NOTES ON LACUNARY INTERPOLATION BY SPLINES. II. (0,2) INTERPOLATION

THARWAT FAWZY
Dept. of Math., Suez-Canal University, Ismailia
(Received November 24, 1984)

(0,2) Interpolation

Abstract. A new method for solving the (0,2) interpolation problem is presented. It has been shown that if $f \in \mathrm{Lip}_M \alpha$, $0 < \alpha \leqq 1$, $f \in C^r[0,1]$ and $r = 2, 3, 4$, then the method is $O(h^{r-i+\alpha})$ in $f^{(i)}(x)$ for all $i = 0, 1, \ldots, r$, where h is the maximum step size. In addition, a stability result for such interpolation is also presented.

1. Introduction. In the recent paper [1] by A. Meir and A. Sharma, error bounds have been developed for (0,2) interpolation of certain functions by deficient splines. Swartz add Varga in [2] have extended the results of [1] to a wider class of functions and have indicated that the extended results are the best possible.

The main results of Swartz and Varga are given in the following theorem.

Theorem 1.1. *Let $f \in C^k[0,1]$, where $0 \leqq k < 6$, let n be an odd integer, and let $\bar{S}_n$ be the unique generalized Meir- Sharma interpolation of f in $S^{(3)}_{n,5}$ (cf.* [1], *Theorem 1). Then there exists a constant K, independent of f and n such that*

$$Kn^{1+j-k}\omega(D^k f;\ 1/n) \geqq \|D^j(f-\bar{S}_n)\|_\infty,\quad 0 \leqq j \leqq \min\ (k,\ 4).$$

In this paper, the values of such constants are completely calculated. Moreover, the boundary conditions of the Meir-Sharma interpolant of f,

$$D^3(f-S_n)(0) = 0 \quad \text{and} \quad D^3(f-S_n)(1) = 0$$

are released.

In the following sections we present our interpolants for each value of r separately and prove the convergence in this case.

Thus, we begin with the first case when $f \in C^2[0, 1]$.

2. Case A. In this case $f \in C^2[0, 1]$ and we consider the partition:

$$\triangle: 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

where for $k = 0, 1, \ldots, n-1$, $h_k = x_{k+1} - x_k$ and $h = \max\limits_k h_k$.

Theorem 2.1. *Given arbitrary numbers* $f^{(p)}(x_k)$, $k = 0(1)n-1$ *and* $p = 0, 2$. *Then there exists a unique spline* $S_\Delta(x)$ *such that*

(2.1) $S_\Delta(x) \in C[0, 1]$,

(2.2) $S_\Delta(x) \in \pi_2$ *on each* $[x_k, x_{k+1}]$, $k = 0(1)n-1$ *and*

(2.3) $S_\Delta^{(p)}(x_k) = f^{(p)}(x_k) = f_k^{(p)}$, $k = 0(1)n$; $p = 0, 2$.

Proof. Let for $x_k \leqq x \leqq x_{k+1}$, $k = 0(1)n-1$,

$$S_\Delta(x) = S_k(x) = f_k + a_k(x-x_k) + \frac{1}{2} f_k''(x-x_k)^2. \tag{2.4}$$

Thus, for $k = 0(1)n-1$, the value

$$a_k = \left[f_{k+1} - f_k - \frac{1}{2} h_k^2 f_k''\right] / h_k \tag{2.5}$$

proves Theorem 2.1. □

Let $\omega_i(h)$ $(i = 2, 3, 4)$ denote the modulus of continuity of $f^{(i)}(x)$.

Theorem 2.2. *Let* $f \in C^2[0, 1]$. *Then for the unique quadratic spline* $S_\Delta(x)$ *associated with* f *and given in Theorem 2.1, we have for all* $x \in [0, 1]$,

$$|S_\Delta(x) - f(x)| \leqq h^2 \omega_2(h),$$

$$|S'_\Delta(x) - f'(x)| \leqq \frac{3}{2} h\, \omega_2(h) \text{ and}$$

$$|S''_\Delta(x) - f''(x)| \leqq \omega_2(h).$$

Proof. Using (2.4), (2.5) and the Taylor expansion of $f(x)$, it is easy to prove it. □

3. Case B. In this case $f \in C^3[0, 1]$ and we consider the partition:

$$\Delta: 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

where for $k = 0(1)n-1$, $h_k = x_{k+1} - x_k$ and $h = \max\limits_k k_k$.

