilde{E}^n \eta I(Z^n_\tau = \infty)$$

where $I(\cdot)$ is an indicator function. If $\tau > 0$ is predictable stopping time then ($\tilde{P}^n$ – a.s.)

$$\tilde{E}^n(\eta | F^n_{\tau-}) = I(Z^n_{\tau-} < \infty) E^n(\eta \alpha^n_\tau | F^n_{\tau-}) + \tilde{E}^n(\eta I(Z^n_\tau = \infty) | F^n_{\tau-}).$$

In the Lemma 3 we formulate the very important Lenglart-Rebolledo inequality.

Lemma 3. (see [16]). *Let $X^n = (X^n_t, F^n_t)_{t\geqq 0}$ and $Y^n = (Y^n_t, F^n_t)_{t\geqq 0}$ be two non-neg ativeprocesses, $X^n_0 = Y^n_0 = 0$, with paths in* **D**, *$Y^n_s \leqq Y^n_t$ (P^n – a.s.) for $s \leqq t$. If for each finite stopping time σ*

$$E^n X^n_\sigma \leqq E^n y^n_\sigma$$

then for every stopping time τ and $a > 0$, $b > 0$

$$P^n(\sup_{0\leqq t\leqq \tau} X^n_t > a) \leqq (b + E^n \sup_{0\leqq t\leqq \tau} \triangle Y^n_t)/a + P^n(Y^n_\tau \geqq b),$$

where $\triangle Y^n_t = Y^n_t - Y^n_{t-}$. In the case of predictable process Y^n

$$P^n(\sup_{0\leqq t\leqq \tau} X^n_t > a) \leqq c/a + P^n(Y^n_\tau \geqq b).$$

Corollary 1. (see [15]). *Let the sequence of processes $(X^n, Y^n)_{n\geqq 1}$ satisfy the conditions of Lemma 3 and the process Y^n is either predictable or*

$$P^n(\sup_{t\geqq 0} \triangle Y^n_t \leqq c) = 1$$

for some $c > 0$. Then for every stopping time τ the following implication holds

$$\{Y^n_\tau \xrightarrow{P^n} 0\} \Rightarrow \{\sup_{0\leqq t\leqq \tau} X^n_t \xrightarrow{P^n} 0\}$$

Now we consider briefly the notion of contiguity and connected notions.

Definition 1. (see [14], [22] and [16]). The sequence of probability measures $(\tilde{P}^n)_{n\geqq 1}$ is called to be contiguous to the sequence of probability measures $(P^n)_{n\geqq 1}$ (denoted by $(\tilde{P}^n)\lhd(P^n)$) iff for any sequence of the sets $(A^n)_{n\geqq 1}$, $A^n\in F^n$ the implication

$$\{P^n(A^n)\to 0,\ n\to\infty\}\Rightarrow\{\tilde{P}^n(A^n)\to 0,\ n\to\infty\}\text{ holds.}$$

If $(\tilde{P}^n)\lhd(P^n)$ and $(P^n)\lhd(\tilde{P}^n)$ then the sequences of probability measures $(\tilde{P}^n)$ and (P^n) are mutually contiguous, i.e. $(\tilde{P}^n)\lhd\rhd(P^n)$. □

The notion of contiguity is closely connected with the notion of tightness (see Theorem 1).

Definition 2. Let ξ^n be an F^n-measurable random variable, $n\geq 1$. The sequence (ξ^n, P^n) is said to be tight iff

$$\lim_{L\to\infty}\overline{\lim_{n\to\infty}}\, P^n(|\xi^n|\geq L) = 0. \quad \square$$

To formulate general criteria of contiguity we also introduce the notion of the Hellinger process. To this end we define the processes

$$(3)\qquad \beta^n = (\beta_t^n,\ F_t^n)_{t>0},\ \lambda^n(x) = (\lambda_t^n(x),\ F_t^n)_{t>0},\quad \tilde{\lambda}^n(x) = (\tilde{\lambda}_t^n(x),\ F_t^n)_{t>0}$$

with

$$(4)\qquad \beta_t^n = ((\zeta_{t-}^n)^{\otimes}+(\tilde{\zeta}_{t-}^n)^{\otimes})^2,$$

$$\lambda_t^n(x) = (1+x(\zeta_{t-}^n)^{\otimes})\vee 0,\quad \tilde{\lambda}_t^n(x) = (1-x(\tilde{\zeta}_{t-}^n)^{\otimes})\vee 0,$$

where $x\in R^1$ and

$$(5)\qquad a^{\otimes} = \begin{cases} 0 & \text{if } a = 0,\\ 1/a & \text{if } a\neq 0,\ |a|\neq\infty,\\ 0 & \text{if } |a| = \infty.\end{cases}$$

Definition 3. (see [16], [10] and [17]). The process $H^n = (H_t^n,\ F_t^n)_{t\geqq 0}$ is said to be the Hellinger process corresponding to P^n, $\tilde{P}^n$ and $\mathbf{F}^n$ iff

$$(6)\qquad H_t^n = \frac{1}{4}(\beta^n\circ\langle\zeta^{nc}\rangle)_t+((\sqrt{\lambda^n(x)}-\sqrt{\tilde{\lambda}^n(x)})^2*\nu_{\zeta^n,Q^n})_t$$

where $\langle\zeta^{nc}\rangle$ is the quadratic characteristic of the continuous martingale part of ζ^n, ν_{ζ^n,Q^n} is the compensator of the jump measure of ζ^n with respect to $(\mathbf{F}^n, Q^n)$. □

Here and further on the notations " ∘ " and " ∗ " are used for two types of Lebesgue-Stiltjes integrals, i.e.

$$(h\circ\langle\zeta^{nc}\rangle)_t = \int_0^t h_s d\langle\zeta^{nc}\rangle_s,$$

$$(g(x)*\nu_{\zeta^n,Q^n})_t = \int_{[0,t]}\int_{R^1\setminus\{0\}} g_s(x)\nu_{\zeta^n,Q^n}(ds,\ dx).$$

The following theorem gives the general criteria of contiguity.

Theorem 1. (see [16], [5]). *Each of the following conditions*

(a) *the sequence* $(Z^n_T, \tilde{P}^n_T)$ *is tight,*
(b) *the sequence* $(\sup_{0\leq t\leq T} Z^n_t, \tilde{P}^n_T)$ *is tight,*
(c) *for every subsequence* $Z^{n_k}_T$ *converging weakly with respect to* P^{n_k} *to* Z' *the equality* $EZ' = 1$ *holds,*
(d) *the sequences* $(Z^n_0, \tilde{P}^n_T)$, $(\sup_{0<t\leq T} \alpha^n_t, \tilde{P}^n_T)$, $(H^n_T, \tilde{P}^n_T)$ *are tight,*

is necessary and sufficient for $(\tilde{P}^n_T) \triangleleft (P^n_T)$.

Parallel with the Hellinger process H^n we consider the process $B^n = (B^n_t, F^n_t)_{t\geq 0}$ with

$$B^n_t = \frac{1}{4}(\beta^n \circ \langle \zeta^{nc}\rangle)_t + \big(I(Z^n_- > 0)(\sqrt{Y^n(x)} - 1)^2 * \nu_{Z^n, P^n}\big)_t \tag{7}$$

where

$$Z^n_- = (Z^n_{t-}, F^n_t)_{t>0},\ Y^n(x) = \big(Y^n_t(x), F^n_t\big)_{t>0},\ Y^n_t(x) = \big(1 + x(Z^n_{t-})^{\circledast}\big) \vee 0,$$

and $I(\cdot)$ is an indicator function.

As we shall see later, there is a close relationship between B^n and H^n. In the following Lemma 4 we give the estimations for them. Let us define the stopping times

$$\tau^n_k = \inf\{t\geq 0: \zeta^n_t \leq\ \leq 1/k\},\ \tilde{\tau}^n_k = \inf\{t\geq 0: \tilde{\zeta}^n_t \leq 1/k\},\ T^n_k = \tau^n_k \wedge \tilde{\tau}^n_k \tag{8}$$

for $k = 1, 2, \ldots$, with $\inf\{\emptyset\} = \infty$.

Lemma 4. *For every* $n\geq 1$, $k\geq 1$ *and* $t\geq 0$ *we have*

$$E^n B^n_{t\wedge T^n_k} \leq E^n H^n_{t\wedge T^n_k} \leq 4k, \tag{9}$$

$$E^n |H^n_{t\wedge T^n_k} - B^n_{t\wedge T^n_k}| \leq 4k\tilde{P}^n(\zeta^n_{t\wedge T^n_k} = 0). \tag{10}$$

Proof. From [23, Theorem 1] it follows that for every $t\geq 0$

$$E^n_Q H^n_{t\wedge T^n_k} = E^n_Q(1 \circ H^n)_{t\wedge T^n_k} \leq kE^n_Q(\sqrt{\zeta^n_{s-}\tilde{\zeta}^n_{s-}} \circ H^n)_{t\wedge T^n_k} \leq$$
$$\leq k\varrho^2(P^n_{t\wedge T^n_k}, \tilde{P}^n_{t\wedge T^n_k}) \leq 2k$$

where E^n_Q is the expectation with respect to Q^n and $\varrho(.,.)$ is the Hellinger distance. So,

$$E^n H^n_{t\wedge T^n_k} = E^n_Q \zeta^n_\infty H^n_{t\wedge T^n_k} \leq 4k, \tag{11}$$

because by Lemma 1, $\zeta^n_\infty = \lim_{t\to\infty} \zeta^n_t \leq 2$.

Now we show that

$$E^n((\sqrt{Y^n(x)} - 1)^2 * \mu_{Z^n})_{T^n_k} < \infty \tag{12}$$

where μ_{Z^n} is the jump measure of Z^n. For this purpose we consider a localizing sequence of stopping times

$$\sigma_r = \inf\{t \geqq 0: ((\sqrt{Y^n(x)}-1)^2 * \mu_{Z^n})_t \geqq r\},\ r = 1, 2, \ldots,$$

with $\inf\{\emptyset\} = \infty$. Because $Z^n_{t-} = \tilde{\zeta}^n_t/\zeta^n_{t-} \geqq 1/(2k)$ for $(\omega, t) \in [0, T^n_k]$ and, hence, $\alpha^n_t = Z^n_t/Z^n_{t-} \leqq 2kZ^n_t$, by Lemma 1 we get

$$E^n((\sqrt{Y^n(x)}-1)^2 * \mu_{Z^n})_{T^n_k \wedge \sigma_r} \leqq r + E^n(1-\sqrt{\alpha^n_{T^n_k \wedge \sigma_r}})^2 \leqq r + 2k + 1.$$

By virtue of $Y^n(\triangle Z^n_s) = \tilde{\lambda}^n(\triangle \zeta^n_s)/\lambda^n(\Delta\zeta^n_s)$ on $(\omega, s) \in [0, T^n_k]$ and Lemma 1, we get from [11, Proposition 3] and [16, Lemma 2.3] that

$$E^n\Big((\sqrt{y^n(x)}-1)^2 * \mu_{Z^n}\Big)_{T^n_k \wedge \sigma} =$$

$$= E^n(I(\lambda^n(x)>0)\Big((\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1)^2 * \mu_{\zeta^n}\Big)_{T^n_k \wedge \sigma_r} =$$

$$= E^n\Big(I(\lambda^n(x)>0)(\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1)^2 * \nu_{\zeta^n, P^n}\Big)_{T^n_k \wedge \sigma_r} =$$

$$= E^n\Big(I(\lambda^n(x)>0)(\sqrt{\tilde{\lambda}^n(x)}-\sqrt{\lambda^n(x)})^2 * \nu_{\zeta^n, Q^n}\Big)_{T^n_k \wedge \sigma_r} \leqq E^n H^n_{T^n_k} \leqq 4k.$$

Since the sequence of stopping times $(\sigma_r)_{r \geqq 1}$ is nondecreasing, there exists $\sigma = \lim_{r \to \infty} \sigma_r$ and by Fatou's lemma we get

$$E^n\Big((\sqrt{y^n(x)}-1)^2 * \mu_{Z^n}\Big)_{T^n_k \wedge \sigma} \leqq 4k.$$

The integrand on the left-hand side of this inequality is equal to infinity on the set $\{\sigma < T^n_k\}$. So, the set $\{\sigma < T^n_k\}$ has P^n-measure zero and we get (12).

For every bounded predictable function f we get from (6) and (7) that

$$E^n(f \circ B^n)_{t \wedge T^n_k} = \frac{1}{4} E^n(f\beta^n \circ \langle \zeta^{nc} \rangle)_{t \wedge T^n_k} +$$

$$+ E^n\Big(fI(Z^n_- > 0)(\sqrt{Y^n(x)}-1)^2 * \nu_{Z^n, P^n}\Big)_{t \wedge T^n_k},$$

$$E^n(f \circ H^n)_{t \wedge T^n_k} = \frac{1}{4} E^n(f\beta^n \circ \langle \zeta^{nc} \rangle)_{t \wedge T^n_k} +$$

$$+ E^n\Big(f(\sqrt{\tilde{\lambda}^n(x)}-\sqrt{\lambda^n(x)})^2 * \nu_{\zeta^n, Q^n}\Big)_{t \wedge T^n_k}.$$

Furthermore, from (12), [11, Proposition 3] and [16, Lemma 2.3] we obtain

$$E^n\Big(fI(Z^n_- > 0)(\sqrt{Y^n(x)}-1)^2 * \nu_{Z^n, P^n}\Big)_{t \wedge T^n_k} =$$

$$= E^n\Big(fI(Z^n_- > 0)(\sqrt{Y^n(x)}-1)^2 * \mu_{Z^n}\Big)_{t \wedge T^n_k} =$$

$$= E^n\Big(fI(\lambda^n(x)>0)(\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1)^2 * \mu_{\zeta^n}\Big)_{t \wedge T^n_k} =$$

$$= E^n\Big(fI(\lambda^n(x)>0)(\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1)^2 * \nu_{\zeta^n, P^n}\Big)_{t \wedge T^n_k} =$$

$$= E^n\Big(fI(\lambda^n(x)>0)(\sqrt{\tilde{\lambda}^n(x)}-\sqrt{\lambda^n(x)})^2 * \nu_{\zeta^n, Q^n}\Big)_{t \wedge T^n_k}.$$

Taking $f \equiv 1$ in the above equalities we get (9) from (11).

To prove (10) we choose f to be equal to the predictable function f^* such that $|f^*| \leqq 2$,

$$E^n|H^n_{t\wedge T^n_k} - B^n_{t\wedge T^n_k}| \leqq E^n(f^* \circ (H^n - B^n))_{t\wedge T^n_k}.$$

Then from the above equalities we have

$$E^n|H^n_{t\wedge T^n_k} - B^n_{t\wedge T^n_k}| \leqq 2E^n(I(\lambda^n(x) = 0)\tilde{\lambda}^n(x) * \nu_{\zeta^n, Q^n})_{t\wedge T^n_k} \leqq$$
$$\leqq 4E^n_Q(I(\lambda^n(x) = 0)\tilde{\lambda}^n(x) * \nu_{\zeta^n, Q^n})_{t\wedge T^n_k} =$$
$$= 4E^n_Q(I(\lambda^n(x) = 0)\tilde{\lambda}^n(x) * \mu_{\zeta^n})_{t\wedge T^n_k} \leqq$$
$$\leqq 4kE^n_Q\big(\tilde{\zeta}^n I(\lambda^n(x) = 0) * \mu_{\zeta^n}\big)_{t\wedge T^n_k} \leqq 4k\tilde{P}^n(\tilde{\zeta}^n_{t\wedge T^n_k} = 0).$$

The inequality (10) is proved. □

2. Functional limit theorems for likelihood ratio processes of the simplest statistical models

In this paragraph we consider the simplest statistical models $\mathscr{E}^n_s = \{(\Omega^n, F^n, \mathbf{F}^n), \tilde{P}^n, P^n\}_{n\geqq 1}$ where (Ω^n, F^n) are measurable spaces, $\mathbf{F}^n$ are right-continuous filtrations, $\bigvee_{t\geqq 0} F^n_t = F^n$, completed by the Q^n-null sets of F^n, and $\tilde{P}^n$, P^n are probability measures on (Ω^n, F^n). Throughout this paragraph we use the notation of **1.** We apply the expression $\mu_n \xrightarrow{P^n} \mu$ to indicate that $P^n(|\mu_n - \mu| > \varepsilon) \to 0$ as $n \to \infty$, $\forall \varepsilon > 0$, and by $I(\cdot)$ we denote an indicator function.

