

(0, 1, 4) LACUNARY INTERPOLATION BY SPLINES

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1. Introduction. Recently, Th. Fawzy ([1]–[4]) presented some local methods to solve (0,3), (0, 2, 3) and (0, 2) lacunary interpolation problems using g -splines. A. Meir and A. Sharma [6], B. K. Swartz and R. S. Varga [9], A. K. Varma [10], R. S. Mishra and K. K. Mather [7] presented global methods for solving such lacunary interpolation problems.

In this paper, we follow the methods introduced by Th. Fawzy and present a spline solution for the (0, 1, 4) lacunary interpolation problem. We are given the mesh points:

$$\Delta: 0 = x_0 < x_1 < \dots < x_n = 1$$

with $x_{k+1} - x_k = h$, $k = 0, 1, \dots, n-1$, and real numbers

$$\{f_k, f'_k, f_k^{(4)}\}_{k=0}^n.$$

associated with the knots.

We are going to construct spline interpolant $S_\Delta(x)$ for which $S_\Delta^{(q)}(x_i) = f_i^{(q)}$ $i = 0, 1, \dots, n$ and $q = 0, 1, 4$. This construction is given in the following two cases.

2. Case A. In this case we consider the situation when $Y \in C^4[0, 1]$ and then we define the spline interpolant as follows:

$$(2.1) \quad \begin{aligned} S_\Delta(x) = S_k(x) = & y_k + y'_k(x - x_k) + \frac{1}{2}a_k(x - x_k)^2 + \\ & + \frac{1}{3!}b_k(x - x_k)^3 + \frac{1}{4!}y_k^{(4)}(x - x_k)^4, \end{aligned}$$

where $x_k \leq x \leq x_{k+1}$ and $k = 0, 1, \dots, n-1$. If we require that $S_\Delta(x) \in C^1[0, 1]$, then it is easy to prove that formula (2.1) assures that this spline polynomial

exists and is unique. That is clear from the continuity conditions of $S_{\Delta}(x)$ and $S'_{\Delta}(x)$ from which we get:

$$(2.2) \quad y_{k+1} = y_k + y'_k h + \frac{1}{2} a_k h^2 + \frac{1}{3!} b_k h^3 + \frac{1}{4!} y_k^{(4)} h^4.$$

and

$$(2.3) \quad y'_{k+1} = y' + a_k h + \frac{1}{2} b_k h^2 + \frac{1}{3!} y_k^{(4)} h^3.$$

Hence,

$$(2.4) \quad a_k = \frac{6}{h^2} A_k - \frac{2}{h} B_k,$$

$$(2.5) \quad b_k = \frac{6}{h^2} B_k - \frac{12}{h^3} A_k,$$

$$(2.6) \quad A_k = y_{k+1} - y_k - h y'_k - \frac{h^4}{4!} y_k^{(4)} \text{ and}$$

$$(2.7) \quad B_k = y'_{k+1} - y'_k - \frac{h^3}{3!} y_k^{(4)}.$$

Theorem 1. Let $S_k(x)$ be the spline interpolant solving the lacunary case $(0, 1, 4)$ for which $y(x) \in C^4[0, 1]$. Then for all $x \in [0, 1]$ the inequality

$$|y^{(i)}(x) - S_{\Delta}^{(i)}(x)| \leq c_i h^{4-i} \omega(h)$$

holds for all $i = 0, 1, 2, 3$ and 4 , where $\omega(h)$ is the modulus of continuity of $y^{(4)}(x)$, $c_0 = 7/12$, $c_1 = 3/2$, $c_2 = 31/12$, $c_3 = 5/2$ and $c_4 = 1$.