Theorem 3.1. *Given arbitrary numbers* $f^{(p)}(x_k)$, $k = 0(1)n$; $p = 0, 2$, *then there exists a unique spline* $S_\Delta(x)$ *such that:*

(3.1) $S_\Delta(x) \in \pi_3$ *on each* $[x_k, x_{k+1}]$, $k = 0(1)n-1$,

(3.2) $S_\Delta(x) \in C^{(0,2)}[0, 1]$, *i.e. both* $S_\Delta(x)$ *and* $S''_\Delta(x)$ *is continuous for all* $x \in [0, 1]$, *and*

(3.3) $S_\Delta^{(p)}(x_k) = f^{(p)}(x_k)$, $k = O(1)n$ *and* $p = 0, 2$.

Proof. Let for $x_k \leqq x \leqq x_{k+1}$, $k = 0(1)n-1$

$$S_\Delta(x) = S_k(x) = f_k + a_k(x-x_k) + \frac{1}{2} f_k''(x-x_k)^2 + \frac{1}{3!} c_k(x-x_k)^3. \tag{3.4}$$

Then for all $k = 0(1)n-1$, the values

$$(3.5)\qquad a_k = \left[f_{k+1} - f_k - {}^{1/2}f_k''h_k^2 - \frac{1}{3!}h_k^2 f_{k+1}'' + \frac{1}{3!}h_k^2 f_k''\right]h_k.$$

and

$$(3.6)\qquad c_k = [f_{k+1}'' - f_k'']/h_k$$

prove Theorem 3.1. □

Theorem 3.2. *Let $f \in C^2[0, 1]$. Then for the unique cubic spline $S_\Delta(x)$ associated with f and given in Theorem 3.1, we have for all $x \in [0, 1]$,*

$$|S_\Delta^{(i)}(x) - f^{(i)}(x)| \leqq K_{3,i}h^{3-i}\omega_2(h),\ i = 0, 1, 2, 3,$$

where $$K_{3,0} = \frac{1}{3},\ K_{3,1} = \frac{1}{6},\ K_{3,2} = 1 \text{ and } K_{3,3} = 1.$$

Proof. The proof is obvious for $i = 3$, using (3.6).

If $i=0$, 1 and 2, then we consider the Taylor expansion for $x_k \leqq x \leqq x_{k+1}$, $k = 0(1)n-1$,

$$(3.7)\qquad f^{(i)}(x) = \sum_{j=i}^{2}\frac{f^{(j)}(x_k)}{(j-i)!}(x-x_k)^{(j-i)} + \frac{f^{(3)}(\xi_i^{(k)})}{(3-i)!}(x-x_k)^{(3-i)}.$$

where $x_k < \xi_i^{(k)} < x_{k+1}$.

Using the above equation (3.7) with (3.4), (3.5) and (3.6) it will be easy to complete the proof. □

4. Case C. In this case $f \in C^4[0, 1]$ and we consider the partition:

$$\Delta\colon 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

where $x_{k+1} - x_k = h$ and $k = 0(1)n-1$.

Theorem 4.1. *Given arbitrary numbers $f^{(p)}(x_k) = f_k^{(p)}$, $k = 0(1)n$; $p = 0, 2$. Then, there exists a unique spline $S_\Delta(x)$ such that*

$$(4.1)\qquad S_\Delta(x) \in \pi_4 \text{ on each } [x_k, x_{k+1}],\ k = 0, 1, \ldots, n-1,$$

$$(4.2)\qquad S_\Delta(x) \in C^{(0,2)}[0, 1],$$

$$(4.3)\qquad S_\Delta^{(p)}(x_k) = f^{(p)}(x_k) = f_k^{(p)},\ k = 0(1)n;\ p = 0, 2,$$