Theorem 2. *Let A_t be a nonnegative continuous deterministic function, $0 \leq t \leq T$, $A_0 = 0$, and let $N = (N_t)_{t\geqq 0}$ be a Gaussian martingale with $\langle N\rangle_t = 4A_t$. Suppose that the measures $(\tilde{P}^n_T)$ and (P^n_T) are mutually contiguous, i.e. $(\tilde{P}^n_T) \lhd \rhd (P^n_T)$, and for every $0 < t \leq T$ and $\varepsilon > 0$*

(A) $Z^n_0 \xrightarrow{P^n} 1$,

(B) $\big((\sqrt{Y^n(x)} - 1)^2 I(|\sqrt{Y^n(x)} - 1| \geq \varepsilon) * \nu_{Z^n, P^n}\big)_t \xrightarrow{P^n} 0$,

(C) $B^n_t \xrightarrow{P^n} A_t$.

Then for $Z^n = (Z^n_t, F^n_t)_{t\geqq 0}$ we have

$$Z^n \xrightarrow{\mathbf{D}_T(P^n)} \exp\left(N - \frac{1}{2}\langle N\rangle\right)$$

where $\xrightarrow{\mathbf{D}_T(P^n)}$ denotes a weak convergence in the Skorokhod space $\mathbf{D}[0, T]$ with respect to P^n.

Remark 1. Under the mutual contiguity $(\tilde{P}_T^n) \lhd \rhd (P_T^n)$, the condition (C) is equivalent to

$$(C') \quad H_t^n \xrightarrow{P^n} A_t.$$

Remark 2. The same theorem for discrete time was obtained in [5]. In a slightly different formulation, Theorem 2 was obtained in [12].

Proof of Theorem 2. For $n \geqq 1$ let us introduce the stopping times

$$(13) \qquad \tau^n = \inf\{t \geqq 0: \zeta_t^n \leqq 1/n\}, \ \tilde{\tau}^n = \inf\{t \geqq 0: \tilde{\zeta}_t^n \leqq 1/n\} \ \text{and}$$
$$T^n = \tau^n \wedge \tilde{\tau}^n \ \text{with} \ \inf\{\emptyset\} = \infty.$$

The proof of Theorem 2 will be given according to the following plan. We shall show that under conditions (A), (B) and (C) the following conclusions hold:

$$\text{(i)} \qquad P^n(\sup_{0 \leqq t \leqq T} |Z_t^n - Z_{t \wedge T^n}^n| \geqq \varepsilon) \to 0, \ n \to \infty, \ \forall \varepsilon > 0,$$

$$\text{(ii)} \qquad P^n(\sup_{0 \leqq t \leqq T} |Z_{t \wedge T^n}^n - \exp(V_t^n)| \geqq \varepsilon) \to 0, \ n \to \infty, \ \forall \varepsilon > 0,$$

$$\text{(iii)} \qquad V^n \xrightarrow{D_T(P^n)} N - \frac{1}{2}\langle N \rangle$$

where $V^n = (V_t^n, F_t^n)_{t \geqq 0}$ is the process built in some special way. It is evident that (i), (ii), and (iii) imply Theorem 2.

We show that $(P_T^n) \lhd (\tilde{P}_T^n)$ implies the relation (i). From (13) we have

$$(14) \qquad P^n(\sup_{0 \leqq t \leqq T} |Z_t^n - Z_{t \wedge T^n}^n| \geqq \varepsilon) \leqq P^n(T^n \leqq T) \leqq P^n(\tau^n \leqq T) + P^n(\tilde{\tau}^n \leqq T).$$

By Lemma 1 $\zeta_\tau^n + \tilde{\zeta}_\tau^n = 2$ (Q^n – a.s.) for every stopping time τ, and, hence, $Z^n = 2/\zeta_\tau^n - 1$ (P^n – a.s.). From the same Lemma 1

$$P^n(\tau^n \leqq T) \leq P^n(\zeta_{T \wedge \tau^n}^n \leqq 1/n) = P^n(Z_{T \wedge \tau^n}^n \geqq 2n - 1) \leqq$$
$$\leqq P^n(\sup_{0 \leqq t \leqq T} Z_t^n \geqq 2n - 1) \leqq 1/(2n - 1)$$

whence $\lim_{n \to \infty} P^n(\tau^n \leq T) = 0$. In the same way

$$\tilde{P}^n(\tilde{\tau}^n \leq T) \leq \tilde{P}^n(\sup_{0 \leqq t \leqq T} \tilde{Z}_t^n \geq 2n - 1) \leq 1/(2n - 1)$$

which together with the contiguity $(P_T^n) \lhd (\tilde{P}_T^n)$ gives $\lim_{n \to \infty} P^n(\tilde{\tau}^n \leq T) = 0$. From here

$$(15) \qquad \lim_{n \to \infty} P^n(T^n \leq T) = 0$$

which together with (14) implies (i).

To prove the relation (ii) for every $t>0$ we set

$$\alpha_t^n = Z_t^n/Z_{t-}^n \text{ and } \tilde{\alpha}_t^n = \tilde{Z}_t^n/\tilde{Z}_{t-}^n$$

putting $0/0 = 0$ and $\infty/\infty = \infty$. Since Q^n—a.s. $\tilde{Z}_t^n = 1/Z_t^n$, $\tilde{\alpha}_t^n = 1/\alpha_t^n$ for every $t>0$ and $Y_t^n(\triangle Z_t^n) = \tilde{\lambda}_t^n(\triangle\zeta_t^n)/\lambda_t^n(\triangle\zeta_t^n)$ for $(\omega, t)\in[0, T^n]$ from [16, Theorem 2.1] substituting

$$\tilde{Z}_t^n,\ \tilde{\zeta}_t^n,\ \tilde{\alpha}_t^n,\ \tilde{\lambda}_t^n(-x),\ \lambda_t^n(-x) \text{ and } P^n \text{ for } Z_t^n,\ \zeta_t^n,\ \alpha_t^n,\ \lambda_t^n(x),\ \tilde{\lambda}_t^n(x) \text{ and } \tilde{P}^n,$$

respectively, for $t>0$ and $L>0$ (P^n—a.s.) we get:

$$\begin{aligned} Z_{t\wedge T^n}^n = Z_0^n \exp\Big\{\sum_{0<s\leqq t<T^n}(\ln\alpha_s^n - U_L(\ln\alpha_s^n)) + 2M_t^{nc} - 2G_{t\wedge T^n}^n + \\ +\big(U_L(\ln(\tilde{\lambda}^n(x)/\lambda^n(x)))*(\mu_{\zeta^n} - \nu_{\zeta^n, P^n})\big)_{t\wedge T^n} - \\ -\big(G_L(\tilde{\lambda}^n(x)/\lambda^n(x))*\nu_{\zeta^n, P^n}\big)_{t\wedge T^n}\Big\} = Z_0^n \exp\Big\{\sum_{0<s\leqq t\wedge T^n}(\ln\alpha_s^n - U_L(\ln\alpha_s^n)) + \\ +2M_t^{nc} - 2G_{t\wedge T^n}^n + \big(U_L(\ln Y^n(x))*(\mu_{Z^n} - \nu_{Z^n, P^n})\big)_{t\wedge T^n} - \\ -\big(G_L(Y^n(x))*\nu_{Z^n, P^n}\big)_{t\wedge T^n}\Big\} \end{aligned} \tag{16}$$

where $M^{nc} = (M_t^{nc}, F_t^n)_{t\geqq 0}$ is a continuous P^n-martingale with

$$\langle M^{nc}\rangle_t = G_{t\wedge T^n}^n = \frac{1}{4}(\beta^n \circ \langle\zeta^{nc}\rangle)_{t\wedge T^n}, \tag{17}$$

$$U_L(x) = \begin{cases} x \text{ if } |x|\leq L, \\ L \operatorname{sign} x \text{ if } |x|>L, \end{cases}$$

$$G_L(x) = x - 1 - U_L(\ln x).$$

Let us introduce the process $V^n = (V_t^n, F_t^n)_{t\geqq 0}$ with

$$V_t^n = 2M_t^{nc} + 2\big((\sqrt{Y^n(x)} - 1)^2*(\mu_{Z^n} - \nu_{Z^n, P^n})\big)_{t\wedge T^n} - 2B_{t\wedge T^n}^n. \tag{18}$$

From Lemma 4

$$E^n\big((\sqrt{Y^n(x)} - 1)^2*\nu_{Z^n, P^n}\big)_{t\wedge T^n} \leqq E^n B_{t\wedge T^n}^n \leqq 4n$$

and from [11, Theorem 2] we conclude that the integral $\big((\sqrt{Y^n(x)} - 1)*(\mu_{Z^n} - \nu_{Z^n, P^n})\big)_{\cdot\wedge T^n}$ is well-defined and it is a square-integrable martingale. So, the process V^n is well-defined.

Now we continue the proof of (ii). For this purpose let us set $\mathcal{F}_\delta = \{|1 - Z_0^n|\leqq\delta\}\cap\{\sup_{0<t\leqq T}|1 - \sqrt{\alpha_t^n}|\leqq\delta\}$ with $0<\delta<1$. Then for every $\varepsilon>0$, $A>0$

$$P^n\big(\sup_{0\leqq t\leqq T}|Z_{t\wedge T^n}^n - \exp(V_t^n)|\geqq\varepsilon\big)\leqq$$

$$\leqq P^n(\{\sup_{0\leqq t\leqq T}|\ln Z^n_{t\wedge T^n}-V^n_t|\geqq\delta(\varepsilon,\ A)\}\cap\mathcal{F}_\delta)+$$

$$+P^n(\sup_{0\leqq t\leqq T}Z^n_t\geqq A)+P^n(|1-Z^n_0|\geqq\delta)+P^n(\sup_{0\leqq t\leqq T}|1-\sqrt{\alpha^n_t}|\geqq\delta)$$

where $\delta(\varepsilon, A) = \min\{\ln(1+\varepsilon/A), |\ln(1-\varepsilon/A)|\}$.

According to Lemma 1

$$P^n(\sup_{0\leqq t\leqq T}Z^n_t\geqq A)\leq 1/A,\ \forall A>0,$$

and by the condition 1)

$$P^n(|1-Z^n_0|\geqq\delta)\to 0,\ n\to\infty.$$

Hence, for (ii) it is sufficient so show that

$$(19)\qquad P^n(\sup_{0<t\leqq T}|1-\sqrt{\alpha^n_t}|\geqq\delta)\to 0,\ n\to\infty,$$

and

$$(20)\qquad \lim_{A\to\infty}\overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}P^n(\{\sup_{0\leqq t\leqq T}|\ln Z^n_{t\wedge T^n}-V^n_t|\geqq\delta(\varepsilon,\ A)\}\cap\mathcal{F}_\delta)=0.$$

To prove (19) we note that

$$(21)\qquad P^n(\sup_{0<t\leqq T}|1-\sqrt{\alpha^n_t}|\geqq\delta)\leqq P^n(\sup_{0<t\leqq T\wedge T^n}|1-\sqrt{\alpha^n_t}|\geqq\delta)+P^n(T^n\leqq T).$$

Further, because $\alpha^n_t = Y^n_t(\triangle Z^n_t)$ for $(\omega, t)\in[0, T^n]$, we get

$$P^n(\sup_{0<t\leqq T\wedge T^n}|1-\sqrt{\alpha^n_t}|\geqq\delta)\leqq P^n(\sum_{0<t\leqq T\wedge T^n}I(|1-\sqrt{\alpha^n_t}|\geq\delta/2)\geqq 1)\leqq$$

$$(22)\qquad \leqq P^n(\sum_{0<t\leqq T\wedge T^n}(1-\sqrt{\alpha^n_t})^2 I(|1-\sqrt{\alpha^n_t}|\geqq\delta/2)\geqq\delta^2/4)=$$

$$=P^n\Big(\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta/2)*\mu_{Z^n}\big)_{T\wedge T^n}\geqq\delta^2/4\Big).$$

From [11, Proposition 3] and (12) we find that for every finite stopping time τ

$$(23)\qquad E^n\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\leqq\delta/2)*\mu_{Z^n}\big)_{\tau\wedge T^n}=$$

$$=E^n\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1)|\geqq\delta/2)*\nu_{Z^n,P^n}\big)_{\tau\wedge T^n}.$$

Using Lemma 3 we obtain for every $\gamma>0$

$$P^n\Big(\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta/2)*\mu_{Z^n}\big)_{T\wedge T^n}\geqq\delta^2/4\Big)\leqq$$

$$\leq\gamma+P^n\Big(\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geq\delta/2)*\nu_{Z^n,P^n}\big)_{T\wedge T^n}\geqq\gamma\delta^2/4\Big).$$

Therefore, because of (21), (22), (15) and condition (B), after taking $\lim_{\gamma\to 0}\overline{\lim}_{n\to\infty}$ in the above inequality (19) follows.

To establish (20) we use Taylor formula and (16), (18) and also

$$G_L(x)+U_L^2(x)\leqq q_L(1-\sqrt{x})^2,\ x\geqq 0,$$

where the constant q_L depends on L. Then for $L>2\max(\ln(1+\delta),\ |\ln(1-\delta)|)$ on the set $\mathcal{F}_\vartheta$ we have

$$|\ln Z^n_{t\wedge T^n}-V^n_t|\leqq|\ln Z^n_0|+$$
$$+(1-\delta)^{-2}\left|(\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\leqq\delta)*(\mu_{Z^n}-\nu_{Z^n,P^n})_{t\wedge T^n}\right|+$$
$$+\delta(1-\delta)^{-3}\Big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\leqq\delta)*\nu_{Z^n,P^n}\Big)_{t\wedge T^n}+$$
$$+C_1(L,\delta)\Big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta)*\mu_{Z^n}\Big)_{t\wedge T^n}+\tag{24}$$
$$+C_2(L,\delta)\Big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta)*\nu_{Z^n,P^n}\Big)_{t\wedge T^n}$$

where

$$C_1(L,\delta)=L/\delta^2+2/\delta,\ C_2(L,\delta)=L/\delta^2+L^2/\delta^2+2/\delta+q_L+2.$$

By condition (A)

$$\lim_{n\to\infty}P^n(|\ln Z_0^n|\geqq\delta(\varepsilon,A)/5)=0.\tag{25}$$

In order to estimate the second term on the right-hand side of (24) we note that for every finite stopping time τ

$$E^n\Big(\Big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\leqq\delta)*(\mu_{Z^n}-\nu_{Z^n,P^n})\Big)_{\tau\wedge T^n}\Big)^2=$$
$$=E^n\langle(\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\leqq\delta)*(\mu_{Z^n}-\nu_{Z^n,P^n})\rangle_{\tau\wedge T^n}\leqq\delta^2E^nB^n_{\tau\wedge T^n}.$$

Hence, from Lemma 3

$$P^n\Big(\sup_{0\leqq t\leqq T}\left|\Big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\leqq\delta)*(\mu_{Z^n}-\nu_{Z^n,P^n})\Big)_{t\wedge T^n}\right|\geqq$$
$$\geqq\delta(\varepsilon,A)/5\Big)\leqq\delta(\varepsilon,A)+P^n\big(B^n_T\geqq\delta^3(\varepsilon,A)/(25\delta^2)\big).\tag{26}$$

As the sequence (B^n_T, P^n) is tight and $\delta(\varepsilon,A)\to 0$ when $A\to\infty$, from (26) we get

$$\lim_{A\to\infty}\overline{\lim}_{\delta\to 0}\overline{\lim}_{n\to\infty}P^n\Big(\sup_{0\leqq t\leqq T}\Big|\Big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\leqq\delta)*\tag{27}$$
$$*(\mu_{Z^n}-\nu_{Z^n,P^n})\Big)_{t\wedge T^n}\Big|\geqq\delta(\varepsilon,A)/5\Big)=0.$$

For the third term on the right-hand side of (24) we have

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, P^n\Big(\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\leqq\delta)*\nu_{Z^n,P^n}\big)_{T\wedge T^n}\geqq$$

$$\geqq\delta(\varepsilon, A)/(5\delta)\Big)\leqq\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, P^n(B^n_T\geqq\delta(\varepsilon, A)/(5\delta)) = 0, \qquad (28)$$

because of the tightness of (B^n_T, P^n).