Proof. From (2.4) we find

$$|a_k - y''_k| = \left| \frac{6}{h^2} \left(y_{k+1} - y_k - h y'_k - \frac{h^4}{4!} y_k^{(4)} \right) - \frac{2}{h} \left(y'_{k+1} - y'_k - \frac{h^3}{3!} y_k^{(4)} \right) - y''_k \right|,$$

and taking:

$$y_{k+1} = y_k + h y'_k + \frac{h^2}{2!} y''_k + \frac{h^3}{3!} y'''_k + \frac{h^4}{4!} y^{(4)}(\eta_k),$$

where $x_k < \eta < x_{k+1}$ and

$$y'_{k+1} = y'_k + y''_k \cdot h + \frac{h^2}{2} y'''_k + \frac{h^3}{3!} y^{(4)}(\xi_k),$$

where $x_k < \xi_k < x_{k+1}$, it is easy to get:

$$(2.8) \quad |a_k - y''_k| \leq \frac{7}{12} h^2 \omega(h).$$

Similarly, from (2.7) we get

$$(2.9) \quad |b_k - y_k'''| = \left| \frac{6}{h^2} \left(y_{k+1}' - y' - \frac{h^3}{3!} y_k^{(4)} \right) - \frac{12}{h^3} \left(y_{k+1} - y_k - h y_k' - \frac{h}{4!} y_k^{(4)} \right) - y_k''' \right| \leq \frac{3}{2} h \omega(h),$$

and using (2.1), (2.8) and (2.9) we get, for $x_k \leq x \leq x_{k+1}$ and $k = 0, 1, \dots, n-1$:

$$\begin{aligned} |S_k(x) - y(x)| &= \left| y_k + y_k'(x - x_k) + \frac{1}{2} a_k(x - x_k)^2 + \right. \\ &\quad \left. + \frac{1}{3!} b_k(x - x_k)^3 + \frac{1}{4!} y_k^{(4)}(x - x_k)^4 - y_k - y_k'(x - x_k) - \right. \\ &\quad \left. - \frac{1}{2} y_k''(x - x_k)^2 - \frac{1}{3!} y_k'''(x - x_k)^3 - \frac{1}{4!} y^{(4)}(\eta_k)(x - x_k)^4 \right| \leq \\ &\leq \frac{h^2}{2} |a_k - y_k''| + \frac{h^3}{3!} |b_k - y_k'''| + \frac{h^4}{3!} h^3 \omega(h) \leq \frac{7}{12} h^4 \omega(h). \end{aligned}$$

Similarly:

$$\begin{aligned} |S_k'(x) - y'(x)| &\leq h |a_k - y_k''| + \frac{h^2}{2} |b_k - y_k'''| + \frac{h^3}{3!} h^3 \omega(h) \leq \frac{3}{2} h^3 \omega(h), \\ |S_k''(x) - y''(x)| &\leq |a_k - y_k''| + h |b_k - y_k'''| + \frac{h^2}{2} \omega(h) \leq \frac{31}{12} h^2 \omega(h), \\ |S_k'''(x) - y_k'''(x)| &\leq |b_k - y_k'''| + h \omega(h) \leq \frac{5}{2} h \omega(h), \quad \square \end{aligned}$$

and finally

$$|S_k^{(4)}(x) - y^{(4)}(x)| = |y_k^{(4)} - y^{(4)}(x)| \leq \omega(h). \quad \square$$

3. Case B. In this case $y \in C^5[0, 1]$ and we define the spline interpolant as

$$(3.1) \quad \begin{aligned} S_{\Delta}(x) = S_k(x) &= y_k + y_k'(x - x_k) + \frac{1}{2} a_k(x - x_k)^2 + \frac{1}{3!} b_k(x - x_k)^3 + \\ &\quad + \frac{1}{4!} y_k^{(4)}(x - x_k)^4 + \frac{1}{5!} c_k(x - x_k)^5, \end{aligned}$$

where $x \in [x_k, x_{k+1}]$ and $k = 0, 1, \dots, n-1$.

Let

$$(3.2) \quad c_k = \frac{1}{h} (y_{k+1}^{(4)} - y_k^{(4)}).$$

Now if $S_{\Delta}(x) \in C^1[0, 1]$, then the existence and uniqueness of $S_{\Delta}(x)$ is easy to be proved, since in this case a_k and b_k are uniquely determined by

$$(3.3) \quad a_k = \frac{6}{h^2} A_k - \frac{2}{h} B_k,$$

$$(3.4) \quad b_k = \frac{12}{h^3} \left(\frac{h}{2} B_k - A_k \right),$$

$$(3.5) \quad A_k = y_{k+1} - y_k - h y_k' \frac{h^4}{4!} y_k^{(4)} - \frac{h^5}{5!} b_k \text{ and}$$

$$(3.6) \quad B_k = y_{k+1}' - y_k' - \frac{h^3}{3!} y_k^{(4)} - \frac{h^4}{4!} c_k.$$

