ON STABILITY AND CONVERGENCE OF FINITE DIFFERENCE SCHEME FOR FILTRATION EQUATION

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1. Introduction

Kalashnikov has proved in [1] that the following Cauchy-problem has a unique generalized solution in the half-plane

(1.0) $$\mathbf{R}^2_+ = \{(x,t)\colon 0 \le t < \infty\,,\ x \in \mathbf{R}^1\}$$

$$u_t = [\Phi(u)]_{xx} - \Psi(u),\ u(x,\,0) = u^\circ(x),\ x \in \mathbf{R}^1$$

$u^\circ(x) \in C(\mathbf{R}^1)$ (where $\Phi(u) \ge 0$, $\Psi(u) \ge 0$ are continuous for $u \ge 0$, $\Phi'(u) > 0$ for $u > 0$, $\Phi'(0) \ge 0$, $\Phi(0) = 0$, $\Psi(0) = 0$).

In this paper we give a stable and convergent explicit finite-difference scheme applied for a certain type of Cauchy-problem (1.0).

Consider the condition $u(x,\,0) = u^0_1(x) \in C^r(\mathbf{R}^1)$ $r \le 4$, instead of the initial condition of the problem (1.0) where $|u^0(x) - u^0_1(x)| \le \varepsilon_{u^0}$. That is we approximate the function $u^0(x)$ (continuous in $-\infty < x < \infty$) by a function $u^0_1(x) \in C^r(\mathbf{R}^1)$.

Assume that $u^0(x) \to 0$ and $\dfrac{du_0(x)}{dx} \to 0$ as $|x| \to \infty$. In this case the following approximation of $u^0(x)$ is usual:

$u^\circ_1(x) = e^{-dx^2} \sum\limits_{i=0}^{S} a_i x^i$ (d, a_i are constants). Further we assume that $\Phi^{(l)}(u)$ ($l \le 5$) exist, or if it does not exist then we approximate the function $\Phi(u)$ by a function $\Phi_1(u) = \sum\limits_{i=1}^{N} \Phi_{1i}(u)$ where $\Phi^{(l)}_{1i}(u)$ ($l \le 5$) ($i = 1, \ldots, N$) exist and $|\Phi(u) - \Phi_1(u)| \le \varepsilon_\Theta$, if $|u| \le U$ (U is a constant).

We deal with the following Cauchy-problem because of generality:

$$u_t(x,\,t) = (g_2(u(x,\,t)))_{xx} + (g_1(u(x,\,t)))_x + g_0(u(x,\,t)) \equiv Au(x,\,t),$$

(1.1) $$u(x,\,0) = u^\circ(x) \in C^r(\mathbf{R}^1),$$

where $r \leq 4$;

$$\left|\frac{\partial^{l_2}}{\partial u^{l_2}} g_2(u)\right|, \ (0 \leq l_2 \leq 5); \left|\frac{\partial^{l_1}}{\partial u^{l_1}} g_1(u)\right|, \ (0 \leq l_1 \leq 5);$$

$$\left|\frac{\partial^{l_0}}{\partial u^{l_0}} g_0(u)\right|, \ (0 \leq l_0 \leq 5)$$

are bounded. Our difference scheme is based on the splitting of the operator A:

$$Au = B_2 u + B_1 u + B_0 u,$$

where $B_l u = \dfrac{\partial^l}{\partial u^l}(g_l(u(x, t)))$ $l = 0, 1, 2.$

In the following part we give some methods for the estimates of the l-th differences ($l \leq 4$) of the approximate solution of the equation (1.1). Therefore we give estimates for the schemes established for the simpler equations $u_t = B_2 u$ in 2.1, $u_t = B_1 u$ in 2.2, $u_t = B_0 u$ in 2.3. Thereafter we easily get the estimates for the equation (1.1) in 2.4. Using these estimates we obtain stability theorems for the problems (1.0) and (1.1). In **3.** we give convergence theorems for the strong solution, too.

2. Stability estimations

2.1. At first we deal with the problem:

$$u_t = (g(u))_{xx} \ (g(u) = g_2(u)), \ g'(u) \geq 0,$$
$$u(x, 0) = u^0(x).$$

We establish the usual difference scheme on the mesh $H_{h,k} = (ih, nk)$

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{g(u_{i+1}^n) - 2g(u_i^n) + g(u_{i-1}^n)}{h^2} \equiv Dg(u_i^n). \tag{2.1}$$

We seek the approximate solution in a triangle with equal legs, base $\bar{A}$, altitude $T\left(|x| \leq \dfrac{\bar{A}}{2}, \ 0 \leq t \leq T, \ \dfrac{k}{h} = \dfrac{2T}{\bar{A}}, \ k = \dfrac{T}{N}, \ h = \dfrac{\bar{A}}{2N}\right)$.

Theorem 1. *Assume that* $\triangle_h^{(l)} u_i^n$ *and* $\triangle_k^{(l)} u_i^n$ *are the l-th difference (symmetrical if l is even) of* u_i^n *according to* $x\,(x = ih)$ *and* $t\,(t = nk)$; $|\triangle_h^{(l)} u_i^0| \leq \leq V_l$ (V_l *are constants,* $l = 0, 1, 2, 3, 4$) *for all* i; $\dfrac{k}{h^2} \leq \dfrac{1}{2|g'|_m}$ where

$$|g'|_m \equiv \max_{|z| \leq \max |u^0(x)|} g'(z), \ g'(z) \geq 0 \text{ if } |z| \leq \max |u^0(x)|.$$

Then the scheme (2.1) is stable with respect to the initial condition and

$$|u_i^n| \leq \max_{x \in R^1} |u^\circ(x)|, \quad \triangle_h^{(l)} u_i^n \leq U_l \ (l = 1, 2, 3, 4),$$

$$\triangle_k^{(m)} u_i^n \leq U_m (m = 1, 2) \text{ where } U_l, U_m$$

depend essentially only on the initial function $u^0(x)$.