$$(4.4)\qquad S_\Delta(x) = \begin{cases} S_0(x), & x_0 \leqq x \leqq x_1, \\ S_k(x), & x_k \leqq x \leqq x_{k+1},\ k = 1(1)n-1, \end{cases}$$

where

$$(4.5)\qquad S_k(x) = f_k + a_k(x-x_k) + \frac{1}{2}f_k''(x-x_k)^2 + \frac{1}{3!}c_k(x-x_k)^3 + \frac{1}{4!}d_k(x-x_k)^4,$$

$$(4.6)\qquad d_k = [f_{k+1}'' - 2f_k'' + f_{k-1}'']/h^2,\ k = 1(1)n-1.$$

and

(4.7) $$S_0(x) = f_0 + a_1(x-x_0) + \frac{1}{2} f_0''(x-x_0)^2 + \frac{c_1}{3!}(x-x_0)^3 + \frac{d_1}{4!}(x-x_0)^4.$$

Proof. Using (4.2), (4.3) and (4.5), then we easily get,

(4.8) $$c_k = [f_{k+1}'' - f_k'' - \frac{1}{2} h^2 d_k]/h \text{ and}$$

(4.9) $$a_k = [f_{k+1} - f_k - \frac{1}{2} h^2 f_k'' - \frac{h^3}{3!} c_k - \frac{h^4}{4!} d_k]/h,$$

and this determines uniquely $S_k(x)$ and $S_0(x)$.
Hence the proposition of Theorem 4.1. □

Theorem 4.2. *Let $f \in C^4[0, 1]$. Then for the unique spline $S_\Delta(x)$ given in Theorem* 4.1, *we have for all $x \in [0,1]$, $k = 1(1)n-1$*

(4.10) $$|S_\Delta^{(i)}(x) - f^{(i)}(x)| \leqq K_{4,i} h^{4-i} \omega_4(h), \quad i = 0(1)4$$

and for all $x \in [x_0, x_1]$,

(4.11) $$S|_0^{(i)}(x) - f^{(i)}(x)| \leq K_{4,i}^* h^{4-i} \omega_4(h), \quad i = 0(1)4,$$

where $K_{4,0} = 3/8$, $K_{4,1} = 13/16$, $K_{4,2} = 3/2$, $K_{4,3} = 9/4$, $K_{4,4} = 3/2$,
$K_{4,0}^* = 5/12$, $K_{4,1}^* = 47/48$, $K_{4,2}^* = 2$, $K_{4,3}^* = 13/4$, $K_{4,4}^* = 5/2$.

Before proving this theorem, we state and prove some lemmas which will help us in arriving at the proof.

Lemma 4.1. *For d_k given in* (4.6), *we have*

$$|d_k - f^{(4)}(x)| \leqq (3/2)\omega_4(h)$$

which holds for all $x \in [x_k, x_{k+1}]$ and all $k = 1(1)n-1$.

Proof. Using (4.6), the Taylor expansion of f_{k+1}'' and f_k'' and the definition of the modulus of continuity, we can easily prove this lemma. □

Lemma 4.2. *For c_k given in* (4.8), *we have*

$$|c_k - f^{(3)}(x_k)| \leq \frac{3}{4} h\omega_4(h)$$

which holds for all $k = 1(1)\, n-1$.

Proof. Using (4.8), the Taylor expansion of f_{k+1}'' and Lemma 4.1, it will be easy to prove it. □

Lemma 4.3. *For a_k given in* (4.9), *the inequality*

$$|a_k - f'(x_k)| \leq \frac{9}{2(4!)} h^3 \omega_4(h).$$

holds for all $k = 1(1)n-1$.