Theorem 2. Let $S_{\Delta}(x)$ be the spline interpolant, given in (3.1), and solving the lacunary case $(0, 1, 4)$ for which $y \in C^5[0, 1]$. Then for all $x \in [0, 1]$ the inequality:

$$|S_{\Delta}^{(4)} - y^{(4)}(x)| \leq c_i h^{5-i} \omega(h)$$

holds true for all $i = 0, 1, \dots, 5$, where $\omega(h)$ is the modulus of continuity of $y^{(5)}(x)$ and

$$c_0 = 2/15, c_1 = 19/60, c_2 = 13/20, c_3 = 17/20, c_4 = 1, c_5 = 1.$$

Proof. From (3.3), (3.5) and (3.6) we get

$$(3.7) \quad |a_k - y_k''| \leq \frac{2}{15} h^3 \omega(h)$$

and from (3.4), (3.5) and (3.6) we can get

$$(3.8) \quad |b_k - y_k'''| \leq 7/20 h^2 \omega(h).$$

Thus:

$$\begin{aligned} |S_k(x) - y(x)| &= |y_k - y_k'(x - x_k) + \frac{1}{2} a_k (x - x_k)^2 + \frac{1}{3!} b_k (x - x_k)^3 - \\ &\quad + \frac{1}{4!} y_k^{(4)} (x - x_k)^4 + \frac{1}{5!} c_k (x - x_k)^5 - y_k - y_k'(x - x_k) - \\ &\quad - \frac{1}{2} y_k''(x - x_k)^2 - \frac{1}{3} y_k'''(x - x_k)^3 - \frac{1}{4!} y_k^{(4)}(x - x_k) - \\ &\quad - \frac{1}{5!} y^{(5)}(\xi_k)(x - x_k)^5| \leq \frac{h^2}{2} |a_k - y_k''| + \frac{h^3}{3!} |b_k - y_k'''| + \\ &\quad + \frac{2}{15} h^5 \omega(h). \end{aligned}$$

Similarly, it is easy to complete the proof for $|S_k^{(i)}(x) - y^{(i)}(x)|$ where $i = 1, 2, \dots, 5$. $\square$

REFERENCES

- [1] Fawzy Th.: Note on lacunary interpolation with splines. I. (0,3)-lacunary interpolation. *Annales. Univ. Sci. Budapest, Sectio Math.* **28** (1985), 17–28.
- [2] Fawzy Th.: Notes on lacunary interpolation by splines., II. *Annales Univ. Sci. Budapest., Sectio Computatorica* **6** (1985). 117–123.
- [3] Fawzy Th.: Notes on lacunary interpolation by splines. III. (0,2) case. *Acta Math. Acad. Sci. Hungar.* (submitted).
- [4] Fawzy Th.: Lacunary interpolation by g-splines. (0, 2, 3) case. *Studia Sci. Math. Hungar.* (submitted).
- [5] Guo Zhu Rui: Note on lacunary interpolation by splines. *Math. Num. Sinica* **3** (2) (1981), 175–178.
- [6] Meir A. and Sharma A.: Lacunary interpolation by splines. *SIAM J. Num. Anal* **10** (3) (1973).
- [7] Mishra R. S. and Mathur K. K.: Lacunary interpolation by splines. (0, 2, 3) and (0, 2, 4) cases. *Acta Math. Acad. Sci. Hungar.* **36** (1980), 251–260.
- [8] Schumaker L.: Spline Functions. Basic Theory. John Wiley and Sons, New York, Toronto, 1981.
- [9] Swartz B. K. and Varga R. S.: A note on lacunary interpolation by splines. *SIAM J. Num. Anal.* **10** (1) (1973), 433–447.
- [10] Varma A. K.: Lacunary interpolation by splines. *Acta Math. Acad. Sci. Hungar.* **31** (3–4) (1978), 183–192.