Proof. From (2.1):

$$u_i^{n+1} = u_i^n + \frac{k}{h^2}(g(u_{i+1}^n) - 2g(u_i^n) + g(u_{i-1}^n)).$$

Let $\Phi_1(y) = y - 2\frac{k}{h^2}g(y)$, $\Phi_2(y) = \frac{k}{h_2}g(y)$, $\Phi_3(y) = \frac{k}{h^2}g(y)$.
Φ_1, Φ_2 and Φ_3 are nondecreasing functions of y.
Let $U = \sup_i |u_i^n|$, it follows that

$$\Phi(-U) \leq u_i^{n+1} \leq \Phi(U) \quad \text{for all } i;$$

where

$$\Phi(y) = \Phi_1(y) + \Phi_2(y) + \Phi_3(y) = y.$$

So $|u_i^{n+1}| \leq U = \sup |u_i^n|$ and by mathematical induction:

$$|u_i^{n+1}| \leq \sup_i |u_i^0|.$$

The estimation of $\triangle u_i^n = \dfrac{u_{i+1}^n - u_i^n}{h} \equiv \triangle_h^{(1)} u_i^n$.

We give three methods of the estimation (2.1/a, 2.1/b, 2.1/c) and apply the method which is suitable for the initial function and the equation of the problem.

Let $|\ldots|_m = \max |\ldots|$; $|g'|_m = \max_u |g'(u)|$; $|\triangle u_i^n|_m =$

$$= \max_i |\triangle u_i^n|; \lambda = \frac{k}{h^2};$$

$$\delta g(u_i^n) = \frac{g(u_{i+1}^n) - g(u_i^n)}{u_n^{i+1} - u_i^n} = g'(\tilde{u}_i^n)(u_i^n \leq \tilde{u}_i^n \leq u_{i+1}^n); \ \delta^2 g(u_i^n) =$$

$$= \frac{\delta g(u_{i+1}^n) - \delta g(u_i^n)}{u_{i+1}^n - u_i^n} = g''(\tilde{\tilde{u}}_i^n)$$

$$\delta^3 g(u_i^n) = \frac{\delta^2 g(u_{i+1}^n) - \delta^2 g(u_i^n)}{u_{i+1}^n - u_i^n} = g^{(3)}(\approx{u}_i^n); \ d^l u_i^n \equiv \triangle_x^{(l)} u_i^n.$$

2.1/a. From (2.1), we deduce:

$$\triangle u_i^{n+1} \equiv \frac{u_{i+1}^{n+1} - u_i^{n+1}}{h} = \frac{u_{i+1}^n - u_i^n}{h} + G, \text{ where} \tag{2.1.1}$$

$$G = \frac{k}{h^2}\left[\delta g(u_{i+1}^n)\frac{u_{i+2}^n - u_{i+1}^n}{h} - 2\delta g(u_i^n)\frac{u_{i+1}^n - u_i^n}{h} + \delta g(u_{i-1}^n)\frac{u_i^n - u_{i-1}^n}{h}\right].$$

I. Assume that $|\triangle u_i^n| \geq |G|$; $\triangle u_i^n \geq 0$ and $G \leq 0$ (or $\triangle u_i^n < 0$ and $G > 0$). Then it is trivial $|\triangle u_i^{n+1}|_m \leq |\triangle u_i^n|_m (\leq \ldots \leq |\triangle u_i^0|_m)$. Let $|\triangle u_i^n| < |G|$ and $\triangle u_i^n \geq 0$ and $G \leq 0$ (or $\triangle u_i^n < 0$ and $G > 0$), and $\frac{k}{h^2} \leq \frac{1}{4|g'|_m}$. Then it is obviously:

$$|\triangle u_i^{n+1}|_m \leq |\triangle u_i^n|_m (\leq \ldots \leq |\triangle u_i^0|_m).$$

II/1. Assume that $\triangle u_i^n > 0$, $G > 0$.

A) $\triangle u_{i+1}^n - \triangle u_i^n > 0$.

Suppose that k and h are such that $|G| \leq |\triangle u_{i+1}^n - \triangle u_i^n|$,

$$|G| = \left|\frac{k}{h^2}\left[\delta g(u_{i+1}^n)\triangle u_{i+1}^n - 2\delta g(u_i^n)\triangle u_i^n + \delta g(u_{i-1}^n)\triangle u_{i-1}^n\right]\right| \leq$$

$$\leq |\triangle u_{i+1}^n - \triangle u_i^n| \equiv h|Du_i^n|$$

thus

$$\left|\frac{k}{h}[\delta^2 g(u_i^n)\triangle u_{i+1}^n \triangle u_i^n + \delta g(u_i^n) Du_i^n -\right.$$

$$\left. -(\delta^2 g(u_{i-1}^n)\triangle u_i^n \triangle u_{i-1}^n + \delta g(u_{i-1}^n) Du_{i-1}^n)]\right| \leq$$

$\leq h|Du_i^n|$, it is sufficient for this: $kC_1 \leq h|Du_i^n|$, where

$$C_1 = |\delta^3 g(u_i^n)|_m |\triangle u_{i\,m}^n|^3 + 3|\delta^2 g(u_i^n)|_m |\triangle u_i^n|_m |Du_i^n|_m +$$

$$+ |\delta g(u_i^n)|_m |d^3 u_i^n|_m.$$

Thus if $\frac{k}{h} \leq \frac{|Du_i^n|}{C_1}$ then

$$|\triangle u_i^{n+1}| = |\triangle u_i^n + G| \leq |\triangle u_i^n + \triangle u_{i+1}^n - \triangle u_i^n| = |\triangle u_{i+1}^n|.$$