Proof. Using (4.9) with the help of Lemma 4.1 and Lemma 4.2, we can easily complete the proof. □

Proof of Theorem 4.2. We have for all $x \in [x_k, x_{k+1}]$ and all $k = 1(1)n-1$, the Taylor expansion,

$$(4.12) \quad f^{(i)}(x) = \sum_{j=1}^{3} [f^{(i)}(x_k)/(j-i)!](x-x_k)^{(j-i)} + \frac{1}{(4-i)!} f^{(4)}(\xi_i^{(k)})(x-x_k)^{(4-i)},$$

where $x_k < \xi_i^{(k)} < x_{k+1}$ and $i = 0, 1, 2, 3$.

Using (4.5), (4.6), (4.8) and (4.9) with the help of Lemmas 4.1–4.3. we can complete the proof of this theorem for $k = 1(1)n-1$ and $i = 0, 1, 2, 3$.

If $i = 4$, then we get the situation of Lemma 4.1 for all $k = 1(1)n-1$. Hence the proposition (4.10).

For $x \in [x_0, x_1]$, we use similar technique and we easily can prove (4.11). Thus the proof of Theorem 4.2 is complete. □

5. Stability. We conclude this note with a stability result concerning the case C, when $f \in C^4[0, 1]$ while it is easy to prove similar stability results for the other two cases when $f \in C^2$ and C^3.

Theorem 5.1. *Let $f \in C^4[0, 1]$ and let $\bar{S}_\Delta(x)$ be the unique spline constructed in the same manner as that of Theorem 4.1 and satisfying the following data:*

$$(5.1) \qquad \bar{S}_\Delta(x_k) = \alpha_k \qquad k = 0(1)n,$$

$$(5.2) \qquad \bar{S}'_\Delta(x_k) = \beta_k \qquad k = 0(1)n$$

where we suppose that there exists a function $F(f, h)$ such that:

$$(5.3) \qquad \omega_4(h)\, h^4 F(f, h) \geqq \max_k |f(x_k) - \alpha_k|$$

and

$$(5.4) \qquad \omega_4(h)\, h^2 F(f, h) \geqq \max_k |f''(x_k) - \beta_k|.$$

Then there exist constants K_i and $\bar{K}_i$ independent of F, f and h such that the inequality

$$\|D^i(f - \bar{S}_\Delta)\| \leqq h^{4-i} \omega_4(h)[\bar{K}_i F + K_i]$$

holds for all $i = 0(1)4$, where $\|\cdot\|_\infty = \|\cdot\|_{L_\infty[0,1]}$.

Proof. The unique spline polynomial $\bar{S}_\Delta(x)$ can be easily constructed in the form:

$$(5.5) \qquad \bar{S}_\Delta(x) = \begin{cases} \bar{S}_0(x), & x_0 \leqq x \leqq x_1 \\ \bar{S}_k(x), & x_k \leqq x \leqq x_{k+1}, \quad k = 1(1)n-1, \end{cases}$$

where

$$(5.6) \quad \bar{S}_0(x) = \alpha_0 + \bar{a}_1(x-x_0) + \frac{1}{2}\beta_0(x-x_0)^2 + \frac{1}{3!}\bar{c}_1(x-x_0)^3 + \frac{1}{4!}\bar{d}_1(x-x_0)^4,$$

(5.7) $$\bar{S}_k(x) = \alpha_k + \bar{a}_k(x-x_k) + \frac{1}{2}\beta_k(x-x_k)^2 + \frac{1}{3!}\bar{c}_k(x-x_k)^3 + \frac{1}{4!}\bar{d}_k(x-x_k)^4,$$

(5.8) $$\bar{d}_k = [\beta_{k+1} - 2\beta_k + \beta_{k-1}]/h^2, \quad k = 1(1)n-1,$$

(5.9) $$\bar{c}_k = [\beta_{k+1} - \beta_k - \frac{1}{2}h^2\bar{d}_k]/h, \; k = 1(1)n-1$$

and

(5.10) $$\bar{a}_k = [\alpha_{k+1} - \alpha_k - \frac{1}{2}h^2\beta_k - \frac{h^3}{3!}\bar{c}_k - \frac{h^4}{4!}\bar{d}_k]/h, \; k = 1(1)n-1.$$

We prove this theorem for $S_k(x)$ only where $k = 1(1)n-1$ while it is easy to prove it for $S_0(x)$.

