ON PROPERTIES OF NETS FOR MODELLING OF SYSTEMS AND GENERALIZED PROCESSES

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The analysis and synthesis of discrete dynamic systems require knowledge of interdependences between structural properties of systems and processes generated by the systems. To study these interdependences it is convenient to specify both systems and processes in terms of the same formalism – in particular, in terms of nets. Such an approach has been initiated by Petri's paper [1] and further developed in a chain of works [2, 3, 4, 5] which formally define parallel processes and some related characteristic properties of nets specifying the processes. This paper continues the chain. Its aim is to generalize the notion of process allowing it to contain alternative actions which mutually exclude each other. The structural properties of nets specifying such generalized parallel (and sequential) processes with alternatives are studied.

1. We assume that an abstract system consists of *events*, *conditions* and dynamic relations between these system elements. The system can generate (abstract) *processes* which consist of process elements called *actions* (event occurrences) and *condition changes* and relations between the elements. In "traditional" definitions of a concurrent process it is assumed that each process element (action or condition change) is unique and occurs in the process exactly once. In our generalization of a process we allow it to contain elements which, though listed as process elements, can occur or can be omitted (in favour of other, alternative elements).

The type of a process is defined by the type of relations which can occur between the process elements. All these relations are derivatives of a basic relation $<$: $x<y$ can be interpreted as "if both x and y occurs in a process then x occurs earlier than y".

Thus, a process is a pair (X, R), where X is a set of elements, R is a finite set of relations in X. Any pair of distinct elements of X belongs precisely to one of the relations of R.

A process (X, R) is *sequential* iff all its elements occur and $R = \{li\}$, where li is a relation of *succession:*

$$x\, li\, y \Leftrightarrow (x < y \vee y < x) \vee (x = y).$$

A process is *concurrent* iff all its elements occur and $R = \{li, co\}$, where co is a relation of *concurrency:*

$$x\, co\, y \Leftrightarrow (\neg (x < y) \wedge \neg (y < x)) \vee (x = y).$$

A process is *sequential-alternative* iff $R = \{li, al\}$, where al is a relation of *alternative:*

$x\, al\, y \Leftrightarrow (x$ occurs $\Rightarrow y$ is omitted$) \wedge (y$ occurs $\Rightarrow x$ is omitted$)$.

A process is *concurrent-alternative* iff $R = \{li, co, al\}$, where the relations li, co, al are defined as above.

2. A *net* is a triple (P, T, F) where P is a non-empty set of *places*, T is a non-empty set of *transitions* and $F \subseteq P \times T \cup T \times P$ is an *incidence relation.* The following conditions are valid for the nets ($X = P \cup T$ is a set of all net elements: transitions and places):

A1. $P \cap T = \emptyset$.
A2. $(F \neq \emptyset) \wedge (\forall x \in X, \forall y \in X: xFy \vee yFx)$, i.e. every element is incidental to at least one element of another type.
A3. $\forall p_1, p_2 \in P: (p_1^{\bullet} = p_2^{\bullet} \wedge {}^{\bullet}p_1 = {}^{\bullet}p_2) \Rightarrow p_1 = p_2$,

where

${}^{\bullet}x = \{y | yFx\}$ is a set of *input elements* for x,
$x^{\bullet} = \{y | xFy\}$ is a set of *output elements* for x.

A Petri net is $N = (P, T, F, M_0)$ where (P, T, F) is a finite net (X is finite) and $M_0 : P \rightarrow \{0, 1, 2, \ldots\}$ is an *initial marking.*

We omit well-known definitions of transition firings, firings sequences, reachable markings, etc. One can find these definitions in [6] or other books and papers on Petri nets.

In a Petri net, modelling discrete system, transitions correspond to system events and places correspond to conditions.

3. Now we introduce nets which will serve as syntactical forms of processes. In a Petri net, specifying a process, transitions correspond to process actions and places correspond to condition changed.

The following additional restrictions will be general for all types of nets representing processes considered below.

Let $H(N) = \{p | p \in P \wedge {}^{\bullet}p = \emptyset\}$ be a set of *head places* of net N, $G(N) = \{p | p \in P \wedge p^{\bullet} = \emptyset\}$ be a set of *tail places* of N.

