

INTERVAL FILLING SEQUENCES

ZOLTÁN DARÓCZY*, ANTAL JÁRAI* and IMRE KÁTAI**

* Dept. of Mathematics, Kossuth University, H-4010 Debrecen, Egyetem tér 1.

** Dept. of General Computer Science, Eötvös University, H-1088 Budapest,
Múzeum krt. 6-8.

(Received July 16, 1984)

1. The notion of interval filling sequence has been introduced in the paper [4]. Let \mathcal{A} denote the set of those real sequences $\{\lambda_n\}$, for which

$$\lambda_n > \lambda_{n+1} > 0 \quad (n \in \mathbf{N}) \text{ and } L := \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Definition 1.1. We call the sequence $\{\lambda_n\} \in \mathcal{A}$ *interval filling*, if for any $x \in [0, L]$ there exists a sequence $\varepsilon_n \in \{0, 1\}$ ($n \in \mathbf{N}$) so that

$$(1.1) \quad x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n.$$

In [4] we have proved the following

Theorem 1.1. *The sequence $\{\lambda_n\} \in \mathcal{A}$ is interval filling if and only if*

$$\lambda_n \leq \sum_{i=n+1}^{\infty} \lambda_i$$

for any $n \in \mathbf{N}$.

In this paper we investigate certain classes of interval filling sequences, in order to clarify the finer structure of the representations (1.1).

2. We shall need the following definitions:

Definition 2.1. Let $\{\lambda_n\} \in \mathcal{A}$ be a fixed sequence and $k \in \mathbf{N}$. We shall call the number $x \in [0, L]$ *k-decomposable* if for any $x_i \in [0, L]$ ($i = 1, 2, \dots, k$) satisfying $x = \sum_{i=1}^k x_i$ there exist sequences $\varepsilon_n(i) \in \{0, 1\}$ ($n \in \mathbf{N}$, $i = 1, 2, \dots, k$) so that

$$(2.1) \quad x_i = \sum_{n=1}^{\infty} \varepsilon_n(i) \lambda_n \quad (i = 1, 2, \dots, k)$$

and $\sum_{i=1}^k \varepsilon_n(i) \in \{0, 1\} \quad (n \in \mathbf{N})$.

Definition 2.2. Let $\{\lambda_n\} \in \mathcal{A}$ and $k \in \mathbb{N}$ fixed. We call the sequence *interval filling of order k* if any $x \in [0, L]$ is k -decomposable.

Remarks. (i) If the sequence $\{\lambda_n\} \in \mathcal{A}$ is interval filling of order $k \geq 2$, then it is also interval filling of order $(k-1)$. As a matter of fact, if any $x \in [0, L]$ is k -decomposable, then choosing the numbers $x_i \in [0, L]$ ($i = 1, 2, \dots, k$) so as to have $x = \sum_{i=1}^{k-1} x_i$ and $x_k := 0$ we obtain that x is $(k-1)$ -decomposable.

(ii) Clearly, the sequence $\{\lambda_n\} \in \mathcal{A}$ is interval filling of the first order iff it is interval filling in the sense of Definition 1.1.

Theorem 2.1. *The sequence $\{\lambda_n\} \in \mathcal{A}$ is interval filling of order k , if and only if*

$$(2.2) \quad k\lambda_n \leq \sum_{i=n+1}^{\infty} \lambda_i$$

holds for any $n \in \mathbb{N}$.

Proof. (i) Let $\{\lambda_n\} \in \mathcal{A}$ be interval filling of order k (for a fixed $k \in \mathbb{N}$), and suppose that in contradiction to our statement (2.2) does not hold for some $n \in \mathbb{N}$, i.e. that

$$k\lambda_n > \sum_{j=n+1}^{\infty} \lambda_j.$$

Then there exist numbers x_1, x_2, \dots, x_{k-1} and x_k such that

$$(2.3) \quad \frac{1}{k} \sum_{j=n+1}^{\infty} \lambda_j < x_i < \min \left\{ \lambda_n, \frac{1}{k} \sum_{j=n}^{\infty} \lambda_j \right\}$$

for $i = 1, 2, \dots, k-1$ and

$$(2.4) \quad \sum_{j=1}^{n-1} \lambda_j + \frac{1}{k} \sum_{j=n+1}^{\infty} \lambda_j < x_k < \min \left\{ \sum_{j=1}^n \lambda_j, \sum_{j=1}^{n-1} \lambda_j + \frac{1}{k} \sum_{j=n}^{\infty} \lambda_j \right\}.$$

Now $x_1 + x_2 + \dots + x_k < L$ and by (2.3) the values $\lambda_1, \lambda_2, \dots, \lambda_n$ cannot occur in the representation (2.1) of the numbers x_1, x_2, \dots, x_{k-1} (i.e. $\varepsilon_j(i) = 0$ for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, k-1$). On the other hand, in view of (2.4) the representation (2.1) of x_k cannot contain all the values $\lambda_1, \lambda_2, \dots, \lambda_n$ so that

$$x_1 + x_2 + \dots + x_k \leq \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \lambda_j$$

follows. If we still add the left-hand sides of the inequalities (2.3) and (2.4), we get

$$x_1 + x_2 + \dots + x_k > \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \lambda_j$$

a contradiction. Thus the number $x := \sum_{i=1}^k x_i$ is not k -decomposable.