For this reason, we use (5.7)–(5.10) and (4.5)–(4.9) and we easily get:

(5.11) $$|\bar{a}_k - a_k| \leqq (62/24)h^3\omega_4(h)F,$$

(5.12) $$|\bar{c}_k - c_k| \leqq h\omega_4(h)F$$

and

(5.13) $$|\bar{d}_k - d_k| \leqq 4\omega_4(h)F.$$

We also have, for all $x_k \leqq x \leqq x_{k+1}$ and $k = 1(1)n-1$,

$$|f(x) - \bar{S}_k(x)| \leqq |f(x) - S_k(x)| + |S_k(x) - \bar{S}_k(x)| \leqq$$

$$\leqq |f(x) - S_k(x)| + |f_k - \alpha_k| + h|a_k - \bar{a}_k| + \frac{1}{2}h^2|f_k'' - \beta_k| +$$

$$+ \frac{h^3}{3!}|c_k - \bar{c}_k| + \frac{h^4}{4!}|d_k - \bar{d}_k|.$$

Using Theorem 4.2, (5.3), (5.4), (5.11), (5.12) and (5.13) we easily get

(5.14) $$|f(x) - S_k(x)| \leqq h^4\omega_4(h)\left[\frac{53}{12}F + \frac{3}{8}\right].$$

Similarly, we can get the following results for the derivatives:

(5.15) $$|f'(x) - \bar{S}_k'(x)| \leqq h^3\omega_4(h)\left[\frac{19}{4}F + \frac{13}{16}\right],$$

(5.16) $$|f''(x) - \bar{S}_k''(x)| \leqq h^2\omega_4(h)\left[4F + \frac{3}{2}\right],$$

(5.17) $$|f^{(3)}(x) - \bar{S}_k^{(3)}(x)| \leqq h\omega_4(h)\left[5F + \frac{9}{4}\right].$$

and

$$|f^{(4)}(x)-\bar{S}_k^{(4)}(x)| \leqq \omega_4(h)\left[4F+\frac{3}{2}\right]. \tag{4.18}$$

Hence the proposition of Theorem 5.1. □

We used the following example to test the method and we got the following results.

Example. We considered $f(x) = 1+xe^x$, $x\in[0, 1]$, $x_k = kh$, $k = 0(1)10$ and $h = 0.1$. The results are given for $x = 0.86$:

The function	Numerical values	Exact values	The error
Case A) $f\in C^2$ [0,1]:			
f	3.032880959	3.032318197	5.627600E−4
f'	4.394415716	4.395478890	1.063170E−3
f''	6.23154600	6.758639584	5.271249E−1
Case B) $f\in C^3$ [0,1]:			
f	3.032304099	3.032318197	1.409800E−5
f'	4.395617486	4.395478890	1.385960E−4
f''	6.772315150	6.758639584	1.367567E−2
$f^{(3)}$	9.013344220	9.121800278	1.084561E−1
Case C) $f\in C^4$ [0,1]:			
f	3.032317366	3.032318197	8.300000E−7
f'	4.395485583	4.395478890	6.693000E−6
f''	6.759480996	6.758639584	8.414120E−4
$f^{(3)}$	9.120296352	9.121800278	1.503926E−3
$f^{(4)}$	10.69521320	11.48496097	7.897478E−1

REFERENCES

[1] *Meir A.* and *Sharma A.:* Lacunary interpolation by splines. *SIAM J. Numer. Anal.*, **10** (3) (1973).

[2] *Swartz B. K.* and *Varga R. S.:* A note on lacunary interpolation by splines. SIAM J. Numer. Anal. **10** (1) (1973), 443–447.

[3] *Varma A. K.:* Lacunary interpolation by splines. *Acta Math. Acad. Sci. Hungar.* **31** (3–4) (1978), 183–192.

[4] *Fawzy Th.:* Notes on lacunary interpolation by splines. I. (0,3) interpolation. *Annales Univ. Sci. Budapest., Sectio Math.* **28** (1985), 17–28.