An ordered sequence of net elements $x_1, x_2, \ldots$ is called a *path* $D(x_1)$ from x_1, if $\forall i \geq 1: x_i F x_{i+1}$, and is called an *inverse path* $D^{-1}(x_1)$, if $\forall i \geq 1$, $x_i F^{-1} x_{i+1}$. A finite (inverse) path $(x, \ldots, y)$ is called an (inverse) *segment* and denoted by $D(x, y)$ $(D^{-1}(x, y))$.

A4. $\forall x, y \in X$: $(x \neq y \wedge xF^+y) \Rightarrow \neg(yF^+x)$, i.e. the net contains no loops.
A5. $(H(N) \neq \emptyset) \wedge (\forall x \in X, \ \forall D^{-1}(x)$: $D^{-1}(x)$ is finite).

This restriction demands that any net representing a process should have a non-emtpy set of head places and shoul not contain infinite inverse paths.

A6. $\forall t \in T$: $(^{\cdot}t \neq \emptyset \wedge t^{\cdot} \neq \emptyset)$, i.e. any transition has at least one input and one output place.

A7. $\forall p \in P: M_0(p) = \begin{cases} 1, \text{ if } p \in H(N), \\ 0, \text{ otherwise.} \end{cases}$

The process nets have standard initial marking: each head place contains one token, other places have no tokens.

Occurrence nets [1] (or *O*-nets) representing concurrent processes are nets (with a standard marking) which in addition to the conditions *A1*–*A7* will satisfy the following restrictions:

A8. $\forall p \in P$: $(|^{\cdot}p| \leq 1 \wedge |p^{\cdot}| \leq 1)$, i.e. every net place has only one input or output transition; all the places which do not belong to the set of head places or the set of tail places have one input and one output transition.

In the general case *O*-nets can be infinite. Any *O*-net is safe because of standard initial marking and restrictions upon the net topology, specified by conditions *A4*, *A5* and *A8*. An example of an *O*-net is shown in Figure 1.

This definition of occurrence net is a particular case of a more general definition given in [1] because of restrictions *A5* and *A6*.

Now we introduce nets for describing processes with alternative. A sequential-alternative net (or S-net) satisfies, in addition to *A1*–*A7*, the following restrictions:

A9. $|H(N)| = 1$, i.e. the net has only one head place.
A10. $\forall t \in T$: $(|^{\cdot}t| = 1) \wedge (|t^{\cdot}| = 1)$, i.e. any transition in the net has only one input and one output place.

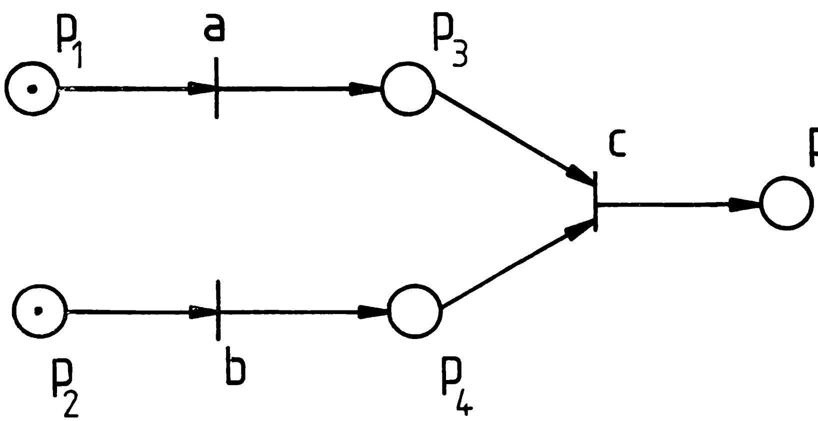

Fig. 1.

It also follows from the conditions *A9 – A10* that *S*-nets are safe and they represent a connected graph. An example of an *S*-net is shown in Figure 2.

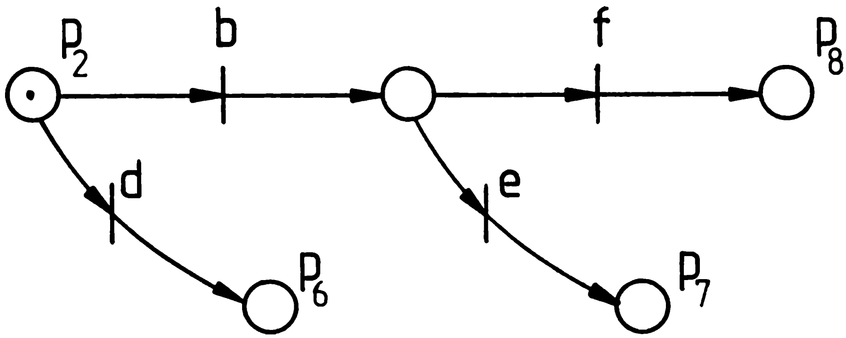

Fig. 2.