From condition (B) and Corollary 1 it follows that

$$\lim_{n\to\infty} P^n\Big(\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta)*\mu_{Z^n}\big)_{T\wedge T^n}\geqq$$

$$\geqq\delta(\varepsilon, A)/(5C_1(L, \delta))\Big) = 0,$$

$$\lim_{n\to\infty} P^n\Big(\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta)*\nu_{Z^n,P^n}\big)_{T\wedge T^n}\geqq$$

$$\geqq\delta(\varepsilon, A)/(5C_2(L, \delta))\Big) = 0.$$

These relations together with (24), (25), (27) and (28) imply (20) and hence, (ii).

Now we prove the relation (iii). From (18) we get

$$V^n_t = 2\hat{M}^n_t - 2B^n_{t\wedge T^n} \qquad (29)$$

where $\hat{M}^n = (\hat{M}^n_t, F^n_t)_{t\geqq 0}$ is a square integrable P^n-martingale with

$$\hat{M}^n_t = M^{nc}_t + \big((\sqrt{Y^n(x)}-1)*(\mu_{Z^n}-\nu_{Z^n,P^n})\big)_{t\wedge T^n}. \qquad (30)$$

Since for every $\varepsilon>0$

$$P^n(\sup_{0\leqq t\leqq T}|B^n_{t\wedge T^n}-A_t|\geqq\varepsilon)\leqq P^n(\sup_{0\leqq t\leqq T}|B^n_t-A_t|\geqq\varepsilon)+P^n(T^n\leqq T),$$

from condition (C), [18, Lemma 1] and (15) we find that

$$P^n(\sup_{0\leqq t\leqq T}|B^n_{t\wedge T^n}-A_t|\geqq\varepsilon)\to 0,\ n\to\infty,\ \forall\,\varepsilon>0. \qquad (31)$$

Hence, by [15, Corollary 2], we need only to show that for every $0<t\leq T$ and $\varepsilon>0$

$$(x^2 I(|x|\geq\varepsilon)*\nu_{\hat{M}^n, P^n})_t\xrightarrow{P^n}0, \qquad (32)$$

$$\langle\hat{M}^n\rangle_t\xrightarrow{P^n}A_t. \qquad (33)$$

First we prove the Lindeberg condition (32). Because of [2, Corollary 9.37], [11, Theorem 2] and Lemma 4

$$E^n(x^2 I(|x|\geqq\varepsilon)*\mu_{\hat{M}^n})_t\leqq E^n\sum_{0<s\leqq t}(\triangle\hat{M}^n_s)^2\leqq E^n\langle\hat{M}^n\rangle_t\leqq E^n B^n_{t\wedge T^n}\leqq 4n,$$

we have

$$(x^2 I(|x|\geqq\varepsilon) * \nu_{\hat{M}^n,\,P^n})_t = \mathrm{comp}_{P^n}(x^2 I(|x|\geqq\varepsilon) * \mu_{\hat{M}^n})_t$$

where comp_{P^n} denotes the operation of the taking of the compensator with respect to $(\mathbf{F}^n, P^n)$.

After setting for $\delta<\varepsilon/2$ and $0<s\leqq T$

$$\triangle M_s^{n,1} = \int\limits_{R^1\setminus\{0\}} (\sqrt{Y_s^n(x)}-1) I(|\sqrt{Y_s^n(x)}-1|\leqq\delta)(\mu_{Z^n}(\{s\}, dx) - \\ - \nu_{Z^n,P^n}(\{s\}, dx))$$

and

$$\triangle M_s^{n,2} = \int\limits_{R^1\setminus\{0\}} (\sqrt{Y_s^n(x)}-1) I(|\sqrt{Y_s^n(x)}-1|>\delta)(\mu_{Z^n}(\{s\}, dx) - \\ - \nu_{Z^n,P^n}(\{s\}, dx))$$

we get from (30) that for $(\omega, s)\in[0, t\wedge T^n]$

$$\triangle\hat{M}_s^n = \triangle M_s^{n,1} + \triangle M_s^{n,2}.$$

Because of $|\triangle M_s^{n,1}|\leqq 2\delta$, from the previous expression we have

$$(x^2 I(|x|\geqq\varepsilon) * \mu_{\hat{M}^n})_t = \sum_{0<s\leqq t\wedge T^n} (\triangle\hat{M}_s^n)^2 I(|\triangle\hat{M}_s^n|\geqq\varepsilon) \leqq$$

$$\leqq 2 \sum_{0<s\leqq t\wedge T^n} (\triangle M_s^{n,2})^2 + 2 \sum_{0<s\leqq t\wedge T^n} (\triangle M_s^{n,1})^2 I(|\triangle\hat{M}_s^n|\geqq\varepsilon) \leqq$$

$$\leqq (2+8\delta^2)/(\varepsilon-2\delta)^2) \sum_{0<s\leqq t\wedge T^n} (\triangle M_s^{n,2})^2 \leqq 2(2+8\delta^2)/(\varepsilon-2\delta)^2)\cdot$$

$$\cdot\Big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta) * (\mu_{Z^n}+\nu_{Z^n,P^n})\Big)_{t\wedge T^n}.$$

According to [4, Lemma 2, §2, Ch. 2] and (12) the same inequality holds (P^n — a.s.) for the compensators of these processes with respect to $(\mathbf{F}, P^n)$. Since the compensator of the right-hand side in the above inequality is equal to

$$4((2+8\delta^2)/(\varepsilon-2\delta)^2)\Big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta) * \nu_{Z^n,P^n}\Big)_{t\wedge T^n},$$

we have (32) from condition (B).

Now we prove (33). According to [11, Theorem 2] we get

$$\langle\hat{M}^n\rangle_t = B^n_{t\wedge T^n} - U_t^n \tag{34}$$

where

$$U_t^n = \sum_{0<s\leqq t\wedge T^n} \Big(\int\limits_{R^1\setminus\{0\}} (\sqrt{Y_s^n(x)}-1)\,\nu_{Z^n,P^n}(\{s\}, dx)\Big)^2. \tag{35}$$

Because of (31) and (34) it is sufficient to show that for every $\varepsilon > 0$

$$P^n(U_T^n \geqq \varepsilon) \to 0, \quad n \to \infty. \tag{36}$$

For this purpose we set

$$\Delta U^n = \{(\omega, t) : U_t^n \neq U_{t-}^n\}$$

Then, by [2, Theorem 6.46] we get

$$\Delta U^n = \bigcup_{j \geqq 1} [S_j^n] \tag{37}$$

where S_j^n, $j \geqq 1$, are predictable stopping times and

$$[S_i^n] \cap [S_j^n] = \emptyset \text{ for } i \neq j.$$

We have from [8, Lemma, p. 229]

$$\begin{aligned} 2I(S_j^n \leqq T \wedge T^n) \int_{R^1 \setminus \{0\}} (\sqrt{Y_{S^n_j}^n(x)} - 1) \nu_{Z^n, P^n}(\{S_j^n\}, dx) &= \\ = 2E^n\big(I(S_j^n \leqq T \wedge T^n)(\sqrt{\alpha_{S^n_j}^n} - 1) \,|\, F_{S^n_j -}^n\big) &= \\ = -I(S_j^n \leqq T \wedge T^n)(\Delta B_{S^n_j}^n + E^n(1 - \alpha_{S^n_j}^n \,|\, F_{S^n_j -}^n)). & \end{aligned} \tag{38}$$

Using Lemma 2 we obtain

$$\begin{aligned} E^n(1 - \alpha_{S^n_j}^n \,|\, F_{S^n_j -}^n) &= \\ = I(Z_{S^n_j -}^n = 0) + I(Z_{S^n_j -}^n > 0) \tilde{E}^n\big(I(\alpha_{S^n_j}^n = \infty) \,|\, F_{S^n_j -}^n\big) & \end{aligned} \tag{38a}$$

which together with (38) gives

$$U_T^n \leqq \sum_{j=1}^{\infty} I(S_j^n \leqq T \wedge T^n)(\Delta B_{S^n_j}^n)^2 +$$

$$+ \sum_{j=1}^{\infty} I(S_j^n \leqq T \wedge T^n)\big(\tilde{E}^n(I(\alpha_{S^n_j}^n = \infty) \,|\, F_{S^n_j -}^n)\big)^2 \leqq$$

$$\leqq (\sup_{0 < s \leqq T} \Delta B_s^n) B_T^n + \sum_{j=1}^{\infty} \tilde{P}^n(\{S_j^n \leqq T \wedge T^n\} \cap \{\alpha_{S^n_j}^n = \infty\} \,|\, F_{S^n_j -}^n).$$

Since A_t is a continuous function, from condition (C) and [18, Lemma 1] it follows that

$$P^n(\sup_{0 < s \leqq T} \Delta B_s^n \geqq \varepsilon) \to 0, \quad n \to \infty \quad \forall \varepsilon > 0.$$

Since the sequence (B_T^n, P^n) is tight, for (36) it remains to show that

$$\Psi^n = \sum_{j=1}^{\infty} \tilde{P}^n(\{S_j^n \leqq T \wedge T^n\} \cap \{\alpha_{S^n_j}^n = \infty\} \,|\, F_{S^n_j -}^n) \xrightarrow{P^n} 0. \tag{39}$$

For every $a>0$ we have from [2, Theorem 5.26]

$$\{\tilde{P}(\{S_j^n \leqq T \wedge T^n\} \cap \{\alpha_{S^n_j}^n = \infty\} \mid F^n_{S^n_j -}) > a\} =$$
$$= \{I(S_j^n \leqq T \wedge T^n)\tilde{P}^n(\alpha^n_{S^n_j} = \infty \mid F^n_{S^n_j -}) > a\} =$$
$$= \{S_j^n \leqq T \wedge T^n\} \cap \{\tilde{P}^n(\alpha^n_{S^n_j} = \infty \mid F^n_{S^n_j -}) > a\} \in F^n_{T \wedge T^n -}.$$

If we exclude from the sum in (39) the addends which are equal to zero then Ψ^n will be F^n_T-measurable. By Lemma 2 for arbitrary $L>0$

$$P^n(\Psi^n \geqq \varepsilon) \leqq L P^n(\Psi^n \geqq \varepsilon) + P^n(\tilde{Z}^n_T \geqq L). \tag{40}$$

By Lemma 3 for $0<\varepsilon<1$ and $0<\delta<1$

$$\begin{aligned}\tilde{P}^n(\Psi^n \geqq \varepsilon) &\leqq \delta + \frac{1}{\varepsilon}\tilde{E}^n \sup_j I(\{S_j^n \leqq T \wedge T^n\} \cap \{\alpha^n_{S^n_j} = \infty\}) + \\ &+ \tilde{P}^n\left(\sum_{j=1}^{\infty} I(\{S_j^n \leqq T \wedge T^n\} \cap \{\alpha^n_{S^n_j} = \infty\}) \geqq \varepsilon\delta\right) \leqq \delta + \\ &+ \frac{1}{\varepsilon}\tilde{E}^n \sup_j I(\{S_j^n \leqq T \wedge T^n\} \cap \{\alpha^n_{S_j^n} = \infty\}) + \\ &+ \tilde{P}^n(\sup_{0<s\leqq T} \alpha^n_s = \infty) \leqq \delta + \left(1 + \frac{1}{\varepsilon}\right)\tilde{P}^n(\sup_{0<s\leqq T} \alpha^n_s = \infty).\end{aligned} \tag{41}$$

Since $P^n(\sup_{0<t\leqq T} \alpha^n_s = \infty) = 0$ and $(\tilde{P}^n_T) \lhd (P^n_T)$, we get that

$$\tilde{P}^n(\sup_{0<s\leqq T} \alpha^n_s = \infty) \to 0, \ n \to \infty.$$

In addition, by $(P^n_T) \lhd (\tilde{P}^n_T)$ and Theorem 1

$$\lim_{L\to\infty} \overline{\lim_{n\to\infty}} P^n(\tilde{Z}^n_T \geqq L) = 0.$$

Hence, (40) and (41) imply (39) after the operations $\lim_{L\to\infty} \overline{\lim_{\delta\to 0}} \overline{\lim_{n\to\infty}}$, that gives (36). So, the relation (33) is proved together with (iii). Theorem 2 is proved. □

Proof of Remark 1. Let us consider the stopping times defined by (8). For every $0<t\leq T$ and $\varepsilon>0$ we have

$$\begin{aligned}P^n(|H^n_t - A_t| \geqq \varepsilon) &\leqq P^n(|H^n_{t\wedge T^n_k} - A_t| \geqq \varepsilon) + P^n(T^n_k \leqq t) \leqq \\ &\leqq P^n(|H^n_{t\wedge T^n_k} - B^n_{t\wedge T^n_k}| \geqq \varepsilon/2) + P^n(|B^n_{t\wedge T^n_k} - A_t| \geqq \varepsilon/2) + \\ &+ P^n(T^n_k \leq t) \leqq (2/\varepsilon)E^n|H^n_{t\wedge T^n_k} - B^n_{t\wedge T^n_k}| + P^n(|B^n_t - A_t| \geqq \varepsilon/2) + \\ &+ 2P^n(T^n_k \leqq t).\end{aligned} \tag{42}$$

In the same way

$$P^n(|B_t^n - A_t| \geqq \varepsilon) \leqq (2/\varepsilon) E^n |H^n_{t\wedge T^n_k} - B^n_{t\wedge T^n_k}| + P^n(|H_t^n - A_t| \geqq \varepsilon/2) + \tag{43}$$
$$+ 2P^n(T_k^n \leqq t).$$

Because of $(P_T^n) \lhd (\tilde{P}_T^n)$ we have, similarly to (15), that

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} P^n(T_k^n \leqq t) = 0. \tag{44}$$

From Lemma 1, $(\tilde{P}_T^n) \lhd (P_T^n)$ and the bound (10) of Lemma 4 we obtain

$$\lim_{n\to\infty} E^n \left| H^n_{t\wedge T^n_k} - B^n_{t\wedge T^n_k} \right| = 0 \tag{44a}$$

which together with (42), (43), (44) gives the equivalence of (C) and (C′).

In the following theorem we prove the necessity of the conditions of Theorem 2.

Theorem 3. *Suppose that*

$$Z^n \xrightarrow{D_T(P^n)} \exp\left(N - \frac{1}{2}\langle N\rangle\right) \tag{45}$$

where $N = (N_t)_{0\leqq t\leqq T}$ is a Gaussian martingale, with $N_0 = 0$ and continuous quadratic characteristic $\langle N\rangle_t$. Then $(P_T^n) \lhd \rhd (\tilde{P}_T^n)$ and conditions (A), (B) *and* (C) *are satisfied.*

Remark 3. For the discrete time case this theorem was proved in [5].

Proof of Theorem 3. The proof will be performed according to the following plan: we establish that

(j) $$(45) \Rightarrow (P_T^n) \lhd \rhd (\tilde{P}_T^n),$$

(jj) $$(45) \Rightarrow (A),\ (B),\ (45) \Rightarrow \left\{V^n \xrightarrow{D_T(P^n)} N - \frac{1}{2}\langle N\rangle\right\},$$

(jjj) $$\left\{V^n \xrightarrow{D_T(P^n)} N - \frac{1}{2}\langle N\rangle\right\} \Rightarrow \{[V^n, V^n]_t \to [N, N]_t,\ 0 < t \leqq T\}$$

and

(jjjj) $$\{[V^n, V^n]_t \xrightarrow{P^n} [N, N]_t,\ 0 < t \leqq T\} \Rightarrow (C)$$

where $[\cdot,\cdot]$ is the quadratic variation process.