We show that if $\frac{k}{h} < \frac{|Du_i^n|}{C_1}$ then for sufficiently small h the first difference can be estimated, too. Since $\lim\limits_{|x|\to\infty} \frac{du^0(x)}{dx} = 0$ thus for any $\varepsilon > 0$ there exists A_1 such that $|\triangle u_i^0|<\varepsilon$ if $i\geq I$ where $Ih = A_1$. It follows from $|Du_i^n|\leq\frac{k}{h}C_1$:

$$|\triangle u_i^n|\leq A_1\frac{k}{h}C_1\leq A_1hC_1\frac{1}{4|g'|_m} \quad \text{(because of I.)}$$

Choose h so that $h\leq\frac{|\triangle u_i^0|_m 4|g'|_m}{A_1C_1}$. Then $|\triangle u_i^1|\leq|\triangle u_i^0|_m$ and by mathematical induction: $|\triangle u_i^n|\leq|\triangle u_i^0|_m$ and $|\triangle u_i^n|\leq\varepsilon$ if $i\geq I$. Therefore $|\triangle u_i^n|$ can be estimated independently of the rate $\frac{k}{h}$. We give the estimations of the second and third differences involved in C_1 later.

B) Assume that $\triangle u_i^n-\triangle u_{i+1}^n>0$; then a) $\triangle u_{i-1}^n-\triangle u_i^n>0$ or b) $\triangle u_{i-1}^n-\triangle u_i^n\leq 0$.

In the case a) if k and h are such that $|G|\leq|\triangle u_i^n-\triangle u_{i-1}^n|$ then similarly to A) we get:

$$|\triangle u_i^{n+1}|=|\triangle u_i^n+G|\leq|\triangle u_i^n+\triangle u_{i-1}^n-\triangle u_i^n| = |\triangle u_{i-1}^n|_m,$$

if $\frac{k}{h}\leq\frac{|Du_{i-1}^n|}{C_1}$. If $\frac{k}{h}>\frac{|Du_{i-1}^n|}{C_1}$ then we also proceed according to A).

In the case *b)* we utilize that the third difference is bounded (see later). So $|Du_i^n|\leq h|d^3u_i^n|_m$ that is $|\triangle u_i^n|\leq A_1h|d^3u_i^n|_m$. Choose h that

$$h\leq\frac{|\triangle u_i^0|_m}{A_1|d^3u_i^n|_m} \quad \text{then } |\triangle u_i^n|\leq|\triangle u_i^0|_m.$$

C) If $\triangle u_i^n = \triangle u_{i+1}^n$ then the estimation is trivial.

II/2: $\triangle u_i^n<0, G<0$; A) $\triangle u_{i+1}^n-\triangle u_i^n<0$. If k and h are such that $|G|\leq|\triangle u_{i+1}^n-\triangle u_i^n|$ then similarly to II/1 *A)*:

$$|\triangle u_i^{n+1}| = |\triangle u_i^n + G|\leq|\triangle u_i^n+\triangle u_{i+1}^n-\triangle_i^n| = |\triangle u_{i+1}^n|.$$

B) $\triangle u_i^n-\triangle u_{i+1}^n<0$, then *a)* $\triangle u_{i-1}^n-\triangle u_i^n<0$ or *b)* $\triangle u_{i-1}^n-\triangle u_i^n\geq 0$. The method and the result are similar to II/1 *B)*. *C)* as above.

Consequently, if $\frac{k}{h^2}\leq\frac{1}{4|g'|_m}$,

$$h\leq\frac{4|g'|_m|\triangle u_i^0|_m}{A_1C_1} \quad\text{and}\quad h\leq\frac{|\triangle u_i^0|_m}{A_1|d^3u_i^n|_m},$$

then for the scheme (2.1): $|\triangle u_i^n|\leq|\triangle u_i^0|_m$ for all i and n.

If u_i^{*n} takes over the role from u_{i+1}^n then by using the above method we obtain the stability in the following sense:

$$|u_i^n-u_i^{*n}|\leq|u_i^0-u_i^{*0}|_m.$$

2.1/b. In this point we get such an estimation for $|\triangle u_i^n|$ in which there is no limitation for h.

We obtained in 2.1/a :

$$\triangle u_i^{n+1} = \triangle u_i^n + \frac{k}{h^2}[h\delta^2 g(u_i^n) \triangle u_{i+1}^n \triangle u_i^n - h\delta^2 g(u_{i-1}^n) \triangle u_i^n \triangle u_{i-1}^n +$$

$$+ \delta g(u_i^n)(\triangle u_{i+1}^n - \triangle u_i^n) - \delta g(u_{i-1}^n)(\triangle u_i^n - \triangle u_{i-1}^n)].$$

First estimate the right side of this equality by $|\triangle u_i^n|_m$ using that $g'>0$ (and so $\delta g(u)>0$) and $\frac{k}{h^2}<\frac{1}{4|g'|_m}$.

$$\triangle u_i^n + \frac{k}{h^2}(\delta g(u_i^n)\triangle u_{i+1}^n - (\delta g(u_i^n)+\delta g(u_{i-1}^n))\triangle u_i^n + \delta g(u_{i-1}^n)\triangle u_{i-1}^n) \leq$$

$$\leq |\triangle u_i^n|_m\left[1+\frac{k}{h^2}(\delta g(u_i^n)-\delta g(u_i^n)+\delta g(u_{i-1}^n)-\delta g(u_{i-1}^n))\right] = |\triangle u_i^n|_m.$$

So we get:

$$|\triangle u_i^{n+1}| \leq |\triangle u_i^n|_m + \frac{k}{h^2}[h\delta^2 g(u_i^n)\triangle u_{i+1}^n \triangle u_i^n -$$

$$- h\delta^2 g(u_i^{i-1})\triangle u_i^n \triangle u_{i-1}^n],$$

$$|\triangle u_i^{n+1}| \leq |\triangle u_i^n|_m + \frac{C}{N}|\triangle u_i^n|^2 \tag{2.1.2}$$

where $\frac{C}{N} = \frac{2k}{h^2}|g''|_m h$, i.e. $C = 2\lambda|g''|_m\frac{A}{2}$ $\left(\lambda = \frac{k}{h^2},\ h = \frac{A}{2N}\right)$.