We will present concurrent-alternative processes with the help of *acyclic nets*, or *A*-nets, which satisfy axioms *A1 – A7* and on additional restriction *A11* guaranteeing the safeness of *A*-nets. The formal definition of the condition *A11* will be given below. An *A*-net transition can have more than one input and one output place and a place in its turn can be incidental to several transitions, An example of an *A*-net is shown in Figure 3. Note that *O*-nets and *S*-nets form particular subclasses of *A*-nets.

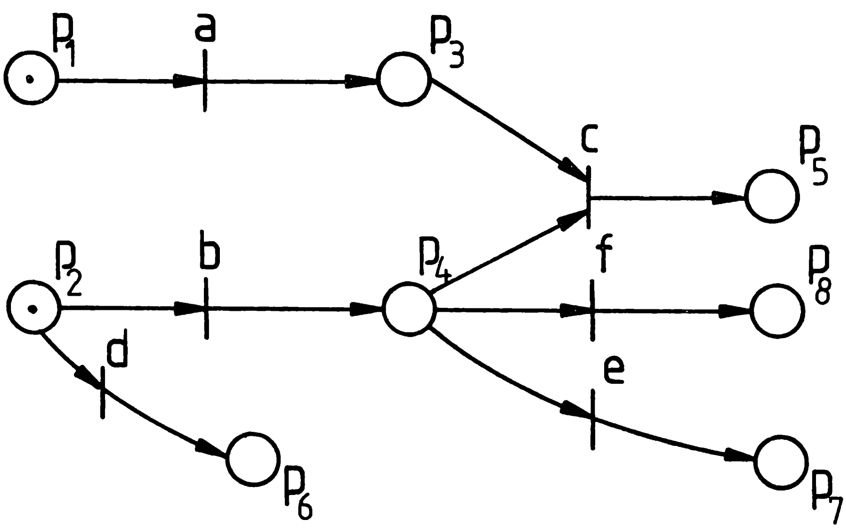

Fig. 3.

4. All the relations defined here for process net elements are introduced with the help of the basic relation *precedence* which is denoted by symbol $<$ and is defined for an arbitrary O-net N in the following way:

$$\forall x,\ y \in X : (x<y) \Leftrightarrow (x \neq y) \wedge xF^{+}y,$$

i.e. element x is previous to element y $(x<y)$, if x differs from y and there is a segment $D(x, y)$ in N. The relation of *succession li* in the O-net is defined as follows:

$$\forall x, y \in X : x\, li\, y \Leftrightarrow (x<y) \vee (x = y) \vee (y<x),$$

i.e. two elements x and y are in relation of succession iff they are equal or one of them precedes the other.

The relation of *concurrency co* for the O-net elements is defined in the following way:

$$x\, co\, y \Leftrightarrow \neg (x\, li\, y) \vee (x = y),$$

i.e. two elements are concurrent, iff they are equal or they are not bounded by the relation of succession. For example, Figure 1 shows $p_1 < p_3$, a *co* b, c *li* p_2. Due to the reflexivity of the relations *li* and *co* any element precedes itself and is concurrent with itself.

A set $L \subseteq X$ will be called a *li*-section, iff

(1) $\forall x, y \in L: x\, li\, y;$
(2) $\forall y \in X \backslash L,\ \exists x \in L : \neg (x\ li\ y).$

A set of $C \subseteq X$ will be called *co*-section iff

(1) $\forall x, y \in C : x\, co\, y;$
(2) $\forall y \in X \backslash C,\ \exists x \in C : \neg (x\ co\ y).$

Figure 1 shows a net with $\{p_1, a, p_3, c, p_5\}$ as a *li*-section, and $\{a, b\}$ as a *co*-section.

As proposed by Petri [1] the property of K-density for O-nets is a property which characterizes their adequacy as net description of concurrent processes.