To prove $(\tilde{P}_T^n) \lhd (P_T^n)$ we note that a Gaussian martingale N is by definition a martingale with Gaussian finite dimensional distributions. From this it follows that N is a Gaussian process with independent increments and $EN_t = EN_0 = 0$, $DN_t = \langle N\rangle_t$. Because of the continuity of the quadratic characteristic $\langle N\rangle$ of the Gaussian martingale N, it is also continuous with probability 1. This fact implies that the projection $Z \to Z_t$ is a continuous function for the limit process in (45) for every $0 \leq t \leq T$. In particular, according to [1, Theorem 5.1]

$$Z_T^n \xrightarrow{d(P^n)} \exp\left(N_T - \frac{1}{2}\langle N\rangle_T\right) \tag{46}$$

where $\xrightarrow{d(P^n)}$ denotes convergence in distribution.

Since $E\exp\left(N_T - \frac{1}{2}\langle N\rangle_T\right) = 1$, from (46) and Theorem 1 we get that $(\tilde{P}_T^n) \lhd (P_T^n)$.

To prove $(P_T^n) \lhd (\tilde{P}_T^n)$ we note that $P\left(\exp\left(N_T - \frac{1}{2}\langle N\rangle_T\right) = 0\right) = P(N_T = -\infty) = 0$. This relation together with (46), [1, Theorem 5.1] and $\tilde{Z}_T^n = 1/Z_T^n$ (Q^n – a.s.) gives

$$\tilde{Z}_T^n \xrightarrow{d(P^n)} \exp\left(N_T + \frac{1}{2}\langle N\rangle_T\right).$$

In particular, for $L > \exp(\langle N\rangle_T/2)$ we have $\lim_{n\to\infty} P^n(\tilde{Z}_T^n \geqq L) = P\left(\exp\left(N_T + \frac{1}{2}\langle N\rangle_T\right) \geqq L\right) = P(N_T \geqq \ln L - \langle N\rangle_T/2) \leqq \langle N\rangle_T/(\ln L - \langle N\rangle_T/2)^2$.
From this inequality it follows that

$$\lim_{L\to\infty} \overline{\lim_{n\to\infty}} P^n(\tilde{Z}_T^n \geqq L) = 0,$$

which, according to Theorem 1, implies $(P_T^n) \lhd (\tilde{P}_T^n)$.

Now we prove the implications (jj). Condition (A) follows from (45) by virtue of measurability in **D** and continuity (at all points of continuity of the limit process) of projection $Z \to Z_t$, in particular at $t = 0$.

To obtain condition (B) let us consider the functional

$$\tau(\delta, Z) = \inf\{0 < t \leqq T : |\Delta Z_t| \geqq \delta\}$$

with $\inf\{\emptyset\} = T + 1$, $\delta > 0$. The functional $\tau(\delta, Z)$ is measurable in **D** and continuous on the set $\{Z : |\Delta Z_t| \neq \delta,\ 0 \leqq t \leqq T\}$. Since the limit process in (45) is continuous with probability 1, for every $\delta > 0$

$$\tau(\delta, Z^n) \xrightarrow{d(P^n)} \tau(\delta, Z)$$

where $Z = \exp(N - 1/2\langle N\rangle)$. But $\tau(\delta, Z) = T + 1$ with probability 1, and, hence, for every $0 \leqq t \leqq T$ and every $\delta > 0$

$$\lim_{n\to\infty} P^n(\tau(\delta, Z^n) \leqq t) = 0. \tag{47}$$

Since $Y^n_s(\Delta Z^n_s)=\alpha^n_s$ for $(\omega, s)\in[0, T^n]$ and $\tilde{Z}^n_s=1/Z^n_s$ $(Q^n-\text{a.s.})$, for every $\delta>0$, $\varepsilon>0$, $A>0$ and $0<t\leqq T$ we have

$$P^n\Big(\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta)*\mu_{Z^n}\big)_{t\wedge T^n}\geqq\varepsilon\Big)\leqq$$

$$\leqq P^n(\sup_{0<s\leqq t\wedge T^n}|1-\sqrt{\alpha^n_s}|\geqq\delta)\leqq P^n(\sup_{0<s\leqq t\wedge T^n}|1-\alpha^n_s|\geqq\delta)\leqq$$

$$\leqq P^n(\sup_{0<s\leqq t}\tilde{Z}^n_s\leqq A)+P^n(\tau(\delta/A, Z^n)\leqq t). \tag{48}$$

From the contiguity $(P^n_T)\lhd(\tilde{P}^n_T)$ and Theorem 1 we conclude that the sequence $(\sup_{0<s\leqq t}\tilde{Z}^n_s, P^n)$ is tight, i.e.

$$\lim_{A\to\infty}\overline{\lim_{n\to\infty}}\, P^n(\sup_{0<s\leqq t}\tilde{Z}^n_s\geqq A) = 0.$$

Now, from (47), (48) after the operation $\lim_{A\to\infty}\overline{\lim_{n\to\infty}}$ we obtain

$$\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta)*\mu_{Z^n}\big)_{t\wedge T^n}\xrightarrow{P^n}0. \tag{49}$$

We consider the stopping times T^n_k defined by (8). By (49) for every $k\geq 1$

$$\big((\sqrt{Y^n(x)}-1)^2 I(|\sqrt{Y^n(x)}-1|\geqq\delta)*\mu_{Z^n}\big)_{t\wedge T^n_k}\xrightarrow{P^n}0.$$

Because of $Y^n_s(\Delta Z^n_s) = \tilde{\lambda}^n_s(\Delta\zeta^n_s)/\lambda^n_s(\Delta\zeta^n_s)$ and $0<\lambda^n_s(\Delta\zeta^n_s) = \zeta^n_s/\zeta^n_{s-}\leqq 2k$ for $(\omega, s)\in[0, t\wedge T^n_k]$, we have from the above expression, that

$$\big((\sqrt{\lambda^n(x)}-\sqrt{\tilde{\lambda}^n(x)})^2 I(|\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1|\geqq\delta)*\mu_{\zeta^n}\big)_{t\wedge T^n_k}\xrightarrow{P^n}0 \tag{50}$$

and by contiguity $(\tilde{P}^n_T)\lhd(P^n_T)$ the same relation holds with respect to Q^n.

From the proof of Lemma 4, $E^n_Q H^n_{t\wedge T^n_k}\leqq 2\text{k}$. Moreover, in the same way as the proof of (12) we get

$$E^n_Q\big((\sqrt{\lambda^n(x)}-\sqrt{\tilde{\lambda}^n(x)})^2 I(|\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1|\geqq\delta)*\mu_{\zeta^n}\big)_{T^n_k}<\infty.$$

This relation together with [11, Proposition 3] gives that for every finite stopping time τ

$$E^n_Q\big((\sqrt{\lambda^n(x)}-\sqrt{\tilde{\lambda}^n(x)})^2 I(|\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1|\geqq\delta)*\mu_{\zeta^n}\big)_{\tau\wedge T^n_k}=$$

$$=E^n_Q\big((\sqrt{\lambda^n(x)}-\sqrt{\tilde{\lambda}^n(x)})^2 I(|\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1|\geqq\delta)*\nu_{\zeta^n, Q^n}\big)_{\tau\wedge T^n_k}.$$

Then, by Lemma 3 we have for every $\varepsilon>0$, $\gamma>0$

$$Q^n\Big(\big((\sqrt{\lambda^n(x)}-\sqrt{\tilde{\lambda}^n(x)})^2 I(|\sqrt{\tilde{\lambda}^n(x)/\lambda^n(x)}-1|\geqq\delta)*\nu_{\zeta^n, Q^n}\big)_{t\wedge T^n_k}\geqq\varepsilon\Big)\leqq\gamma+$$

$$+\frac{1}{\varepsilon}E^n_Q\Big\{\sup_{0<s\leq t\wedge T^n_k}(\sqrt{\lambda^n_s(\Delta\zeta^n_s)}-\sqrt{\tilde\lambda^n_s(\Delta\zeta^n_s)})^2\times$$

$$\times I(|\sqrt{\tilde\lambda^n_s(\Delta\zeta^n_s)/\lambda^n_s(\Delta\zeta^n_s)}-1|\geq\delta)\Big\}+Q^n\Big(\big((\sqrt{\lambda^n(x)}-$$

$$-\sqrt{\tilde\lambda^n(x)})^2\, I(|\sqrt{\tilde\lambda^n(x)/\lambda^n(x)}-1|\geq\delta)*\mu_{\zeta^n}\big)_{t\wedge T^n_k}\geq\varepsilon\gamma\Big).$$

Since $\lambda^n_s(\Delta\zeta^n_s)=\zeta^n_s/\zeta^n_{s-}\leq 2k$, $\tilde\lambda^n_s(\Delta\zeta^n_s)=\tilde\zeta^n_s/\tilde\zeta^n_{s-}\leq 2k$ for $(\omega,s)\in$ $\in[0,t\wedge T^n_k]$, from this inequality, the relation (50) with respect to Q^n and the Lebesgue dominated convergence theorem we have after $\lim\limits_{\gamma\to 0}\overline{\lim\limits_{n\to\infty}}$:

(51) $$\big((\sqrt{\lambda^n(x)}-\sqrt{\tilde\lambda^n(x)})^2\, I(|\sqrt{\tilde\lambda^n(x)/\lambda^n(x)}-1|\geq\delta)*\nu_{\zeta^n,Q^n}\big)_{t\wedge T^n_k}\xrightarrow{Q^n}0.$$

This implies, of course, the same relation with respect to the measure P^n. Furthermore, for every $\varepsilon>0$

$$P^n\Big(\big((\sqrt{Y^n(x)}-1)^2\, I(|\sqrt{Y^n(x)}-1|\geq\delta)*\nu_{Z^n,P^n}\big)_{t\wedge T^n_k}\geq\varepsilon\Big)\leq$$

$$\leq P^n\Big(\big((\sqrt{\lambda^n(x)}-\sqrt{\tilde\lambda^n(x)})^2\, I(|\sqrt{\tilde\lambda^n(x)/\lambda^n(x)}-1|\geq\delta)*\nu_{\zeta^n,Q^n}\big)_{t\wedge T^n_k}\geq\varepsilon/2\Big)+$$

$$+(2/\varepsilon)E^n\Big|\big((\sqrt{Y^n(x)}-1)^2\, I(|\sqrt{Y^n(x)}-1|\geq\delta)*\nu_{Z^n,P^n}\big)_{t\wedge T^n_k}-$$

$$-\big((\sqrt{\lambda^n(x)}-\sqrt{\tilde\lambda^n(x)})^2\, I(|\sqrt{\tilde\lambda^n(x)/\lambda^n(x)}-1|\geq\delta)*\nu_{\zeta^n,Q^n}\big)_{t\wedge T^n_k}\Big|.$$

The first term on the right-hand side of this inequality tends to zero as $n\to\infty$ by (51), and the second one can be estimated — as in Lemma 4 — by $(8k/\varepsilon)\tilde P^n(\zeta^n_{t\wedge T^n_k}=0)$ which in turn tends to zero as $n\to\infty$ by Lemma 1 and $(\tilde P^n_T)\lhd(P^n_T)$. So, the left-hand side of the above inequality tends to zero as $n\to\infty$, which, by (44), implies condition (B).

To finish the proof of the implications (jj) we have to show that the sequence (B^n_T, P^n) is tight. For every $L>1$ and $k\geq 1$ we have

$$P^n(B^n_T\geq L)\leq P^n(B^n_{T\wedge T^n_k}\geq L)+P^n(T^n_k\leq T)\leq P^n(H^n_T\geq L-1)+$$

$$+E^n|B^n_{T\wedge T^n_k}-H^n_{T\wedge T^n_k}|+P^n(T^n_k\leq T).$$

By the contiguity $(P^n_T)\lhd(\tilde P^n_T)$ and Theorem 1 we get that the sequence (H^n_T, P^n) is tight. By Lemmas 1 and 4 and $(\tilde P^n_T)\lhd(P^n_T)$ we have also that

$$E^n|B^n_{T\wedge T_k{}^n}-H^n_{T\wedge T^n_k}|\to 0 \text{ and } n\to\infty.$$

According to these facts and (44) we find after $\lim\limits_{L\to\infty}\overline{\lim\limits_{k\to\infty}}\,\overline{\lim\limits_{n\to\infty}}$ that the sequence (B^n_T, P^n) is tight.

From conditions (A) and (B), the tightness of the sequence (B^n_T, P^n) and from the proof of the relations (i) and (ii) in Theorem 2 we find that for every $\varepsilon>0$

$$P^n(\sup_{0\leq t\leq T}|Z^n_t-\exp(V^n_t)|\geq\varepsilon)\to 0,\ n\to\infty.$$

This relation and (45) give by virtue of the continuity of the mapping $Z \to \ln Z$ on the set $\{Z>0\}$ that

$$V^n \xrightarrow{\mathrm{D}_T(P^n)} N - \frac{1}{2}\langle N\rangle,$$

and the implications (jj) are proved.

To prove (jjj) we represent the process V^n as the sum of three components

$$V_t^n = N_t^n(a) + \check{V}_t^n(a) + F_t^n(a)$$

where $a>0$ and

$$N_t^n(a) = 2M_t^{nc} + 2\Big(\big(\sqrt{Y^n(x)}-1\big)I\big(\,|\sqrt{Y^n(x)}-1\,| \leqq a\big) * (\mu_{Z^n} - \nu_{Z^n,P^n})\Big)_{t\wedge T^n},$$

$$\check{V}_t^n(a) = 2\Big(\big(\sqrt{Y^n(x)}-1\big)I\big(\,|\sqrt{Y^n(x)}-1\,| > a\big) * \mu_{Z^n}\Big)_{t\wedge T^n},$$

$$F_t^n(a) = -2B_{t\wedge T^n}^n - 2\Big(\big(\sqrt{Y^n(x)}-1\big)I\big(\,|\sqrt{Y^n(x)}-1\,| > a\big) * \nu_{Z^n,P^n}\Big)_{t\wedge T^n}.$$

According to [9, Theorem 1.8] it is sufficient to show that for every $a>0$ and $0<t\leqq T$

(i) $$\sup_n E^n\big(\sup_{0<s\leqq t} |\Delta N_s^n(a)|\big) < \infty,$$

(ii) $$\lim_{a\to\infty} \sup_n P^n\big(\sup_{0<s\leqq t} |\check{V}_s^n(a)| > 0\big) = 0,$$

(iii) $$\lim_{b\to\infty} \sup_n P^n\big(\mathrm{Var}(F^n(a))_t \geqq b\big) = 0.$$

The relation (i) is evident. To establish (ii) we note that for $a>1$

$$P^n\big(\sup_{0<s\leqq t} |\check{V}_s^n(a)| > 0\big) \leqq$$

$$\leqq P^n\Big(\Big(\big(\sqrt{Y^n(x)}-1\big)^2 I\big(\,|\sqrt{Y^n(x)}-1\,| > a\big) * \mu_{Z^n}\Big)_{t\wedge T^n} > 0\Big) =$$

$$= P^n\Big(\Big(\big(\sqrt{Y^n(x)}-1\big)^2 I\big(\,|\sqrt{Y^n(x)}-1\,| \geqq a\big) * \mu_{Z^n}\Big)_{t\wedge T^n} > 1/2\Big)$$

and the last probability tends to zero as $n\to\infty$ by (49). So, for every $\varepsilon>0$ there exists $n_0 = n_0(\varepsilon)$ such that for every $n \geqq n_0$ and $a>1$

(52) $$P^n\big(\sup_{0<s\leqq t} |\check{V}_s^n(a)| > 0\big) < \varepsilon.$$

For $n<n_0$ and $a>1$ we have from Lemma 4 and (23)

$$P^n\big(\sup_{0<s\leqq t} |\check{V}_s^n(a)| > 0\big) \leqq$$

(53) $$\leqq P^n\Big(\Big(\big(\sqrt{Y^n(x)}-1\big)^2 I\big(\,|\sqrt{Y^n(x)}-1\,| \geqq a\big) * \mu_{Z^n}\Big)_{t\wedge T^n} \geqq a/2\Big) \leqq$$

$$\leqq (2E^n B_{t\wedge T^n}^n)/a \leqq 8n_0/a.$$

Hence, for every $a > \max(8n_0(\varepsilon)/\varepsilon, 1)$ we obtain from (52) and (53)

$$\sup_n P^n(\sup_{0<s\leq t} |\check{V}_s^n| > 0) < \varepsilon,$$

i.e., condition (ii) holds.