In (2.1.2) $|\triangle u_i^{n+1}|_m$ increases in n. Let K be a given constant such that $|\triangle u_i^0|_m < K$. We can compute all n $(n \leq n_1)$ such that $|\triangle u_i^n| \leq K$. Since

$$|\triangle u_i^{n+1}|_m \leq |\triangle u_i^n|_m\left(1+\frac{C}{N}K\right) \leq |\triangle u_i^{n-1}|_m\left(1+\frac{C}{N}K\right)^2 \leq \ldots \leq$$

$$\leq |\triangle u_i^0|_m\left(1+\frac{C}{N}K\right)^{n+1}$$

thus we find

$$|\triangle u_i^0|_m e^{\frac{n}{N}CK} \leq K.$$

We get a greater n_1 if we use the expression (2.1.2) succesively from $n = 0$ to a suitable n_1 such that

$$|\triangle u_i^{n_1}|_m \leq K \text{ but } |\triangle u_i^{n_1+1}|_m > K.$$

The proof of stability is entirely similar, (now $\bar{u}_i^{*n}$ takes over the role from u_{i+1}^n). So we get:

$$|u_i^n - \bar{u}_i^{*n}|_m \leq |u_i^0 - \bar{u}_i^{*0}|_m e^{\frac{n}{N}CK} .$$

After the following estimation of Du_i^n in 2.1/c we give such an estimation for $\triangle u_i^n$ which can be applied for all n.

The estimation of Du_i^n. Let

$$\delta^3 g(u_i^n) = \frac{\delta^2 g(u_{i+1}^n) - \delta^2 g(u_i^n)}{u_{i+1}^n - u_i^n} = g^{(3)}(\tilde{\tilde{u}}_i^n) \quad (u_i^n \leq \tilde{\tilde{u}}_i^n \leq u_{i+1}^n).$$

From (2.1), we deduce:

$$Du_i^{n+1} = Du_i^n + \frac{k}{h^2}\{\delta^2 g(u_{i+1}^n)(\triangle u_{i+1}^n)^2 + \delta g(u_{i+1}^n) Du_{i+1}^n -$$

$$-2(\delta^2 g(u_i^n)(\triangle u_i^n)^2 + \delta g(u_i^n) Du_i^n) + \delta^2 g(u_{i-1}^n)(\triangle u_{i-1}^n)^2 +$$

$$+\delta g(u_{i-1}^n) Du_{i-1}^n\} \leq [1 + \lambda(\delta g(u_{i+1}^n) - 2\delta g(u_i^n) +$$

$$+\delta g(u_{i-1}^n))] |Du_i^n|_m + \lambda\{\delta^3 g(u_i^n)(\triangle u_{i+1}^n)^2 \triangle u_i^n +$$

$$+h\delta^2 g(u_i^n) Du_i^n(\triangle u_{i+1}^n + \triangle u_i^n) - (h\delta^3 g(u_{i-1}^n)(\triangle u_i^n)^2 \triangle u_{i-1}^n +$$

$$+h\delta^3 g(u_{i-1}^n) Du_{i-1}^n(\triangle u_i^n + \triangle u_{i-1}^n))\},$$

$$|Du_i^{n+1}| \leq |1 + \lambda(h\delta^2 g(u_i^n)\triangle u_i^n - h\delta^2 g(u_{i-1}^n)\triangle u_{i-1}^n +$$

$$+4h|\delta^2 g||\triangle u_i^n|_m)||Du_i^n| + hC_2 \leq (1 + C_1 h)|Du_i^n|_m + hC_2,$$

where $C_1 = 6\lambda\ C_3 C_4$, $C_2 = 2\lambda C_5 C_3^3$, and

$$C_3 = \max_{i,n} |\triangle u_i^n|,\ C_4 = \max_{i,n} |\delta^2 g(u_i^n)| \leq |g''|_m,\ C_5 =$$

$$= \max_{i,n} |\delta^3 g(u_i^n)| \leq |g^{(3)}|_m.$$

Furthermore:

$$|Du_i^{n+1}| \leq (1 + C_1 h)((1 + C_1 h)|Du_i^{n-1}|_m + hC_2) + hC_2,$$

and by mathematical induction

$$\left(h = \frac{A}{2N}\right): |Du_i^{n+1}| \leq (1 + C_1 h)^n |Du_i^0|_m + hC_2 \frac{(1 + C_1 h)^n - 1}{C_1 h} \leq e^{\frac{AC_1}{2}} |Du_i^0|_m +$$

$$+\frac{C_2}{C_1}|e^{\frac{AC_1}{2}} - 1| \equiv C_6$$

Let

$$C_7 = \max |\delta g(u_i^n)|, \; C_8 = \max |Du_i^0| \text{ thus } |Dg(u_i^n)| \leq$$
$$\leq C_4 C_3^2 + C_7 C_8 \equiv C$$

If we take

$$|h\delta^3 g(u_i^n)(\triangle u_{i+1}^n)^2 \triangle u_i^n - h\delta^3 g(u_{i-1}^n)(\triangle u_i^n)^2 \triangle u_{i-1}^n| \leq$$
$$\leq h[\delta^4 g(u_i^n)|(\triangle u_i^n)^3 + |\delta^3 g(u_i^n|_m |Du_i^n|_m]$$

into consideration then we can improve the estimation of Du_i^n in the case of great C_2:

$$|Du_i^{n+1}| \leq (1+\overline{C}_1 h)|Du_i^n|_m + h\overline{C}_2,$$

where

$$\overline{C}_1 = \lambda(6C_3C_4 + 3hC_3^2) \text{ and } \overline{C}_2 = \lambda h |g^{(4)}{}_m| C_3^3.$$

If $h \leq |Du_i^n|$ for all i and n then $|Du_i^{n+1}| \leq (1+\overline{\overline{C}}_1 h)|Du_i^n|_m$ (where $\overline{\overline{C}}_1 = \overline{C}_1 + \overline{C}_2/\mathrm{h}$), otherwise $|Du_i^{n+1}| \leq h$.