An O-net is called *K-dense,* iff the intersection of any *li*-section and any *co*-section in the net contains exactly one element. The O-net shown in Figure 1 is K-dense; the O-net in Figure 4 is not K-dense, since the intersection of *li*-section $\{p_1, a, p_2, a_2, \ldots\}$ and *co*-section $\{b_1, b_2, b_3, \ldots\}$ is empty.

5. Sequential-alternative nets of S-nets are used when describing sequential-alternative processes. The precedence relation and the relation of succession *li* are defined in the same way as in the case of O-nets. The alternative relation *al* for S-net elements is defined in the following way:

$$\forall x, y \in X, x\, al\, y \Leftrightarrow \neg (x\, li\, y) \vee (x = y),$$

i.e. the elements x and y of an S-net are alternative iff they are equal or are not successive.

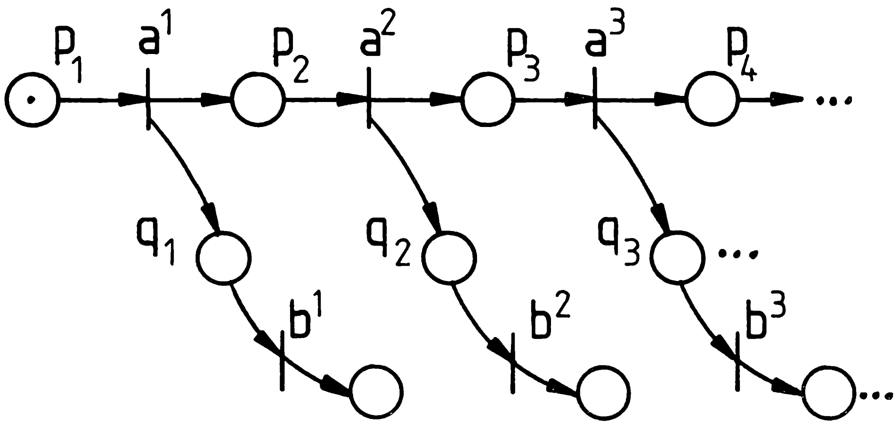

Fig. 4.

The definition of a *li*-section fully coincides with that of a *li*-section for *O*-nets.

The set $A \subseteq X$ is called an *al*-section iff

1. $\forall x, y \in A : x\, al\, y$;
2. $\forall y \in X \setminus A,\ \exists x \in A$: $\neg (x\ al\ y)$.

Figure 2 shows an example of an *S*-net in which the set $\{p_2, b, p_4, e, p_7\}$ is a *li*-section, $\{d, e, f\}$ is an *al*-section.

It follows immediately from the definitions of *li*- and *al*-sections that the intersection of any of the *li*- section and *al*-section of an *S*-net contains at most one element.

Similarly to the case of *O*-nets there arises the problem of *S*-net adequacy. If intepreted as descriptions of sequential-alternative processes they

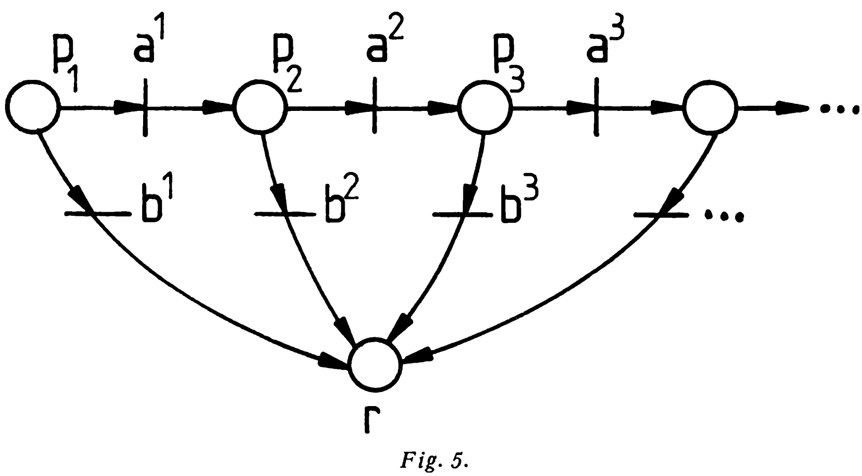

Fig. 5.

can be unacceptable as specifications of real "reasonable" processes. Among O-nets there exist K-dense O-nets which represent reasonable concurrent processes. For the same purposes the property of L-density is introduced for S-nets. The latter is defined in the following way: an S-net is *L-dense,* iff the intersection of any *li- al-* section pair contains at least one element. The S-net in Figure 2 is L-dense whereas the S-net in Figure 5 is not because its infinite *li*-section $\{p_1, a_1, p_2, a_2, \ldots\}$ does not intersect with the infinite *al*-section $\{b_1, b_2, b_3, \ldots\}$.