Since the sequence (B_T^n, P^n) is tight and

$$\mathrm{Var}\,(F^n(a))_t \leq 2B^n_{t\wedge T^n} + 4\big(|\sqrt{Y^n(x)}-1\,|\,I(\,|\sqrt{Y^n(x)}-1\,| > a) * \nu_{Z^n, P^n}\big)_{t\wedge T^n} \leq$$

(54) $$\leq (2+4/a)B^n_{T\wedge T^n} \leq (2+4/a)B^n_T,$$

we get for every $0 < t \leq T$

(55) $$\lim_{b\to\infty} \overline{\lim_{n\to\infty}} P^n\big(\mathrm{Var}\,(F^n(a))_t \geq b\big) = 0.$$

Moreover, by Lemma 4 and (54)

$$P^n(\mathrm{Var}(F^n(a))_t \geq b) \leq P^n(B^n_{T\wedge T^n} \geq b/(2+4/a)) \leq 4(2+4/a)n/b$$

and that, together with (55), gives condition (iii). The implication (jjj) is proved.

To prove the implication (jjjj) we calculate the quadratic variation $[V^n, V^n]_t$ of the process V^n. From (18) we have

$$[V^n, V^n]_t = 4\langle M_t^{nc}\rangle + 4\sum_{0<s\leq t\wedge T^n} [\int_{R^1\setminus\{0\}} (\sqrt{Y_s^n(x)}-1)\cdot$$

(56) $$\cdot(\mu_{Z^n}(\{s\}, dx) - \nu_{Z^n, P^n}(\{s\}, dx)) - \Delta B_s^n]^2 =$$

$$4B^n_{t\wedge T^n} + 4\big((\sqrt{Y^n(x)}-1)^2 I(\,|\sqrt{Y^n(x)}-1\,| \leq \varepsilon) * (\mu_{Z^n} - \nu_{Z^n, P^n})\big)_{t\wedge T^n} +$$

$$+4\big((\sqrt{Y^n(x)}-1)^2 I(\,|\sqrt{Y^n(x)}-1\,| \geq \varepsilon) * \mu_{Z^n}\big)_{t\wedge T^n} -$$

$$-4\big((\sqrt{Y^n(x)}-1)^2 I(\,|\sqrt{Y^n(x)}-1\,| \geq \varepsilon) * \nu_{Z^n, P^n}\big)_{t\wedge T^n} -$$

$$-8\sum_{0<s\leq t\wedge T^n} [\int_{R^1\setminus\{0\}} (\sqrt{Y_s^n(x)}-1)\mu_{Z^n}(\{s\}, dx)][\int_{R^1\setminus\{0\}} (\sqrt{Y_s^n(x)}-1)\cdot$$

$$\cdot \nu_{Z^n, P^n}(\{s\}, dx)] + 4\sum_{0<s\leq t\wedge T^n} [\int_{R^1\setminus\{0\}} (\sqrt{Y_s^n(x)}-1)\,\nu_{Z^n, P^n}(\{s\}, dx)]^2 +$$

$$+4\sum_{0<s\leq t\wedge T^n} (\Delta B_s^n)^2 - 8\sum_{0<s\leq t\wedge T^n} \Delta B_s^n \int_{R^1\setminus\{0\}} \sqrt{Y_s^n(x)}-1)(\mu_{Z^n}(\{s\}, dx) -$$

$$-\nu_{Z^n, P^n}(\{s\}, dx))$$

where the summations are taken over the jumps s of the process V^n.

Now we show that all terms on the right-hand side of (56) except $4B^n_{t\wedge T^n}$, tend in P^n-probability to zero as $n\to\infty$. Indeed, the second term on the right-hand side of (56) tends in P^n-probability to zero as $\varepsilon\to 0$, $n\to\infty$ like in (26) by tightness of the sequence (B^n_T, P^n), the third term tends to zero as $n\to\infty$ by virtue of (49) and the fourth term tends to zero as $n\to\infty$ by condition (B) proved earlier. The fifth, sixth, seventh and eighth terms on the right-hand side of (56) tend to zero as $n\to\infty$ by virtue of the Cauchy-Schwarz inequality, the tightness of the sequence (B^n_T, P^n) and by the relations (36) and

$$\text{(56a)}\qquad \sup_{0\leqq s\leqq T} \Delta B^n_s \xrightarrow{P^n} 0$$

which will be proved later.

First suppose for the moment that (36) and (56a) are proved, then from (56) we get

$$[V^n, V^n]_t - 4B^n_{t\wedge T^n} \xrightarrow{P^n} 0.$$

Furthermore, since $[N, N] = \langle N\rangle$, for every $0\leqq t\leqq T$ we have

$$P^n(|4B^n_t - \langle N\rangle_t| \geqq \varepsilon) \leqq P^n(|[V^n, V^n]_t - 4B^n_{t\wedge T^n}| \geqq \varepsilon/2) +$$
$$+ P^n(|[V^n, V^n]_t - \langle N\rangle_t| \geqq \varepsilon/2) + P^n(T^n \leqq T),$$

that gives (jjjj) by virtue of the implication (jjj) and the relation (15) following from $(P^n_T) \lhd (\tilde{P}^n_T)$.

To prove the relations (36) and (56a) we set

$$\Delta[V^n, V^n] = \{(\omega, t): [V^n, V^n]_t \neq [V^n, V^n]_{t-}\}$$

and note that according to [2, Theorem 6.46 and Theorem 5.16],

$$\triangle[V^n, V^n] \subseteq \bigcup_{k=1}^{\infty} [\sigma^n_k] \cup [S^n_k]$$

where $(\sigma^n_k)_{k\geqq 1}$ and $(S^n_k)_{k\geqq 1}$ are the sequences of totally inaccessible and predictable stopping times, respectively, with disjoint graphs. Then from (35), (38) and (38a) we obtain

$$U^n_T = \sum_{k=1}^{\infty} I(S^n_k \leqq T\wedge T^n)\Big[\int_{R^1\setminus\{0\}} (\sqrt{Y^n_{S^n_k}(x)} - 1)\nu_{Z^n, P^n}(\{S^n_k\}, dx)\Big]^2 \leqq$$
$$\leqq \sum_{k=1}^{\infty} I(S^n_k \leqq T\wedge T^n)\Big[E^n(\sqrt{\alpha^n_{S^n_k}} - 1 \,|\, F^n_{S^n_k-})\Big]^2 \leqq$$
$$\text{(57)}\qquad \leqq \sup_{1\leqq k<\infty} \{I(S^n_k \leqq T\wedge T^n)|E^n(\sqrt{\alpha^n_{S^n_k}} - 1 \,|\, F^n_{S^n_k-})|\}\cdot\{B^n_T + \Psi^n\}$$

where Ψ^n is defined by (39) and tends in P^n-probability to zero by $(\tilde{P}^n_T) \lhd\rhd (P^n_T)$.

Let us show that

$$\text{(58)}\qquad \sup_{1\leqq k<\infty} \{I(S^n_k \leqq T\wedge T^n)|E^n(\sqrt{\alpha^n_{S^n_k}} - 1 \,|\, F^n_{S^n_k-})|\} \xrightarrow{P^n} 0.$$

By the implication (jjj) and [18, Lemma 1] we get

(59) $$\sup_{0<t\leqq T} |[V^n, V^n]_t - \langle N\rangle_t| \xrightarrow{P^n} 0$$

and hence

(60) $$\sup_{1\leqq k<\infty} I(S_k^n \leqq T\wedge T^n)|\sqrt{\alpha_{S^n_k}^n} - 1 - E(\sqrt{\alpha_{S^n_k}^n} - 1 \mid F_{S^n_k -}^n) - \Delta B_{S^n_k}^n| \xrightarrow{P^n} 0.$$

By (38) and (38a) again we obtain

$$\sup_{1\leqq k<\infty} I(S_k^n \leqq T\wedge T^n)\Delta B_{S^n_k}^n \leqq 2 \sup_{0<s\leqq T\wedge T^n} |\sqrt{\alpha_s^n} - 1| + \Psi^n.$$

Now, because of (22), (49), (39) and (15) from (60) it follows (56a) and, hence, (58). From (57), (58), (39) and the tightness of the sequence (B_T^n, P^n) we have (36) and that proves the implication (jjjj). Theorem 3 is proved. □

3. Weak convergence of finite dimensional distributions of the likelihood ratio processes of general statistical models

We consider a sequence of general statistical parametric models $\mathcal{E}^n = \{(\Omega^n, F^n, \mathbf{F}^n), P_\vartheta^n, \vartheta\in\Theta\}$, $\Theta\subseteq R^m$, $n\geqq 1$, where (Ω^n, F^n) are measurable spaces, P_ϑ^n are probability measures on (Ω^n, F^n) which depend on m-dimensional parameter ϑ belonging to some open convex subset Θ of R^m, $m\geqq 1$, and $\mathbf{F}^n = (F_t^n)_{t\geqq 0}$ are families of nondecreasing right-continuous σ-algebras $(F_t^n)_{t\geqq 0}$ such that $\bigvee_{t>0} F_t^n = F^n$, $F_0^n = \{\emptyset, \Omega^n\}$. We suppose $\mathbf{F}^n$ to be P_ϑ^n-complete for $\vartheta\in\Theta$.

For an increasing sequence of positive numbers $(c_n)_{n\geqq 1}$ and $\vartheta\in\Theta$ we set $U_\vartheta^n = \{u\in R^m: \vartheta + c_n^{-1}u\in\Theta\}$ and $Q_u^n = (P_{\vartheta+c_n^{-1}u}^n + P_\vartheta^n)/2$ for $u\in U_\vartheta^n$.

We define the likelihood ratio process $Z^n(u, 0) = (Z_t^n(u, 0), F_t^n)_{t\geqq 0}$ as the process with paths in the space $\mathbf{D}$ of right-continuous functions with left-hand limits and in a way that for every $\mathbf{F}^n$-stopping time τ P_ϑ^n – a.s.

$$Z_\tau^n(u, 0) = \left(\frac{dP_{\vartheta+c_u^{-1}u,\tau}^n}{dQ_{u,\tau}^n}\right)\Big\backslash\left(\frac{dP_{\vartheta,\tau}^n}{dQ_{u,\tau}^n}\right)$$

where $P_{\vartheta,\tau}^n$, $P_{\vartheta+c_n^{-1}u,\tau}^n$, $Q_{u,\tau}^n$ are the restrictions of the measures P_ϑ^n, $P_{\vartheta+c_n^{-1}u}^n$, Q_u^n to the σ-algebra F_τ^n.

In our theorem an essential role will be played by the Hellinger process $H^n(u, v) = (H_t^n(u, v), F_t^n)_{t\geqq 0}$ corresponding to $P_{\vartheta+c_n^{-1}u}^n$, $P_{\vartheta+c_n^{-1}u,\tau}^n$ and $\mathbf{F}^n$ and also by the process

$$y^n(x, u) = (y_t^n(x, u), F_t^n)_{t>0} \text{ with } y_t^n(x, u) = \left(1 + x(Z_{t-}^n(u, 0))^{\otimes}\right)\vee 0$$

where the pseudoinversion $\otimes$ is defined by (5), $x\in R^1$.

As usual we shall denote by ν_{Z^n, P^n_ϑ} the compensator of the jump measure of $Z^n(u, 0)$ with respect to $(\mathbf{F}^n, P^n_\vartheta)$ and by $I(\cdot)$ an indicator function.

Theorem 4. *Let for some $T>0$ a sequence $(c^n)_{n\geq 1}$ exist such that $0<c_n\leq \leq c_{n+1}$, $c_n \uparrow \infty$ as $n\to\infty$, $(P^n_{\vartheta+c_n^{-1}u,\,T}) \triangleleft \triangleright (P^n_{\vartheta,\,T})$ for every $u\in U^n_\vartheta$, and*

(I) *for every $\varepsilon>0$, $u\in U^n_\vartheta$ and $0<t\leq T$*

$$\left(\left(\sqrt{y^n(x, u)}-1\right)^2 I\left(\left|\sqrt{y^n(x, u)}-1\right|\geq\varepsilon\right) * \nu_{Z^n, P^n_\vartheta}\right)_t \xrightarrow{P^n_\vartheta} 0,$$

(II) *for every $u, v\in U^n_\vartheta$ and $0<t\leq T$*

$$H^n_t(u, 0)+H^n_t(v, 0)-H^n_t(u, v) \xrightarrow{P^n_\vartheta} \frac{1}{2}\langle N(u), N(v)\rangle_t.$$

Then, for every set $(u_1, \dots, u_k)$ with $u_i\in U^n_\vartheta$, $1\leq i\leq k$ and $k\geq 1$, we have

$$(Z^n_T(u_1, 0), \dots, Z^n_T(u_k, 0))^\top \xrightarrow{d(P^n_\vartheta)} (Z_T(u_1), \dots, Z_T(u_k))^\top$$

where $Z_T(u)$ is defined by (2) and $\xrightarrow{d(P^n_\vartheta)}$ denotes convergence in distribution.

Proof of Theorem 4. *We put* $Q^n_u = (P^n_{\vartheta+c_n^{-1}u}+P^n_\vartheta)/2$, $\varkappa^n = (Q^n_u+Q^n_v)/2$. Let us denote by $\zeta^n(u, 0) = (\zeta^n_t(u, 0), F^n_t)_{t\geq 0}$, $\bar\zeta^n(u) = (\bar\zeta^n_t(u), F^n_t)_{t\geq 0}$ and $\bar Z^n(u, v) = (\bar Z^n_t(u, v), F^n_t)_{t\geq 0}$ the processes with paths in $\mathbf{D}$ and such that, for every $\mathbf{F}^n$-stopping time τ

$$\text{(61)}\qquad \zeta^n_\tau(u, 0) = \frac{dP^n_{\vartheta,\tau}}{dQ^n_{u,\tau}},\ \bar\zeta^n_\tau(u) = \frac{dP^n_{\vartheta+c_n^{-1}u,\,\tau}}{d\varkappa^n_\tau},\ \bar Z^n_\tau(u, v) = \frac{\bar\zeta^n_\tau(u)}{\bar\zeta^n_\tau(v)}$$

where we put $0/0 = 0$, $a/0 = \infty$ for every $a>0$.

According to the notations of **2** for $P^n = P^n_\vartheta$ and $\tilde P = P^n_{\vartheta+c_n^{-1}u}$, $u\in U^n_\vartheta$, let us define the stopping times $T^n(u)$ and the processes $H^n(u, 0)$, $B^n(u, 0)$, $V^n(u)$, $M^{nc}(u)$ and $\hat M^n(u)$ by the formulas (13), (6), (7), (18), (17) and (30) respectively substituting of $\zeta^n_t(u, 0)$ for ζ^n_t, $(2-\zeta^n_t(u, 0))$ for $\tilde\zeta^n_t$, $Z^n_t(u, 0)$ for Z^n_t and $y^n_s(x, u)$ for $Y^n_s(X, U)$.