2.1/c. Form the inequality obtained in 2.1/b we have

$$|\triangle u_i^{n+1}| \leq |\triangle u_i^n|_m \frac{k}{h^2} h[\delta^2 g(u_{i+1}^n) \triangle u_{i+1}^n \triangle u_i^n -$$
$$- \delta^2 g(u_i^n) \triangle u_i^n \triangle u_{i-1}^n] \leq |\triangle u_i^n|_m + h\lambda[2h|\delta^2 g(u_i^n) Du_i^n \triangle u_i^n|_m +$$
$$+ h|\delta^3 g(u_i^n)(\triangle u_i^n)^3|_m].$$

Assume that

$$h \leq \frac{K|\triangle u_i^n|_m}{2|\delta^2 g(u_i^n)|_m |Du_i^n|_m |\triangle u_i^n|_m + |\delta^3 g(u_i^n)|_m^3 |\Delta u_i^n|_m^3},$$

where K is a constant. Then

$$|\triangle u_i^{n+1}| \leq |\triangle u_i^n| + h\lambda K |\triangle u_i^n|_m = |\triangle u_i^n|_m \left(1 + \frac{A}{2N}\lambda K\right) \leq |\triangle u_i^0|_m e^{\frac{\lambda AK}{2}}$$

We can use this estimation of $|\triangle u_i^{n+1}|$ in the previous estimation of Du_i^n. Now the condition on h does not contain $|d^3 u_i^n|$ as that in 2.1/a.

The estimation of $d^3 u_i^n = \dfrac{Du_{i+1}^n - Du_i^n}{h}$. The method is similar to the estimation of Du_i^n. So the result is:

$$|d^3 u_i^{n+1}| \leq e^{\frac{AC_1^{(3)}}{2}\frac{n}{N}} |d^3 u_i^0|_m + \frac{C_2^{(3)}}{C_1^{(3)}} |e^{\frac{AC_1^{(3)}}{2}\frac{n}{N}} - 1|.$$

where

$$C_2^{(3)} = 2\lambda(|\delta^4 g|_m |\triangle u_i^n|_m^4 + 7|\delta^3 g|_m |\triangle u_i^n|_m^2 |Du_i^n|_m + 3|\delta^2 g|_m |Du_i^n|_m^2),$$

$$|\delta^4 g|_m = \max_{|u|<U} |g^{(4)}(u)|, \quad C_1^{(3)} = 8|\delta^2 g(u_i^n)|_m |\triangle u_i^n|_m \lambda.$$

The estimation of $d^4 u_i^n = \dfrac{d^3 u_{i+1}^n - d^3 u_i^n}{h}$ is also similar:

$$|d^4 u_i^{n+1}|_m \le e^{\frac{nAC_1^{(4)}}{2N}} |d^4 u_i^0|_m + \frac{C_2^{(4)}}{C_1^{(4)}} |e^{\frac{nAC_1^{(4)}}{2N}} - 1|,$$

where

$$C_2^{(4)} = 2\lambda\{|\delta^5 g|_m(|\triangle u_i^n|_m)^5 + 14|\delta^4 g|_m |Du_i^n|_m (|\triangle u_i^n|_m)^3 +$$
$$+ 6|\delta^3 g|_m (|Du_i^n|_m)^2 4(|\triangle u_i^n|_m)^2 + 6|\delta^3 g|_m (|\triangle u_i^n|_m)^3 |d^3 u_i^n|_m +$$
$$+ 3|\delta^3 g|_m |\triangle u_i^n|_m (|Du_i^n|_m)^2 + 14|\delta^2 g|_m |d^3 u_i^n|_m |Du_i^n|_m +$$
$$+ 4|\delta^3 g|_m |\triangle u_i^n|_m^2 |d^3 u_i^n|_m\}$$
$$C_1^{(4)} = 10|\delta^2 g|_m |\triangle u_i^n|\lambda, \quad |\delta^5 g|_m = \max_{|u|<U} |g^{(5)}(u)|.$$

So the proof is complete. □

Now we establish similar numerical schemes for the equations $u_t = B_1 u$, $u_t = B_0 u$ and we get similar estimations for the differences of the solution and for the stability. For the original equation (1.1) the numerical scheme is:

$$u_i^{n,0} \equiv u_i^n,$$

(2.2)
$$\frac{u_i^{n,1} - u_i^{n,0}}{k} = \overline{B}_2 u_i^{n,0},$$

$$\frac{u_i^{n,2} - u_i^{n,1}}{k} = \overline{B}_1 u_i^{n,1},$$

$$\frac{u_i^{n+1} - u_i^{n,2}}{k} = \overline{B}_0 u_i^{n,2}, \text{ where } \overline{B}_2 u_i^{n,0} = \frac{g_2(u_{i+1}^{n,0}) - 2g_2(u_i^{n,0}) + g_2(u_{i-1}^{n,0})}{h^2},$$

$$\overline{B}_1 u_i^{n,1} = \frac{g_1(u_{i+1}^{n,1}) - g_1(u_i^{n,1})}{h}, \quad \overline{B}_0 u_i^{n,2} = g_0(u_i^{n,2}).$$

We get estimations for the differences of the solutions of the equations $u_t = B_1 u$ and $u_t = B_0 u$ more simply than of the equation $u_t = B_2 u$.