6. We will present concurrent-alternative processes by means of acyclic nets (or A-nets) introduced in Section 3.

It is impossible to define the alternative relation for places and transitions in a topologically uniform way since the natures of condition change and action are different. The condition may be changed if at least one input action occurs, i.e. the place will get a token if at least one its transition fires. The action occurs if all its input conditions have changed, i.e. the transition can fire if all its input places have got tokens.

Two transitions t_1 and t_2 of A-net are *alternative* iff

$$t_1 \text{ al } t_2 \Leftrightarrow ((\cdot t_1 \cap \cdot t_2 \neq \emptyset) \vee (\exists p_1 \in \cdot t_1 (\forall t_1' \in \cdot p_1 \colon t_1' \; al \; t_2)) \vee$$
$$\vee (\exists p_2 \in \cdot t_2 (\forall t_2' \in \cdot p_2 \colon t_1 \; al \; t_2')) \wedge \neg (t_1 \; li \; t_2).$$

Two places p_1 and p_2 are *alternative* iff

$$p_1 \, al \, p_2 \Leftrightarrow (p_1 \neq p_2) \wedge (\forall t_1 \in \cdot p_1, \; \forall t_2 \in \cdot p_2 \colon t_1 \, al \, t_2).$$

The place p and the transition t are *alternative* iff

$$p \, al \, t \Leftrightarrow (\forall t' \in \cdot p \colon t' \, al \, t) \wedge \neg (t \, li \, p).$$

Two A-net elements x and y are *concurrent* iff they are not connected by the relations of succession and alternative:

$$x \, co \, y \Leftrightarrow (x = y) \vee \neg (x \, li \, y \vee x \, al \, y).$$

For example, in the net shown in Figure 3 $p_1 \; co \; p_2$, $b \; al \; d$, and $p_2 \; li \; f$.

The class of acyclic nets is defined by means of above-mentioned conditions *A1 – A7* and the following restriction

A11. $\forall p \in P, \; \forall t_1, t_2 \in T : (t_1, t_2 \in \cdot p \vee t_1 \neq t_2) \Rightarrow t_1 \; al \; t_2.$

If the relations *li, co, al* are considered as "coordinate axes" of some three-dimensional space, then O-nets are in a plane formed by the axes *li* and *co* (there are no alternative elements). Structural restrictions for O-nets, which guarantee an adequate net representation of processes were formulated by means of the notions *li-* and *co*-sections (all of them have to intersect pairwise). Similarly, to represent sequential-alternative processes adequately it was required that all the *li-* and *al*-sections should intersect pairwise. For A-nets adequately formalizing concurrent-alternative processes the following requirements should be satisfied. First, K-density and L-density of its subnets, introduced below, and, second, another property, *M-density,* formulated in terms of the intersection of planes formed on the one hand, by *li*- and *co*-sections and, on other hand, by *li*- and *al*-sections.

The net $N' = (P', T', F')$ is called a *subnet* of the net $N = (P, T, F)$ iff $P' \subseteq P$, $T' \subseteq T$, $F' \subseteq F \cup (P' \times T' \cup T' \times P')$.

Remark. While defining a subnet we do not require it to satisfy condition *A3*; being different in the source net, places in a subnet may have the same set of incidental transitions. It is important however to remember that in the source net they were incidental to different transition sets.

A net N' is called an *O-subnet* of the *A*-net N, iff

1. N' is a subnet of N,
2. N' is *O*-net,
3. $\forall t \in T' : \{p \in P \mid pFt\} \subseteq P'$ and $\forall p \in P' : F'(p, t) = F(p, t)$,

i.e. transition t in the *O*-subnet N' has the same set of input places and the same arcs connecting it with these places as in the *A*-net N.

We will call an *O*-subnet N' of a net N *maximal* iff

1. for any *O*-subnet N'' of N we have $N'' \subseteq N'$;
2. all the head places of N' are head places of N, i.e. $H(N') \subseteq H(N)$.