From conditions (I) and (II) with $u = u_i$, $v = u_i$ and the relations (i) and (ii) of Theorem 2 we get for $1\leq i\leq k$ and every $\varepsilon>0$

$$P^n_\vartheta\left(\sup_{0\leq t\leq T} |Z^n_t(u_i, 0)-\exp(V^n_t(u_i))|\geq\varepsilon\right)\to 0,\ n\to\infty.$$

Hence, by the continuity of the mapping $Z\to e^z$ it is sufficient to show that

$$(V^n_T(u_1), \dots, V^n_T(u_k))^\top \xrightarrow{d(P^n_\vartheta)}$$

$$\left(N_T(u_1)-\frac{1}{2}\langle N(u_1)\rangle_T,\ \dots,\ N_T(u_k)-\frac{1}{2}\langle N(u_k)\rangle_T\right)^\top.$$

For this purpose we use the Cramér-Wold method [1, Theorem 7.7] and show that for every $\lambda_1, \ldots, \lambda_k \in R^1$

$$(62) \qquad \sum_{i=1}^{k} \lambda_i V^n(u_i) \xrightarrow{\mathrm{D}_T(P^n_\vartheta)} \sum_{i=1}^{k} \lambda_i N(u_i) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i \langle N(u_i) \rangle.$$

From (29) we have for $0 \leqq t \leqq T$ that

$$\sum_{i=1}^{k} \lambda_i V^n_t(u_i) = 2 \sum_{i=1}^{k} \lambda_i \hat{M}^n_t(u_i) - 2 \sum_{i=1}^{k} \lambda_i B^n_{t \wedge T^n(u_i)}(u_i, 0).$$

From the contiguity $(P^n_\vartheta, T) \lhd (P^n_{\vartheta + c_n^{-1} u_i}, T)$ we obtain in the same way as (15) that $P^n_\vartheta(T^n(u_i) \leqq T) \to 0$, $n \to \infty$.

Then condition (II) with $u = u_i$, $v = u_i$. Remark 1 of Theorem 2 and [18, Lemma 1] imply that

$$P^n_\vartheta \left(\sup_{0 \leqq t \leqq T} \left| B^n_{t \wedge T^n(u_i)}(u_i, 0) - \frac{1}{4} \langle N(u_i) \rangle_t \right| \geqq \varepsilon \right) \to 0, \ n \to \infty.$$

This relation and [15, Corollary 2] yield that for (62) we only need to prove the Lindeberg condition

$$(63) \qquad (x^2 I(|x| \geqq \varepsilon) * \nu_{\Sigma, P^n_\vartheta}) \xrightarrow{P^n_\vartheta} 0, \ \forall \varepsilon > 0, \ 0 \leqq t \leqq T,$$

where $\nu_{\Sigma, P^n_\vartheta}$ is the compensator of the jump measure of $\sum_{i=1}^{k} \lambda_i \hat{M}^n(u_i)$ with respect to $(\mathbf{F}^n, P^n_\vartheta)$, and also the convergence of the quadratic characteristic of the square-integrable P^n_ϑ-martingale $2 \sum_{i=1}^{k} \lambda_i \hat{M}^n(u_i)$ to the quadratic characteristic of the Gaussian martingale $\sum_{i=1}^{k} \lambda_i N(u_i)$, i.e., for every $0 \leqq t \leqq T$

$$(64) \qquad \langle 2 \sum_{i=1}^{k} \lambda_i \hat{M}^n(u_i) \rangle \xrightarrow{P^n_\vartheta} \langle \sum_{i=1}^{k} \lambda_i N(u_i) \rangle_t.$$

The proof of the Lindeberg condition is similar to that of (32) in Theorem 2. As in Theorem 2 we define $M^{n,1}_s(u_i)$ and $M^{n,2}_s(u_i)$ with $\delta < \varepsilon / (2k \max_i |\lambda_i|)$.

Then for $(\omega, s) = [0, T^n(u_i)]$

$$\Delta \hat{M}^n_s(u_i) = \Delta M^{n,1}_s(u_i) + \Delta M^{n,2}_s(u_i)$$

and we have

$$(x^2 I(|x| \geqq \varepsilon) * \mu_\Sigma)_t = \sum_{0 < s \leqq t} \left(\sum_{i=1}^{k} \lambda_i \Delta \hat{M}^n_s(u_i) \right)^2 I\left(\left| \sum_{i=1}^{k} \lambda_i \Delta \hat{M}^n_s(u_i) \right| \geqq \varepsilon \right) \leqq$$

(65) $$\leqq C' \sum_{i=1}^{k} \sum_{0<s\leqq t\wedge T^n(u_1)} (\Delta M_s^{n,2}(u_i))^2$$

where μ_Σ is the jump measure of $\sum_{i=1}^{k} \lambda_i \hat{M}_s^n(u_i)$ and

$$C' = 2^k \max_i \lambda_i^2 + (8k^4\delta^2 \max_i \lambda_i^4)/(\varepsilon - 2k\delta \max_i |\lambda_i|^2).$$

By [4, Lemma 2, §2, Ch. 2], the same inequality holds for the compensators of both sides of (65). As the compensator of the right-hand side of (65) is not greater than

$$4C' \sum_{i=1}^{k} ((\sqrt{y_s^n(x,\ u_i)} - 1)^2 I(|\sqrt{y_s^n(x,\ u_i)} - 1| \geqq \delta) * \nu_{Z^n,\,P^n\vartheta})_{t\wedge T^n(u_i)}$$

which in turn tends to zero in P_ϑ^n-probability as $n\to\infty$ by condition (I).

To prove (64) we note that

(66) $$\langle 2 \sum_{i=1}^{k} \lambda_i \hat{M}^n(u_i)\rangle_t = 4 \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i\lambda_j \langle \hat{M}^n(u_i), \hat{M}^n(u_j)\rangle_t.$$

According to condition (II) and the relation

$$2\langle \hat{M}^n(u),\ \hat{M}^n(u_j)\rangle_t - H_t^n(u_i, 0) - H_t^n(u_j, 0) + H_t^n(u_i, u_j) \xrightarrow{P^n\vartheta} 0$$

which will be proved later, (64) follows from (66).

It remains only to prove that for all fixed $u, v \in U_\vartheta^n$ and $0 < t \leqq T$

(67) $$2\langle \hat{M}^n(u),\ \hat{M}^n(v)\rangle_t - H_t^n(u,0) - H_t^n(v,0) + H_t^n(u,v) \xrightarrow{P^n\vartheta} 0.$$

To this end, for $k = 1, 2, \ldots$ and $u \in U_\vartheta^n$ we introduce other stopping times $T_k^n(u) = \inf\{t\geqq 0 : \zeta_t^n(u, 0) \leqq 1/k\}$ and $\bar{T}_k^n(u) = \inf\{t \geqq 0: \bar{\zeta}_t^n(u) \leqq 1/k\}$, with $\inf\{\emptyset\} = \infty$.

We set

$$\sigma_k^n = T_k^n(u) \wedge T_k^n(v) \wedge \bar{T}_k^n(u) \wedge \bar{T}_k^n(v) \wedge \bar{T}_k^n(0).$$

By the contiguity $(P_{\vartheta,T}^n) \triangleleft (P_{\vartheta+c_n^{-1}u,\,T}^n)$ and $(P_{\vartheta,T}^n) \triangleleft (P_{\vartheta+c_n^{-1}v,\,T}^n)$ we have

(67a) $$\lim_{k\to\infty} \overline{\lim_{n\to\infty}}\, P_\vartheta^n(\sigma_k^n \leqq T) = 0.$$

Hence, it is sufficient to prove (67) with $t\wedge\sigma_k^n$ in place of t. Then, by (44a), in (67) we can substitute $B_{t\wedge\sigma^n_k}^n(u,\ 0)$ and $B_{t\wedge\sigma^n_k}^n(v,\ 0)$ for $H_{t\wedge\sigma^n_k}^n(u,\ 0)$, $H_{t\wedge\sigma^n_k}^n(v,\ 0)$, respectively.

We also consider the Hellinger process $\overline{H}^n(u, v) = (\overline{H}^n_t(u, v), F^n_t)_{t\geq 0}$ corresponding to $P^n_{\vartheta+c_n^{-1}u}$, $P^n_{\vartheta+c_n^{-1}v}$, and $\mathbf{F}^n$ and built with the help of the measure $\varkappa^n$. Then, by [10, Proposition 3.6] we have for every $0<t\leq T$

$$\overline{H}^n_{t\wedge\sigma^n_k}(u, v) = \overline{G}^n_{t\wedge\sigma^n_k}(u, v) + \overline{D}^n_{t\wedge\sigma^n_k}(u, v)$$

where

$$\overline{G}^n_\cdot(u, v) = \frac{1}{4}[(\overline{\zeta}^n_-(u))^{-2}\circ\langle\overline{\zeta}^{nc}(u)\rangle - 2(\overline{\zeta}^n_-(u)\overline{\zeta}^n_-(v))^{-1}\circ$$

$$\circ\langle\overline{\zeta}^{nc}(u), \overline{\zeta}^{nc}(v)\rangle + (\overline{\zeta}^n_-(v))^{-2}\circ\langle\overline{\zeta}^{nc}(v)\rangle]_\cdot, \tag{68}$$

$$\overline{D}^n_\cdot(u, v) = \text{comp}_{\varkappa^n}\Big(\sum_{0<s\leq\cdot}(\sqrt{\overline{\zeta}^n_s(u)/\overline{\zeta}^n_{s-}(u)} - \sqrt{\overline{\zeta}^n_s(v)/\overline{\zeta}^n_{s-}(v)})^2\Big), \tag{69}$$

and $\overline{\zeta}^n(u), \overline{\zeta}^n(v)$ are defined by (61), $\text{comp}_{\varkappa^n}$ denotes the operation of taking the compensator with respect to $(\mathbf{F}^n, \varkappa^n)$.

By virtue of [10, Theorem 2.13] $\overline{H}^n(u, v) = H^n(u, v)$, $(P^n_{\vartheta+c_n^{-1}u} + P^n_{\vartheta+c_n^{-1}v})/2$ – a.s., and, hence, by the contiguity $(P^n_{\vartheta, T}) \lhd (P^n_{\vartheta+c_n^{-1}u, T})$ we have

$$\overline{H}^n_{t\wedge\sigma^n_k}(u, v) - H^n_{t\wedge\sigma^n_k}(u, v) \xrightarrow{P^n_\vartheta} 0. \tag{70}$$

So, in (67) we can substitute $\overline{H}^n_{t\wedge\sigma^n_k}(u, v)$ for $H^n_{t\wedge\sigma^n_k}(u, v)$.

Furthermore, it is clear that

$$\langle\hat{M}^n(u), \hat{M}^n(v)\rangle = \langle M^{nc}(u), M^{nc}(v)\rangle + \langle M^{nd}(u), M^{nd}(v)\rangle$$

where $M^{nd}(u) = \hat{M}^n(u) - M^{nc}(u)$, and from (7) $B^n(u, 0) = G^n(u, 0) + D^n(u, 0)$ where $G^n(u, 0)$ is defined by (68) substituting $\zeta^n(u, 0)$ and $(2-\zeta^n(u, 0))$ for $\overline{\zeta}^n(u)$ and $\overline{\zeta}^n(v)$, respectively, and

$$D^n_\cdot(u, 0) = \text{comp}_{P^n_\vartheta}\Big(\sum_{0<s\leq\cdot}(1-\sqrt{\alpha^n_s(u, 0)})^2\Big), \tag{71}$$

with $\alpha^n_s(u, 0) = Z^n_s(u, 0)/Z^n_{s-}(u, 0)$.

The facts mentioned above yield that (67) is equivalent to

$$(2\langle M^{nc}(u), M^{nc}(v)\rangle - G^n(u, 0) - G^n(v, 0) + \overline{G}^n(u, v))_{t\wedge\sigma^n_k} \xrightarrow{P^n_\vartheta} 0, \tag{72}$$

$$(2\langle M^{nd}(u), M^{nd}(v)\rangle - D^n(u, 0) - D^n(v, 0) + \overline{D}^n(u, v))_{t\wedge\sigma^n_k} \xrightarrow{P^n_\vartheta} 0. \tag{73}$$

Since

$$2\langle M^{nc}(u), M^{nc}(v)\rangle = \langle M^{nc}(u)\rangle + \langle M^{nc}(v)\rangle - \langle M^{nc}(u) - M^{nc}(v)\rangle =$$
$$= G^n(u, 0) + G^n(v, 0) - \langle M^{nc}(u) - M^{nc}(v)\rangle,$$

to prove (72) we need to show that

$$(\langle M^{nc}(u) - M^{nc}(v - \overline{G)}\rangle^n(u, v))_{t\wedge\sigma^n_k} \xrightarrow{P^n_\vartheta} 0. \tag{74}$$

By Lemma 1 $P^n_\vartheta(\inf_{0\leq t\leq T} \bar{\zeta}^n_t(0)=0)=0$ and from (61) we have for $0\leqq t\leqq T$ and $k\geqq 1$ (P^n_ϑ – a.s.)

$$(75)\qquad \bar{Z}^n_{t\wedge\sigma^n_k}(u,v)=\bar{\zeta}^n_{t\wedge\sigma^n_k}(u)/\bar{\zeta}^n_{t\wedge\sigma^n_k}(v)= \\ =\bar{Z}^n_{t\wedge\sigma^n_k}(u,0)/\bar{Z}^n_{t\wedge\sigma^n_k}(v,0)=Z^n_{t\wedge\sigma^n_k}(u,0)/Z^n_{t\wedge\sigma^n_k}(v,0)$$

where we put $0/0=0$, $a/0=\infty$ for every $a>0$. Since the processes $\bar{Z}^n_{\cdot\wedge\sigma^n_k}(u,v)$ and $Z^n_{\cdot\wedge\sigma^n_k}(u,0)/Z^n_{\cdot\wedge\sigma^n_k}(v,0)$ have their paths in **D**, they are indistinguishable P^n_ϑ – a.s..

For every $L>0$ we have the analog of (16) ($\varkappa^n$ – a.s.)

$$(76)\qquad \bar{\zeta}^n_{t\wedge\sigma^n_k}(u)=\bar{\zeta}^n_0(u)\exp\Big\{\sum_{0<s\leqq t\wedge\sigma^n_k}\big[\ln(1+\Delta\bar{m}^{n,k}_s(u))- \\ -U_L\big(\ln(1+\Delta\bar{m}^{n,k}_s(u))\big)\big]+m^{n,k}_t(u)-\frac{1}{2}\langle m^{n,k}(u)\rangle_t+ \\ +\big(U_L(\ln(1+x/\bar{\zeta}^n_-(u)))*(\mu_{\bar{\zeta}^n(u)}-\nu_{\bar{\zeta}^n(u),\varkappa^n})\big)_{t\wedge\sigma^n_k}- \\ -\big(G_L(1+x/\bar{\zeta}^n_-(u))*\nu_{\bar{\zeta}^n(u),\varkappa^n}\big)_{t\wedge\sigma^n_k}\Big\}$$

where $\bar{m}^{n,k}(u)=(\bar{m}^{n,k}_t(u),F^n_t)_{t\geqq 0}$, $m^{n,k}(u)=(\bar{m}^{n,k}(u))^c$,

$$\bar{m}^{n,k}_t(u)=\int_0^{t\wedge\sigma^n_k} d\zeta^n_s(u)/\zeta^n_{s-}(u).$$

Moreover, from (75) we get for $(\omega,s)\in[0,\sigma^n_k]$ (P^n_ϑ – a.s.) that

$$(77)\qquad \frac{1+\Delta\bar{m}^{n,k}(u)}{1+\Delta\bar{m}^{n,k}(v)}=\frac{\bar{\zeta}^n_s(u)/\bar{\zeta}^n_{s-}(u)}{\bar{\zeta}^n_s(v)/\bar{\zeta}^n_{s-}(v)}=\frac{\bar{Z}^n_s(u,v)}{\bar{Z}^n_{s-}(u,v)}=\frac{\alpha^n_s(n,0)}{\alpha^n_s(v,0)}$$

where we again put $0/0=0$, $a/0=\infty$ for every $a>0$. Then from the representations of the type (16) for $Z^n(u,0)$, $Z^n(v,0)$ and from the representations of the type (76) for $\bar{\zeta}^n(u)$, $\bar{\zeta}^n(v)$ and using the equalities (75) and (77) and also [11, Theorem 10] we obtain

$$1=P^n_\vartheta(\bar{Z}^n_{t\wedge\sigma^n_k}(u,v)=Z^n_{t\wedge\sigma^n_k}(u,0)/Z^n_{t\wedge\sigma^n_k}(v,0),\ 0\leqq t\leqq T)\leqq \\ \leqq P^n_\vartheta(\sigma^n_k\leqq T)+P^n_\vartheta\big(m^{n,k}_t(u)-m^{n,k}_t(v)- \\ -\big((\bar{\zeta}^n_-(0))^{-1}\circ\langle m^{n,k}(u)-m^{n,k}(v),\bar{\zeta}^{nc}(0)\rangle\big)_{t\wedge\sigma^n_k}= \\ =2M^{nc}_{t\wedge\sigma^n_k}(u)-2M^{nc}_{t\wedge\sigma^n_k}(v),\ 0\leqq t\leqq T\big)\leqq \\ \leqq 2P^n_\vartheta(\sigma^n_k\leqq T)+P^n_\vartheta(\bar{G}^n_t(u,v)=\langle M^{nc}(u)-M^{nc}(v)\rangle_t,\ 0\leqq t\leqq T).$$

From this and (67a) we get (74) and, hence, (72).