2.2. In the case of the equation $u_t = (g_1(u))_x$ the estimation of $\triangle u_i^n$ and $|u_i^n - u_i^{*n}|$ is the same as in the case of the equation $u_t = (g_2(u))_{xx}$, only that C is the half of that in 2.1/b.

Furthermore C_1, $C_1^{(3)}$, $C_1^{(4)}$ and C_2, $C_2^{(3)}$, $C_2^{(4)}$ are multiplied by $\frac{1}{2}$ in the case of $|Du_i^{n+1}|$, $|d^3u_i^{n+1}|$, $|d^4u_i^{n+1}|$ and $\lambda = \frac{k}{h}$. Thus we find the estimations: $\left(\frac{k}{h} \leq \frac{1}{|g_1'|_m}\right)$

$$|\triangle g_1(u_i^n)| \leq C_7 |\triangle u_i^0|_m, \quad |Dg_1(u_i^{n+1})| \leq C_4 C_3^2 + C_7 C_6.$$

2.3. Consider the equation $u_t = g_0(u)$. Suppose that $|g_0(u) \leq K_0 u$. Then for the scheme $u_i^{n+1} = u_i^n + kg_0(u_i^n)$ we obtain $|u_i^n| \leq e^{\frac{n}{N}TK_0} |u_i^0|_m$.

If $g_0(u) \leq 0$ and t is not too great, then $|u_i^{n+1}| \leq |u_i^0|_m$.

If $u_i^{n+1} = u_i^n + kg_0(u_i^n)$ then:

$$|\triangle u_i^n| \leq e^{\frac{n}{N}T|g'_0|_m} |\triangle u_i^0|_m, \quad |u_i^n - \overline{u}_i^{*n}| e^{\frac{n}{N}T|g'_0|_m} |u_i^0 - \overline{u}_i^{*0}|,$$

$$|Du_i^n| \leq |Du_i^0| e^{\frac{n}{N}T|g'_0|_m} + |g''_0|_m |\triangle u_i^n|_m^2 \frac{|e^{\frac{n}{N}T|g'_0|_m} - 1}{|g'_0|_m},$$

$$|d^3u_i^n| \leq |d^3u_i^0|_m e^{\frac{n}{N}T|g'_0|_m} + C^{\rm III} \frac{|e^{\frac{n}{N}T|g'_0|_m} - 1|}{|g'_0|_m},$$

where

$$C^{\rm III} = |g^{(3)}_0|_m |\triangle u_i^n|_m^3 + 3|\triangle u_i^n|_m |Du_i^n|_m |g_0''|_m,$$

further

$$|d^4u_i^n| \leq |d^4u_i^0|_m e^{\frac{n}{N}T|g'_0|_m} + C^{\rm IV} \frac{|e^{\frac{n}{N}T|g'_0|_m} - 1|}{|g'_0|_m}.$$

where

$$C^{\rm IV} = |g_0^{(4)}|_m |\triangle u_i^n|_m^4 + 6|\triangle u_i^n|_m^2 |Du_i^n|_m |g_0^{(3)}|_m + 3|Du_i^n|^2 |g_0''|_m +$$
$$+ 4|\triangle u_i^n|_m |d^3u_i^n|_m |g_0''|_m.$$

The constants mean quantities similar to those of the second order equations.

2.4. Theorem 2. *Applying the methods of 2.1/a, 2.1/b to (2.2), by using* 2.2 *and* 2.3 *we find that the following estimations are valid for the numerical scheme* (2.2):

(2.4.0/a) $\quad |u_i^n| \leq e^{\frac{n}{N}TK_0} |u_i^0|_m$, *where* $|g_0(u)| < K_0 |u|$ *or:*

(2.4.0/b) $\quad |u_i^n| \leq |u_i^0|_m$ if $g_0(u) \leq 0$ *and* $k < \dfrac{|u_i^n|}{|g_0(u_i^n)|_m}$.

Further

$$|\triangle u_i^n|_m \leq e^{\frac{n}{N}T|g'_0|_m} |\triangle u_i^0|_m,$$

(2.4.1/a)
$$|u_i^n - \bar{u}_i^{*n}|_m \leq e^{\frac{n}{N}T|g'_0|_m} |u_i^0 - \bar{u}_i^{*0}|_m.$$

and as in 2.1/*b*:

$$|\triangle u_i^n| \leq E|\triangle u_i^0|_m \leq K \text{ for } n \leq n_1, \; |u_i^n - \bar{u}_i^{*n}| \leq E|u_i^0 - \bar{u}_i^{*0}|_m \leq K_h^*$$

(2.4.1/b)

$$\textit{for } n \leq n_1^*; \; \textit{where } |\triangle u_i^0|_m \leq K; \; |u_i^0 - \bar{u}_i^{*0}|_m \leq K_h^*,$$

$$E = e^{\frac{n}{N}\left(\lambda A \left|\frac{\partial^2}{\partial u^2}g_2(u)\right|_m + \frac{k}{h}\frac{A}{2}\left|\frac{\partial^2}{\partial u^2}g_1(u)\right|_m + T\left|\frac{\partial}{\partial u}g_0(u)\right|_m\right)},$$

where $(l = 1, 2; k = 0, 1, 2)$

$$\left|\frac{\partial^l}{\partial u^l}g_k(u)\right|_m = \max_{|u|<U}\left|\frac{\partial^l}{\partial u^l}g_k(u)\right|.$$