The set of maximal *O*-subnets forms the set of all concurrent processes, generated by an *A*-net. The *O*-net shown in Figure 1 is a maximal *O*-subnet of the *A*-net shown in Figure 3.

A N'-net is called an *S*-subnet of an *A*-net N, iff

1. N' is a subnet of N;
2. N' is an *S*-net;
3. $\forall p \in P' : \{t \mid tFp\} \subseteq T'$, $\forall t \in T' : F'(t, p) = F(t, p)$, i.e. any place p in N' has the same set of input transitions and the same arcs connecting it with these places as in the net N.

An *S*-subnet N' of an *A*-net N will be called a maximal *S*-subnet iff

1. for any *S*-subnet N'' of N we have $N'' \subseteq N'$;
2. the head place $H(N')$ belongs to the set $H(N)$ of head places of N.

Remark. This definition of an *S*-subnet is valid for *A*-nets which can be represented as superpositions of *S*-nets, where the superposition operation "," is defined as follows:

Let $N_1 = (P_1, T_1, F_1)$ be $N_2 = (P_2, T_2, F_2)$, then $N = (N_1, N_2) = (P_1 \cup P_2, T_1 \cup T_2, F_1 \cup F_2)$.

In the general case item 3 in the definition of *S*-subnets is recorded somewhat differently.

The *S*-net shown in Figure 2 is a maximal *S*-subnet of the *A*-net shown in Figure 3.

We will call an *A*-net *K-dense*, iff all its maximal *O*-subnets are *K*-dense, and an *A*-net will be called *L*-dense iff all its maximal *S*-subnets are *L*-dense.

We will call an *A*-net N *M-dense* iff the intersection of any maximal *S*-subnet of N with any maximal *O*-subnet of net N results in some (unique) *li*-section of a net N.

The A-net shown in Figure 3 is not M-dense for the intersection of its maximal O-subnet shown in Figure 1 with the maximal S-subnet in Figure 2 results in the set $L = \{p_1, a, p_3\}$ which is not a *li*-section of the A-net. The A-net shown in Figure 6 is neither K-, nor L-, nor M-dense.

The following assertion illustrates the adequacy of A-nets as net representations of concurrent-alternative processes.

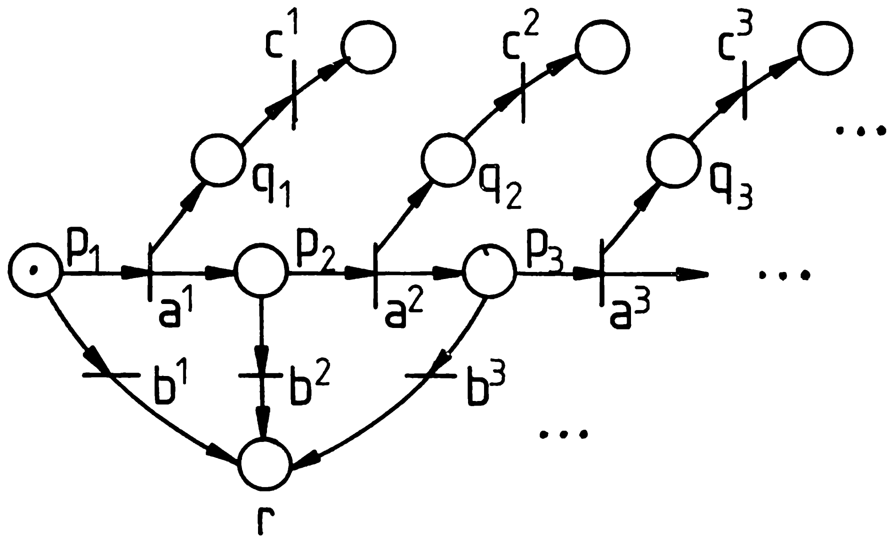

Fig. 6.

We will call a net *correct* iff for any reachable dead marking M (i.e. none of the net transitions can fire at M) and for any place p not belonging to the set of tail places, $M(p) = 0$. Note that in the general case there can be infinite A-nets without tail places and respectively without dead markings. In these cases they are considered to be correct.