To prove (73) we write, using (71):

$$2\langle M^{nd}(u), M^{nd}(v)\rangle_{t\wedge\sigma^n_k} = 2\ \mathrm{comp}_{P^n_\vartheta}\big([M^{nd}(u), M^{nd}(v)]\big)_{t\wedge\sigma^n_k} =$$

$$= 2\ \mathrm{comp}_{P^n_\vartheta}\Big(\big[(\sqrt{y^n_s(x,u)}-1)*(\mu_{Z^n}-\nu_{Z^n, P^n_\vartheta}),$$

(77a)
$$(\sqrt{y^n_s(x,v)}-1)*(\mu_{Z^n}-\nu_{Z^n, P^n_\vartheta})\big]\Big)_{t\wedge\sigma^n_k} =$$

$$= D^n_{\wedge\sigma^n_k}(u,0)+D^n_{t\wedge\sigma^n_k}(v,0) - \mathrm{comp}_{P^n_\vartheta}\Big(\sum_{0<s\le t\wedge\sigma^n_k}(\sqrt{\alpha^n_s(u)}-$$

$$-\sqrt{\alpha^n_s(v,0)})^2 - 2\sum_{0<s\le t\wedge\sigma^n_k}\Big(\int_{R^1\setminus\{0\}}(\sqrt{y^n_s(x,u)}-1)\nu_{Z^n, P^n_\vartheta}(\{s\},dx)\Big)\cdot$$

$$\cdot\Big(\int_{R^1\setminus\{0\}}(\sqrt{Y^n_s(x,v)}-1)\nu_{Z^n, P^n_\vartheta}(\{s\},dx)\Big)\cdot$$

Note that the sum of products of the compensator jumps in (77a) tends to zero as $n\to\infty$ in P^n_ϑ-probability. This fact follows from the Cauchy-Schwarz inequality and (36). If we set

$$A^n_{t\wedge\sigma^n_k}(u,v) = \sum_{0<s\le t\wedge\sigma^n_k}(\sqrt{\alpha^n_s(u,0)}-\sqrt{\alpha^n_s(v,0)})^2,$$

(78)
$$\hat{D}^n_{t\wedge\sigma^n_k}(u,v) = \mathrm{comp}_{P^n_\vartheta}(A^n(u,v))_{t\wedge\sigma^n_k},$$

then it remains only to show that

(79)
$$\hat{D}^n_{t\wedge\sigma^n_k}(u,v) - \overline{D}^n_{t\wedge\sigma^n_k}(u,v) \xrightarrow{P^n_\vartheta} 0.$$

For this purpose we set

$$\overline{A}^n_{t\wedge\sigma^n_k}(u,v) = \sum_{0<s\le t\wedge\sigma^n_k}\Big(\sqrt{\overline{\zeta}^n_s(u)/\overline{\zeta}^n_{s-}(u)}-\sqrt{\overline{\zeta}^n_s(v)/\overline{\zeta}^n_{s-}(v)}\Big)^2.$$

According to Lemma 1, $P^n_\vartheta(\inf_{0<s\le t}\overline{\zeta}^n_s(0)=0)=0$, and by (61), $\overline{Z}^n_s(u,0) = Z^n_s(u,0)(P^n_\vartheta-\text{a.s.})$ for $0<s\le T$. Hence, for $(\omega,s)\in[0,\sigma^n_k]$ we have $(P^n_\vartheta-\text{a.s.})$

$$\triangle A^n_s(u,v) = \triangle\overline{A}^n_s(u,v)\overline{\zeta}^n_{s-}(0)/\overline{\zeta}^n_s(0).$$

Owing to this relation and (69) we have for every bounded predictable function f that

$$E^n_\vartheta(f\circ\hat{D}^n(u,v))_{t\wedge\sigma^n_k} = E^n_\vartheta(f\circ A^n(u,v))_{t\wedge\sigma^n_k} =$$

$$= E^n_\vartheta((fI(\overline{\zeta}^n(0)>0)\overline{\zeta}^n_-(0)/\overline{\zeta}(0))\circ\overline{A}^n(u,v))_{t\wedge\sigma^n_k} =$$

$$= E^n_\varkappa(fI(\overline{\zeta}^n(0)>0)\overline{\zeta}^n_-(0)\circ\overline{A}^n(u,v))_{t\wedge\sigma^n_k} =$$

$$= E^n_\varkappa\overline{\zeta}^n_\infty(0)(f\circ\overline{D}^n(u,v))_{t\wedge\sigma^n_k} - E^n_\varkappa(fI(\overline{\zeta}^n(0)=0)\overline{\xi}^n_-(0)\circ\overline{A}^n(u,v))_{t\wedge\sigma^n_k} =$$

$$= E^n_\vartheta(f\circ\overline{D}^n(u,v))_{t\wedge\sigma^n_k} - E^n_\varkappa(fI(\overline{\zeta}^n(0)=0)\overline{\zeta}^n_-(0)\circ\overline{A}^n(u,v))_{t\wedge\sigma^n_k}$$

where E^n_ϑ and $E^n_\varkappa$ are the expectations with respect to P^n_ϑ and $\varkappa^n$. Hence, for every bounded predictable function f we get

$$E^n_\vartheta(f \circ (\overline{D}(u, v) - \hat{D}(u, v)))_{t \wedge \sigma^n_k} =$$

(80) $$= E^n_\varkappa(f I(\bar{\zeta}^n(0) = 0)\zeta^n_-(0) \circ \overline{A}(u, v))_{t \wedge \sigma^n_k}.$$

It is easy to see that we can find a predictable function f^* such that $|f^*| \leqq 2$ and

$$E^n_\vartheta \,|\, \overline{D}^n_{t \wedge \sigma^n_k}(u, v) - \hat{D}^n_{t \wedge \sigma^n_k}(u, v) \,| \leq E^n_\vartheta(f^* \circ (\overline{D}^n(u, v) - \hat{D}^n(u, v)))_{t \wedge \sigma^n_k}.$$

Then (80) yields

(81) $$E^n_\vartheta \,|\, \overline{D}^n_{t \wedge \sigma^n_k}(u, v) - \hat{D}^n_{t \wedge \sigma^n_k}(u, v) \,| \leqq E^n_\varkappa \,|\, f^* I(\bar{\zeta}^n(0) = 0) \cdot$$
$$\cdot \triangle \bar{\zeta}(0) \triangle \overline{A}^n(u, v) \,|_{t \wedge \sigma^n_k} \leqq 64 k \varkappa^n(\bar{\zeta}^n_{t \wedge \sigma^n_k}(0) = 0),$$

since $\triangle \bar{\zeta}^n_s(0) \leqq 4$, $\triangle \overline{A}^n_s(u, v) \leqq 8k$ on $(\omega, s) \in [0, \sigma^n_k]$.

Because of $\varkappa^n = (Q^n_u + Q^n_v)/2 = (2P^n_\vartheta + P^n_{\vartheta + c_n^{-1}u} + P^n_{\vartheta + c_n^{-1}v})/4$ and $P^n_\vartheta(\inf_{0 \leqq s \leqq T} \bar{\zeta}^n_s(0) = 0) = 0$, we get from (81) and the contiguity $(P^n_{\vartheta + c_n^{-1}u, T}) \lhd (P^n_{\vartheta, T})$ and $(P^n_{\vartheta + c_n^{-1}v, T}) \lhd (P^n_{\vartheta, T})$ the relation (79), and, hence, (73).

Theorem 4 is proved. □

4. A functional limit theorem for the likelihood ratio processes of general statistical models

Throughout this paragraph we use the notation of **3.** We also use the notation $\varphi(y) \in \Phi_1$ for the functions $\varphi(y)$, $y \geqq 0$, which monotonically decrease and are integrable on $[L, \infty)$ for sufficiently large L.

Theorem 5. *Let for some* $T > 0$ *there exist the sequence* $(c_n)_{n \geqq 1}$, $0 < c_n \leqq \leqq c_{n+1}$, $c_n \uparrow \infty$ *as* $n \to \infty$, *and the constants* $\alpha > m$, $d > 0$, $D > 0$, $c \geqq 2$, $C > 0$ *and the function* $\varphi(|u|)$, $|u|^{m-1}[\varphi^{(\alpha - m)/\alpha} \in \Phi_1$ *be such that* $0 < c_n \leqq c_{n+1}$, $c_n \uparrow \infty$ *as* $n \to \infty$, *and for every* $u \in U^n_\vartheta$ *the measure sequences* $(P^n_{\vartheta + c_n^{-1}u, T})$ *and* $(P^n_{\vartheta, T})$ *are mutually contiguous and*

(I) *for every* $\varepsilon > 0$, $u \in U^n_\vartheta$ *and* $0 < t \leq T$

$$((\sqrt{y^n_s(x, u)} - 1)^2 I(\,|\sqrt{y^n_s(x, u)} - 1\,| \geqq \varepsilon) \nu_{Z^n, P^n_\vartheta}(ds, dx))_t \xrightarrow{P^n_\vartheta} 0,$$

(II) *for every* $u, v \in U^n_\vartheta$ *and* $0 < t \leq T$

$$H^n_t(u, 0) + H^n_t(v, 0) - H^n_t(u, v) \xrightarrow{P^n_\vartheta} \frac{1}{2} \langle N(u), N(v) \rangle_t,$$

(III) *for sufficiently small $h>0$ and $|u|\leqq h$*

$$\sup_n \sup_{\vartheta\in\Theta} P^n_\vartheta(H^n_T(u,0)\geqq d|u|^\alpha)\leqq D|u|^{2\alpha},$$

(IV) *for sufficiently large $L>0$ and $|u|\geqq L$*

$$\sup_n P^n_\vartheta(H^n_T(u,0)\geqq c|\ln\varphi(|u|)|)\geqq C\varphi^2(|u|).$$

Then the distributions of a separable modification $\hat{Z}^n_T(u)$, $u\in R^m$, of the likelihood ratio process $Z^n_T(u, 0)$ (which is continued if necessary to the whole of R^m with preservation of the supremum value and the modulus of continuity) weakly converge in $\mathbf{C}(R^m, P^n_\vartheta)$ to the distributions of $Z_T(u)$, $u\in R^m$, as $n\to\infty$, i.e.

$$\hat{Z}^n_T \xrightarrow{\mathbf{C}(R^m, P^n{}_\vartheta)} Z_T.$$

Remark 4. Let us clarify the assumptions figuring in Theorem 5, Conditions (I) and (II) provide weak convergence of finite dimensional distributions of the likelihood ratio process $Z^n_T(\cdot, 0)$ to those of the process $Z_T(\cdot)$. Condition (I) is nothing else but the "predictable" version of the Lindeberg condition. Condition (II) is the convergence condition for mutual quadratic characteristics of martingales generally figuring in martingale limit theorems. The conditions (III) and (IV) which also arise in criteria of c_n-consistency of estimators, guarantee the tightness of the distributions of $\hat{Z}^n_T$.

Remark 5. If instead of the condition $F^n_0 = \{\emptyset, \Omega^n\}$ we require, that for every $u\in U^n_\vartheta$ $Z^n_0(u, 0)\xrightarrow{P^n{}_\vartheta} 1$ and also for sufficiently small $h>0$ and $|u|\leqq h$

(V) $$P^n_\vartheta(|\sqrt{Z^n_0(u,0)}-1|\geqq d|u|^\alpha)\leqq D|u|^\alpha,$$

then Theorem 5 remains true.

Remark 6. Let $\vartheta' = \vartheta + c_n^{-1}u'$ for some fixed $u'\in U^n_\vartheta$. Then from Theorem 5 and [5, Theorem 2] we can find the limit distribution of $\hat{Z}^n_T$ with respect to the measure $P^n_{\vartheta'}$.

Remark 7. Under local absolute continuity $P^n_{\vartheta+c_n^{-1}u} \overset{\text{loc}}{\ll} P^n_{\vartheta+c_n^{-1}v}$, in place of the Hellinger process $H^n(u, v)$ in the conditions of Theorem 5 we may write the process $B^n(u, v)$ introduced in [16].

Corollary 2. *Suppose that the measures P^n_ϑ, $\vartheta\in\Theta$, correspond to processes with independent increments and they are (for simplicity) equivalent. Then according to* [17] *the Hellinger process $H^n(u, v)$ is deterministic for every $u, v\in U^n_\vartheta$. If Condition* (I) *and Condition* (II) *with $\overrightarrow{n\to\infty}$ in place of $\xrightarrow{P^n{}_\vartheta}$ are satisfied, and supposing that*

III′) *for sufficiently small $h>0$ and $|u|\leqq h$*

$$\sup_n \sup_{\vartheta\in\Theta} H^n_T(u,0)<d|u|^\alpha,$$

IV′) *for sufficiently large* $L>0$ *and* $|u|\geq L$

$$\inf_n H_T^n(u,0) > c|\ln\varphi(|u|)|,$$

where α, d, c *and* $\varphi(\cdot)$ *have the same meaning as in Theorem 5, Theorem 5 remains true.*

Remark 8. Let $P_\vartheta^n = \mu_\vartheta^1\times\mu_\vartheta^2\times\ldots\times\mu_\vartheta^n$ and (for simplicity) the measures μ_ϑ^i and $\mu_{\vartheta+c_n^{-1}u}^i$ be equivalent for every $u\in U_\vartheta^n$ and $1\leq i\leq n$. Then

$$H_t^n(u,0) = \sum_{i=1}^{t} \varrho^2(\mu_\vartheta^i, \mu_{\vartheta+c_n^{-1}u}^i),\ t\geq 1,$$

where $\varrho(\cdot,\cdot)$ is the Hellinger distance, and according to Theorem 5 we have a result, analogous to [6, Theorem 1.1, §1, Ch. II] without the usual but inconvenient assumption about the differentiability of $\sqrt{\hat{Z}_t^n(u)}$ in mean square.