Further

(2.4.2)
$$|Du_i^n| \leq e^{\frac{n}{N}\left(\frac{AE_2}{2} + \frac{AE_1}{2} + TE_0\right)} |Du_i^0|_m + \frac{F_m}{E_m}|e^{3A_mE_m\frac{n}{N}} - 1|.$$

where:

$$E_2 = 6\lambda C_{4_2}C_{3_2}, \; E_1 = 3\frac{k}{h}C_{4_1}C_{3_1}, \; E_0 = C_{7_0}.$$

$$E_m = \max(E_2, E_1, E_0), \; F_m =$$

$$= \max(F_2, F_1, F_0), \; A_m = \max\left(\frac{A}{2}, T\right)$$

$$F_2 = 2\lambda C_{5_2}C_{3_2}^3, \; F_1 = \frac{k}{h}C_{5_1}C_{3_1}^3, \; F_0 = C_{4_0}C_{3_0}^2;$$

(2.4.3)
$$|d^3u_i^n| \leq e^{\frac{n}{N}\left(\frac{AE_2}{2} + \frac{AE_1}{2} + TE_0\right)} |d^3u_i^0|_m + \frac{F_m}{E_m}|e^{3A_mE_m\frac{n}{N}} - 1|.$$

where:

$$E_2 = 8\lambda C_{4_2}C_{3_2}, \; E_1 = 4\frac{k}{h}C_{4_1}C_{3_1}, \; E_0 = C_{7_0}.$$

$$E_m = \max(E_2, E_1, E_0), \; F_m = \max(F_2, F_1, F_0), \; A_m =$$

$$= \max\left(\frac{A}{2}, T\right).$$

$$F_2 = 2\lambda C^{(3)}_{2_2}, \; F_1 = \frac{k}{h} C^{(3)}_{2_1}, \; F_0 = C^{\mathrm{III}}_0.$$

(2.4.4) $$|d^4 u_i^n| \leq e^{\frac{n}{N}\left(\frac{AE_2}{2}+\frac{AE_1}{2}+TE_0\right)} |d^4 u_i^0|_m + \frac{F_m}{E_m} |e^{3\frac{n}{N}A_m E_m} - 1|,$$

where

$$E_2 = 10\lambda C_{4_2} C_{3_2}, \; E_1 = 5\frac{k}{h} C_{4_2} C_{4_2}, \; E_0 = C_{7_0}$$

$$E_m = \max (E_2, E_1, E_0), \; F_m = \max (F_2, F_1, F_0), \; A_m = \max\left(\frac{A}{2}, T\right)$$

$$F_2 = 2\lambda C^{(4)}_{2_2}, \; F_1 = \frac{k}{h} C^{(4)}_{2_1}, \; F_0 = C^{\mathrm{IV}}_0.$$

We get the above estimations by using the induction steps on the layers $n,2$; $n,1$; $n,0$ one after the other succesively.

The method of estimation of $d^l u_i^n$ $(l > 4)$ is executed similarly.

3. In **1.** we approximated the original problem

(3.1) $u_t = [\Phi(u)]_{xx} - \Psi(u)$, $u(x, 0) = u^0(x)$ by the problem

(3.2) $u_t = [\Phi_1(u)]_{xx} - \Psi_1(u)$, $u(x, 0) = u_1^0(x)$ where $|\Phi(u) - \Phi_1(u)| \leq \varepsilon_\Phi$

$|\Psi(u) - \Psi_1(u)| \leq \varepsilon_\Psi$ if $|u| < U$ $(\Phi_1^{(l)}(u), \Psi_1^{(l)}(u) (l \leq 5)$

are bounded) and

$$\varepsilon_\Phi = 0\left(\frac{1}{N^2}\right), \; |u^0(x) - u_1^0(x)| \leq \varepsilon.$$

The scheme applied for (3.1) is:

(3.3) $$u_i^{n,1} = u_i^{n,0} + \frac{k}{h^2} (\Phi(u_{i+1}^{n,0}) - 2\Phi(u_i^{n,0}) + \Phi(u_{i-1}^{n,0})),$$

(3.3/*a*) $$u_i^{n+1} = u_i^{n,1} - k\Psi(u_i^{n,1}), \; u_i^{0,0} = u^0(ih), \; u_i^{n,0} \equiv u_i^n.$$

The scheme applied for (3.2) is:

(3.4) $$v_i^{n,1} = v_i^{n,0} + \frac{k}{h^2} (\Phi_1(v_{i+1}^{n,0}) - 2\Phi_1(v_i^{n,0}) + \Phi_1(v_{i-1}^{n,0})).$$

(3.4/*a*) $$v_i^{n+1} = v_i^{n,1} - k\Psi_1(v_i^{n,1}), \; v_i^{0,0} = u^0(ih), \; v_i^{n,0} \equiv v_i^n.$$

The difference of (3.3) and (3.4) is:

$$v_i^{n,1} - u_i^{n,1} = v_i^{n,0} - u_i^{n,0} + G$$

where

$$G = \frac{k}{h^2}\{\Phi_1(v_{i+1}^{n,0}) - \Phi_1(u_{i+1}^{n,0}) - 2(\Phi_1(v_i^{n,0}) - \Phi_1(u_i^{n,0})) +$$

$$+\Phi_1(v_{i-1}^{n,0}) - \Phi_1(u_{i-1}^{n,0}) + \Phi_1(u_{i+1}^{n,0}) - \Phi(u_{i+1}^{n,0}) - 2(\Phi_1(u_i^{n,0}) - \Phi(u_i^{n,0})) +$$

$$+\Phi_1(u_{i-1}^{n,0}) - \Phi(u_{i-1}^{n,0})\}.$$

Assume that

$$\lambda = \frac{k}{h^2} \le \frac{1}{4|\Phi_1'|_m}, \; \Phi_1' \ge 0.$$

If v_i^n and Φ_1 takes over the role from u_{i+1}^n and g then we can apply the method of $2.1/a$:

$$G \le \frac{k}{h^2}\{\delta\Phi_1(u_{i+1}^{n,0})(v_{i+1}^{n,0} - u_{i+1}^{n,0}) - 2\delta\Phi_1(u_i^{n,0})(v_i^{n,0} - u_i^{n,0}) +$$

$$+\delta\Phi_1(u_{i-1}^{n,0})(v_{i-1}^{n,0} - u_{i-1}^{n,0}) + 4\varepsilon_\Phi\}.$$