Theorem 1. *Any S-net and any O-net are correct.*

Theorem 2. *Any M-dense net is correct.*

7. After we have introduced some properties characterizing adequate net interpretations of generalized processes, we want to extend these properties to the case of Petri nets generating these processes. In such a way we distinguish adequate nets modelling "reasonable" systems. For this purpose we should establish some correspondence between Petri nets and A-nets describing their functioning. Also, we would like to find one-to-one correspondence, – to assign to any Petri net a unique generalized process describing its functioning (unfolded process net). First of all, note that we can assign to any acyclic Petri net without loops and with standard initial markings (each head place and only a head place possesses one token), a net-process which is identical with this net. For example, the net shown in Figure 7b generates an occurrence net shown in Figure 4; the net shown in Figure 7a generates an S-net shown in Figure 5; the net shown in Figure 7c generates an A-net shown in Figure 6.

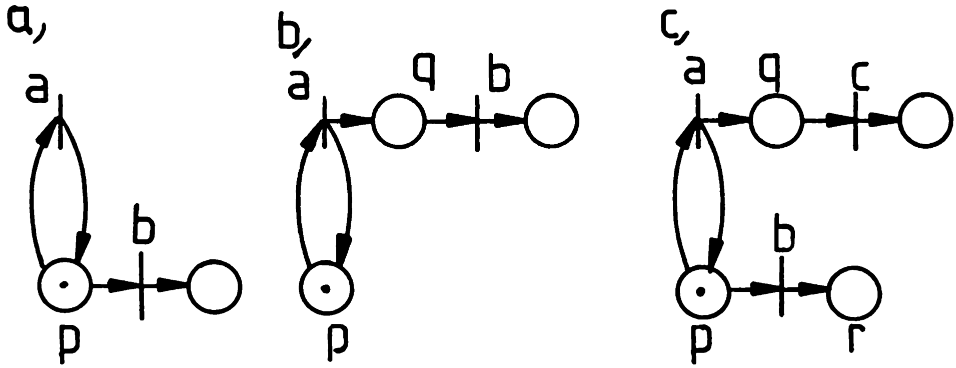

Fig. 7.

Several of the following relations demonstrate the correspondence between a Petri net N and its unfolded process net $\hat{N}$.

1. Let pr^{-1} denote a mapping reverse to the unfolding transformation, i.e. $pr^{-1}: \hat{N} \to N$. Then for any transition $t \in T$, $pr^{-1}(t^{\cdot}) = (pr^{-1}(t)^{\cdot}$; $pr^{-1}({}^{\cdot}t) = {}^{\cdot}(pr^{-1}(t))$.

2. If $x < y$ in $\hat{N}$, then $pr^{-1}(x) < pr^{-1}(y)$ in N.

3. $L(N) = L(\hat{N}, \Sigma)$ where $L(N)$ is the free language of the Petri net $\hat{N}$, $L(\hat{N}, \Sigma)$ is the language of the labelled net $\hat{N}$ in which $\forall t^{i,j} \in \hat{T}: \Sigma(t^{i,j}) = t$, and (i, j) are indices obtained by the transition t from N with unfolding into A-net $\hat{N}$.

The net which generates a unique net-process is *dense (K-dense, L-dense, M-dense)* iff this process is dense (K-dense, L-dense, M-dense).

We will introduce now some additional notions and definitions.

Let a *simple path* in a net be a sequence of net elements $(x_1, x_2, \ldots, x_n)$ such that $x_i F x_{i+1}$ for all i, $1 \leq i \leq n$, and $x_i \neq x_j$ for any two elements except that $x_1 = x_n$ allowed. A simple path is a loop iff $x_1 = x_n$.

A place p is a *loop exit* iff there is a transition in the loop such that $p \in t^{\cdot}$ and there is no loop which contains both p and t.

Theorem 3. *A Petri net N is non-K-dense iff in N there exists an unbounded loop exit p.*

This result is similar to that of Reisig and Goltz [8], for the place p can be primarily unbounded for two reasons:

1. the place p is an output place of a transition which can fire for an infinitely long time because of the initial marking (in [8] this is a transition without any input places) or

2. the place p is an output place of a transition from some loop in which this transition can fire for an infinitely long time, and the place p does not belong to this loop.

Unboundness of the first type is not related to K-density, while the second case characterizes the connection between K-density and the unboundness of a place.

Theorem 4. *If a state-machine net is not L-dense then it is not fair* [7].

The proofs of the theorems stated above will be presented elsewhere.

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