Proof of Theorem 5. Let us consider the (with respect to u) separable modification $\hat{Z}_T^n(u)$, $u\in U_\vartheta^n$, of the likelihood ratio process $Z_T^n(u,0)$, $u\in U_\vartheta^n$. First we show that under conditions (III) and (IV) of Theorem 5 the process $\hat{Z}_T^n(u)$, $u\in U_\vartheta^n$, is continuous P_ϑ^n — a.s. and, hence, it belongs to the space $\mathbf{C}(U_\vartheta^n, P_\vartheta^n)$. For this purpose we establish that

$$P_\vartheta^n\Big(\overline{\lim_{h\to 0}}\sup_{|u-v|\leq h}\Big|\sqrt{\hat{Z}_T^n(u)}-\sqrt{\hat{Z}_T^n(v)}\Big|>0\Big)=0. \tag{82}$$

We have

$$\begin{aligned}
&P_\vartheta^n\Big(\overline{\lim_{h\to 0}}\sup_{|u-v|\leq h}\Big|\sqrt{\hat{Z}_T^n(u)}-\sqrt{\hat{Z}_T^n(v)}\Big|>0\Big)=\\
&=\lim_{k\to\infty}P_\vartheta^n\Big(\overline{\lim_{h\to 0}}\sup_{|u-v|\leq h}\Big|\sqrt{\hat{Z}_T^n(u)}-\sqrt{\hat{Z}_T^n(v)}\Big|\geq\frac{1}{k}\Big)\leq\\
&\leq\lim_{k\to\infty}\overline{\lim_{h\to 0}}P_\vartheta^n\Big(\sup_{|u-v|\leq h}\Big|\sqrt{\hat{Z}_T^n(u)}-\sqrt{\hat{Z}_T^n(v)}\Big|\geq\frac{1}{k}\Big).
\end{aligned} \tag{83}$$

Further, for every $L>0$

$$\begin{aligned}
&P_\vartheta^n\Big(\sup_{|u-v|\leq h}\Big|\sqrt{\hat{Z}_T^n(u)}-\sqrt{\hat{Z}_T^n(v)}\Big|\geq\frac{1}{k}\Big)\leq\\
&\leq P_\vartheta^n\Big(\sup_{|u-v|\leq h,\ |u|\leq L,\ |v|\leq L}\Big|\sqrt{\hat{Z}_T^n(u)}-\sqrt{\hat{Z}_T^n(v)}\Big|\geq\frac{1}{k}\Big)+\\
&+P_\vartheta^n\Big(\sup_{|u|\geq L-h}\hat{Z}_T^n(u)\geq\frac{1}{(2k)^2}\Big)+P_\vartheta^n\Big(\sup_{|v|\geq L-h}\hat{Z}_T^n(v)\geq\frac{1}{(2k)^2}\Big).
\end{aligned} \tag{84}$$

Let us consider the covering of the set $\{|u|\leq L\}$ by the balls S_i, $1\leq i\leq N_h$, of radius h centered at the points u_i, such that the number N_h of the balls will be minimal. By $\bar{S}_i$, $1 \le i \le N_h$, we denote the balls of radius $2h$ centered at the points u_i. Then we have

$$P_\vartheta^n\left(\sup_{|u-v|\le h,\ |u|\le L,\ |v|\le L} |\sqrt{\hat{Z}_T^n(u)} - \sqrt{\hat{Z}_T^n(v)}| \ge \frac{1}{k}\right) \le$$

$$\le \sum_{j=1}^{N_h} P_\vartheta^n\left(\sup_{u,\, v\in \bar{S}_j} |\sqrt{\hat{Z}_T^n(u)} - \sqrt{\hat{Z}_T^n(v)}| \ge \frac{1}{k}\right). \tag{85}$$

By virtue of condition (III) and [23, Theorem 3] we get for $|u-v| \le 4h$ and sufficiently small $h > 0$

$$E_\vartheta^n(\sqrt{\hat{Z}_T^n(u)} - \sqrt{\hat{Z}_T^n(v)})^2 \le \varrho^2(P^n_{\vartheta+c_n^{-1}u,\,T}, P^n_{\vartheta+c_n^{-1}v,\,T}) \le$$

$$\le 4d|u-v|^\alpha + 2\left(P^n_{\vartheta+c_n^{-1}u}(H_T^n(v-u,0) \ge d|u-v|^\alpha)\right)^{1/2} \le$$

$$\le (4d+2D^{1/2})|u-v|^\alpha \le (4d+2D^{1/2})(4h)^\alpha$$

where $\varrho(\cdot,\cdot)$ is the Hellinger distance between the corresponding measures. Then from [18, Lemma 7.4] we get $P_\vartheta^n\left(\sup_{u,\, v\in \bar{S}_j} \left|\sqrt{\hat{Z}_T^n(u)} - \sqrt{\hat{Z}_T^n(v)}\right| \ge \frac{1}{k}\right) \le$ $\le C'h^\alpha$ where C' depends on k, d, D and the dimensionality m of the parametric space Θ. The above inequality together with (85) implies

$$P_\vartheta^n\left(\sup_{|u-v|\le h,\ |u|\le L,\ |v|\le L} |\sqrt{\hat{Z}_T^n(u)} - \sqrt{\hat{Z}_T^n(v)}| \ge \frac{1}{k}\right) \le \tag{86}$$

$$\le C'N_h h^\alpha \le C''L^m h^{\alpha-m}$$

where C'' is some positive constant.

From the conditions (III) and (IV) and [24, Theorems 3,4] we get for sufficiently small $h > 0$ and $|u| \le h$, $\vartheta \in \Theta$, $n \ge 1$,

$$\varrho^2(P^n_{\vartheta,T}, P_{\vartheta+c_n^{-1}u,\,T}) \le 4d|u|^\alpha + 2(P_\vartheta^n(H_T^n(u,0) \ge d|u|^\alpha)^{1/2} \le$$

$$\le (4d+2D^{1/2})|u|^\alpha,$$

and also for sufficiently large L and $|u| \ge L$, $n \ge 1$,

$$h(P^n_{\vartheta,T},\ P^n_{\vartheta+c_n^{-1}u,\,T}) \le \varphi(|u|)^{c/2} +$$

$$+\left(P_\vartheta^n(H_T^n(u,0) \le c|\ln \varphi(|u|)|)\right)^{1/2} \le \varphi(|u|)^{c/2} + C^{1/2}\varphi(|u|)$$

where $h(\cdot,\cdot)$ is the Hellinger integral for the corresponding measures.

From these inequalities and the proof of [25, Theorem 1] we obtain that

$$P_\vartheta^n\left(\sup_{|u|\ge L-h} \hat{Z}_T^n(u) \ge \frac{1}{(2k)^2}\right) \le C(k) \int_{(L-h)/2}^{\infty} y^{m-1}[\varphi(y)]^{(\alpha-m)/\alpha} dy \tag{87}$$

with some constant $C(k) > 0$.

From (83), (84), (86) and (87) after the operations $\lim_{k\to\infty} \overline{\lim}_{L\to\infty} \overline{\lim}_{h\to 0}$ we get (82).

Now we continue the process $\hat{Z}_T^n(u)$, $u\in U_\vartheta^n$, on all R^m with the preservation of $\sup_{u\in U^n_\vartheta} \hat{Z}_T^n(u)$ and modulus of continuity. Such a continuation may be obtained in the following way. We choose some ϑ_0 belonging to the interior of Θ and consider the rays starting from the point ϑ_0. For a given ray $\mathcal{R}$ we denote by ϑ_Γ the intersection point of the boundary $\partial\Theta$ of Θ with $\mathcal{R}$ and set $\hat{Z}_T^n(u)$ to be equal to $\lim_{k\to\infty} \hat{Z}_T^n(u_k)$ where $u_k = c_n(\vartheta_k-\vartheta_0)$, $\vartheta_k\in\mathcal{R}\cap\Theta$, $\lim_{k\to\infty}\vartheta_k = \vartheta_\Gamma$, on the part of this ray which belongs to $R^m\setminus U_\vartheta^n$.

Since the finite dimensional distributions of the processes $\hat{Z}_T^n(u)$ and $Z_T^n(u, 0)$ are equal, from Theorem 4 we get that for every set $(u_1, \ldots, u_k)$ with $u_i\in U_\vartheta^n$, $1\le i\le k$ and $k\ge 1$

(88) $$(\hat{Z}_T^n(u_1), \ldots, \hat{Z}_T^n(u_k))_s^T \xrightarrow{d(P^n_\vartheta)} (Z_T(u_1), \ldots, Z_s^T(u_k))^T$$

where $Z_T(u)$ is defined by (2). Hence, the finite dimensional distributions of $\sqrt{\hat{Z}_T^n(\cdot)}$ weakly converge to the ones of $\sqrt{Z_T(\cdot)}$.

We prove a weak convergence of the distributions of the process $\sqrt{\hat{Z}_T^n(\cdot)}$ to the distributions of $\sqrt{Z_T(\cdot)}$ in $\mathbf{C}(R^m, P_\vartheta^n)$. For this purpose we show that the family of distributions of $\sqrt{\hat{Z}_T^n(\cdot)}$ is tight. Our proof of tightness will be similar to [3, p. 522].

For given $H>0$, $\delta'>0$ and $L'>0$ we consider a subset

$$K(H, \omega_\delta, \gamma_L) = \{Z(u)\in\mathbf{C}(R^m)\colon |Z(0)|\le H;\ \sup_{|u-v|\le\delta,\ u,\,v\in R^m} |Z(u)-Z(v)|\le\omega_\delta,$$
$$0<\delta<\delta';\ \sup_{|u|\ge L} |Z(u)|\le\gamma_L,\ L>L'\}$$

where ω_δ is a function of δ, $\omega_\delta\to 0$ ad $\delta\to 0$, γ_L is a function of L, $\gamma_L\to 0$ as $L\to\infty$. According to [6, Theorem 18] a subset $K(H, \omega_\delta, \gamma_L)$ is compact in $\mathbf{C}(R^m)$.

We show that for every $\eta>0$ there are $H>0$, ω_δ and γ_L such that

$$\sup_n P_\vartheta^n(\sqrt{\hat{Z}_T^n}\notin K(H, \omega_\delta, \gamma_L))\le\eta.$$

Because $\hat{Z}_T^n(0)\xrightarrow{d(P^n_\vartheta)} Z_T(0)$, there is an $H>0$ such that

$$\sup_n P_\vartheta^n(\sqrt{\hat{Z}_T^n(0)}>H)\le\frac{\eta}{3}.$$

To define ω_δ we consider the sequence $(\varepsilon_r)_{r\ge 1}$, $\varepsilon_r\to 0$ as $r\to\infty$. Because of (84), (86) and (87) we get

$$\lim_{h\to 0}\sup_n P_\vartheta^n(\sup_{|u-v|\le h} |\sqrt{\hat{Z}_T^n(u)}-\sqrt{\hat{Z}_T^n(v)}|\ge\varepsilon) = 0,\ \forall\,\varepsilon>0.$$

Hence, for every ε_r there is an $h_r < h_{r-1}$, $h_0 = 1$, $h_r \to 0$ as $r \to \infty$, such that

$$\sup_n P_\vartheta^n\Big(\sup_{|u-v| \leqq h_r} |\sqrt{\hat{Z}_T^n(u)} - \sqrt{\hat{Z}_T^n(v)}\,| \geqq \varepsilon_r\Big) \leqq \frac{\eta}{3 \cdot 2^r}.$$

We set $\omega_\delta = \varepsilon_r$ for $\delta \in [h_{r+1}, h_r)$, $r \geqq 1$, and $\delta' = h_1$.

Further, from (87) we conclude that for every ε_r there is an L_r, $L_r > L_{r-1}$, $L_0 = 1$, $L_r \to \infty$ as $r \to \infty$, and

$$\sup_n P_\vartheta^n\Big(\sup_{|u| \geqq L_r} \sqrt{\hat{Z}_T^n(u)} \geqq \varepsilon_r\Big) \leqq \frac{\eta}{3 \cdot 2^r}.$$

We set $\gamma_L = \varepsilon_r$ for $L \in [L_r, L_{r+1})$, $r \geqq 1$, and $L' = L_1$.
The for properly chosen H, ω_δ and γ_L we have

$$\sup_n P_\vartheta^n(\sqrt{\hat{Z}_T^n} \notin K(H, \omega_\delta, \gamma_L)) \leqq \sup_n P_\vartheta^n(\sqrt{\hat{Z}_T^n(0)} > H) +$$

$$+ \sup_n \sum_{r=1}^{\infty} P_\vartheta^n\Big(\sup_{|u-v| \leqq h_r} |\sqrt{\hat{Z}_T^n(u)} - \sqrt{\hat{Z}_T^n(v)}| \geqq \varepsilon_r\Big) +$$

$$+ \sup_n \sum_{r=1}^{\infty} P_\vartheta^n\Big(\sup_{|u| \geqq L_r} \sqrt{\hat{Z}_T^n(u)} \geqq \varepsilon_r\Big) \leqq \frac{\eta}{3} + \frac{2}{3} \sum_{r=1}^{\infty} \frac{\eta}{2^r} = \eta.$$

So, the family of distributions of $\sqrt{\hat{Z}_T^n(\cdot)}$ is tight and according to Prohorov's theorem [1, Theorems 6.1, 6.2]

$$\sqrt{\hat{Z}_T^n} \xrightarrow{C(R^m,\, P^n{}_\vartheta)} \sqrt{Z_T},$$

which implies Theorem 5. □

5. Sufficient conditions for the asymptotic normality of maximum likelihood estimators

Theorem 5 provides an easy way to obtain "predictable" conditions for the asymptotic normality of maximum likelihood estimators. For this purpose in Theorem 5 we set $N_t(u) = (u, V_t)$, $u \in R^m$, $0 \leqq t \leqq T$, where $V = (V_t)_{t \geqq 0}$ is an m-dimensional Gaussian process with independent increments, $V_0 = 0$, $EV_t = 0$, $EV_tV_s^{\mathsf{T}} = \min(t, s)\Gamma(\vartheta)$ and $\Gamma(\vartheta)$ is a positive definite matrix of size $m \times m$.

We consider the maximum likelihood estimator $\hat{\vartheta}_T^n$ of the parameter ϑ built on the observations up to time $T > 0$, i.e.

$$\hat{\vartheta}_T^n = c_n^{-1} \arg\big(\sup_{u \in U^n{}_\vartheta} \hat{Z}_T^n(u)\big) + \vartheta$$

where $\vartheta_T^n = \infty$ if $\sup\limits_{u \in U^n{}_\vartheta}$ is not achieved.

Theorem 6. *Suppose that the conditions of Theorem 5 are satisfied by* $N_t(u) = (u, V_t)$. *Then the distributions of the normalized maximum likelihood estimators* $c_n(\hat{\vartheta}_T^n - \vartheta)$ *weakly converge to the normal distribution with parameters* $(0, (\Gamma^{\top}(\vartheta))^{-1}/T)$ *as* $n \to \infty$.

Proof. For the given Borel set A we consider the functional

$$\Psi(Z) = I\big(\sup_{u \in A} Z(u) - \sup_{u \in A^c} Z(u) \geqq 0\big)$$

where $Z \in \mathbf{C}(R^m)$ and A^c is the complement of A. The functional $\Psi(Z)$ is continuous on the set of $Z \in \mathbf{C}(R^m)$ for which the supremum with respect to u is achieved in a unique point not belonging to the boundary ∂A of A. In order to show the uniqueness of the maximum of $Z_T(u)$ we calculate $\hat{u}_T = \arg\big(\sup_{u \in R^m} Z_T(u)\big)$. Write

$$(u, V_T) - \frac{T}{2}\big(u, \Gamma(\vartheta)u\big) = -\frac{1}{2}\left|\left|\frac{1}{\sqrt{T}} J^{-1}(\vartheta)V_T - \sqrt{T}\, J^{\top}(\vartheta)u\right|\right|^2 +$$

$$+\frac{1}{2T}\|J^{-1}(\vartheta)V_T\|^2$$

where $\Gamma(\vartheta) = J(\vartheta)J_s^{\top}(\vartheta)$. From this we conclude that the point of maximum of $Z_T(u)$ is unique and

$$\hat{u}_T = \frac{1}{T}\Gamma^{-1}(\vartheta)V_T.$$

From [1, Theorem 5.1] and Theorem 5 we get

$$P_\vartheta^n\big(c_n(\hat{\vartheta}_T^n - \vartheta) \in A\big) = E_\vartheta^n I\big(c_n(\hat{\vartheta}_T^n - \vartheta) \in A\big) =$$

$$= E_\vartheta^n \Psi(\hat{Z}_T) \xrightarrow[n \to \infty]{} E\Psi(Z_T) = P(\hat{u}_T \in A)$$

for every A such that $P(\hat{u}_T \in \partial A) = 0$. Theorem 6 is proved. □

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