Therefore it follows similarly as in $2.1/a$

$$|v_i^{n,1} - u_i^{n,1}| \le |v_i^n - u_i^n| + 4\lambda\varepsilon_\Phi .$$

The difference of $(3.3/a)$ and $(3.4/a)$ is:

$$v_i^{n+1} - u_i^{n+1} = v_i^{n,1} - u_i^{n,1} - k[\Psi_1(v_i^{n,1}) - \Psi_1(u_i^{n,1}) + \Psi_1(u_i^{n,1}) - \Psi(u_i^{n,1})].$$

Assume that $\Psi'(u) \ge 0$, $|\Psi(u) - \Psi_1(u)| \le \varepsilon_\Psi$ if $|u| \le U$, $k \le \frac{1}{|\Psi_1'|_m}$. Then it follows: $|v_i^{n+1} - u_i^{n+1}| \le |v_i^n - u_i^n| + 4\lambda\varepsilon_\Phi + k\varepsilon_\Psi$ and by mathematical induction:

$$|v_i^{n+1} - u_i^{n+1}| \le |v_i^1 - u_i^1| + t_n\varepsilon_\Psi + nO\left(\frac{1}{N^2}\right)$$

Apply the initial function $u_1^0(ih)$ instead of $u^0(ih)$ to the scheme (3.4). Denote by $\bar{v}_i^n$ the solution in this case. Then because of $|u^0(x) - u_1^0(x)| \le \varepsilon$ we have

$$|\bar{v}_i^n - u_i^n| \le |\bar{v}_i^n - v_i^n| + |v_i^n - u_i^n| \le \varepsilon E + |v_i^1 - u_i^1| + t_n\varepsilon_\Psi + nO\left(\frac{1}{N^2}\right)$$

(see E in $2.4.1/b$).
If $n, N \to \infty$, respectively $h \to 0$ $(k \to 0)$ then $\varepsilon_\Psi \to 0$, $\varepsilon \to 0$. Applying the method $2.1/b$ and $2.1/c$, we get similar results.

One can obtain similar estimations for $\bar{v}_i^n$, $\triangle_x\bar{v}_i^n$ and $\triangle_t\bar{v}_i^n$ as in (2.2). These estimations in absolute value norm are valid in weak sense, too, so we can apply Theorem 6.1 of [2].

Theorem 3. *Let $\{h\}$ be an infinite sequence which converges to zero. Assume that the conditions for $\frac{k}{h^2}\left(\text{or } \frac{k}{h}\right)$ are satisfied for each h. Then there exist a subsequence of $\{h\}$ such that $\bar{v}_i^n$ converges uniformly in any subdomain $G\subset H$ to a solution U of problem* (1.0).

We can obtain limitations for $\triangle_x^{(l)}\bar{v}_i^n(l\leq 4)$ and $\triangle_t^{(m)}\bar{v}_i^n(m\leq 2)$ for (3.2). So we get similarly to Theorem 3:

Theorem 4. *$\triangle_x\bar{v}_i^n\to U_x(x, t)$ uniformly, $\triangle_{xx}\bar{v}_i^n\to U_{xx}(x, t)$ uniformly, $\triangle_t\bar{v}_i^n\to U_t(x, t)$ uniformly. If we know the uniqueness then the convergence of the complete sequence follows.*

Proof. The convergence of $\triangle_x\bar{v}_i^n$ follows from the boundedness of $|\triangle_{xx}\bar{v}_i^n|$ and $|\triangle_{xt}\bar{v}_i^n|$, similarly the convergence of $\triangle_{xx}\bar{v}_i^n$ from the boundedness of $|\triangle_{xxx}\bar{v}_i^n|$ and $|\triangle_{xxt}\bar{v}_i^n|$, and the convergence of $\triangle_t\bar{v}_i^n$ from the boundedness of $|\triangle_{tx}\bar{v}_i^n|$ and $|\triangle_{tt}\bar{v}_i^n|$.

From the equation $u_t = [\Phi(u)]_{xx}-\Psi(u)$ we get:

$$u_t = \Phi_{uu}u_x^2-\Phi_u u_{xx}-\Psi,$$

$$u_{tx} = \Phi_u^{(3)}u_x^2+3u_xu_{xx}\Phi_{uu}+\Phi_u u_x^{(3)}-\Psi_u u_x,$$

$$u_{txx} = \Phi_u^{(4)}u_x^3+5\Phi_u^{(3)}u_xu_{xx}+3u_{xx}^2\Phi_{uu}+4u_xu_x^{(3)}\Phi_{uu}+u_x^{(4)}\Phi_u-$$

$$-\Psi_{uu}u_x^2-\Psi_u u_{xx},$$

$$u_{tt} = \Phi_u^{(3)}u_x^2u_t+2u_xu_{xt}\Phi_{uu}+\Phi_{uu}u_{xx}u_t+u_{xxt}\Phi_u-\Psi_u u_t.$$

The boundedness of the expressions on the left sides result from the boundedness of the expressions on right sides.

So the solution of (3.2) (respectively 1.1) exists in strong sense, too, and is arbitrarily near to the solution of (3.1) (respectively (1.0)).

We deal with the difference of the exact and the approximate solution, the rate of the convergence and the uniqueness in other papers.

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