(ii) Let us suppose that the sequence $\{\lambda_n\} \in \mathcal{A}$ satisfies the inequalities (2.2). Let moreover $x_i \in [0, L]$ ($i = 1, 2, \dots, k$) be arbitrary numbers for which $\sum_{i=1}^k x_i \leq L$.

We now define the numbers $\varepsilon_n(i)$ ($n \in \mathbf{N}$, $i = 1, 2, \dots, k$) inductively as follows:

$$\varepsilon_n(i) = \begin{cases} 1 & \text{if } \varepsilon_n(j) = 0 \text{ for } j < i \text{ and} \\ & \sum_{i=1}^{n-1} \varepsilon_i(i) \lambda_i + \lambda_n \leq x_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from the definition that $\sum_{n=1}^k \varepsilon_n(i) \in \{0, 1\}$ ($n \in \mathbf{N}$) and

$$(2.5) \quad \sum_{n=1}^{\infty} \varepsilon_n(i) \lambda_n \leq x_i \quad (i = 1, 2, \dots, k).$$

We are going to show that in (2.5) equality holds. We distinguish three cases:

$$(a) \quad \sum_{i=1}^k \varepsilon_n(i) = 1 \quad \text{for every } n \in \mathbf{N}.$$

If, for some $i \in \{1, 2, \dots, k\}$, strict inequality would hold in (2.5), then condition (a) would imply

$$L = \sum_{n=1}^{\infty} \lambda_n < x_1 + x_2 + \dots + x_k$$

in contradiction to the choice of the numbers x_i .

(b) There exist finitely many values $n \in \mathbf{N}$ for which

$$\sum_{i=1}^k \varepsilon_n(i) = 0.$$

Let us show that this is impossible. Indeed, suppose that (b) is valid, and let N be the greatest natural number for which $\sum_{i=1}^k \varepsilon_N(i) = 0$. Then

$$\sum_{i=1}^{N-1} \varepsilon_i(i) \lambda_i + \lambda_N > x_i$$

for $i = 1, 2, \dots, k$ and from this

$$\begin{aligned} \sum_{i=1}^k \sum_{l=1}^{\infty} \varepsilon_l(i) \lambda_l &= \sum_{i=1}^k \sum_{l=1}^{N-1} \varepsilon_l(i) \lambda_l + \sum_{l=N+1}^{\infty} \lambda_l \cong \\ &\cong \sum_{i=1}^k \left(\sum_{l=1}^{N-1} \varepsilon_l(i) \lambda_l + \lambda_N \right) > x_1 + x_2 + \dots + x_k \end{aligned}$$

follows, a contradiction.

(c) There exist infinitely many values $n \in \mathbb{N}$ for which

$$\sum_{i=1}^k \varepsilon_n(i) = 0.$$

For these values of n

$$x_i < \sum_{l=1}^{n-1} \varepsilon_l(i) \lambda_l + \lambda_n$$

and if $n \rightarrow \infty$ then by $\lambda_n \rightarrow 0$ this implies

$$x_i \leq \sum_{l=1}^{\infty} \varepsilon_l(i) \lambda_l.$$

i.e. in (2.5) equality holds. \square

Remark. For $k = 1$ Theorem 2.1 is clearly equivalent with Theorem 1.1.

Examples. (1) Let $1 < q \leq 2$. Then the sequence $\lambda_n := \frac{1}{q^n}$ belongs to \mathcal{A} .

Proposition. The sequence $\left\{ \frac{1}{q^n} \right\} \in \mathcal{A}$ is interval filling of order k if and only if

$$1 < q \leq \frac{k+1}{k}.$$

Proof. In view of Theorem 2.1 we must investigate, when will the inequality

$$\frac{k}{q^n} \leq \sum_{j=n+1}^{\infty} \frac{1}{q^j} = \frac{1}{q^n} \frac{1}{q-1}$$

be satisfied for any $n \in \mathbb{N}$: (2.6) immediately follows. \square

(2) Let $N \in \mathbb{N}$ be fixed, and consider the sequence

$$\lambda_n^{(N)} := \frac{1}{(N+n)^2} \quad (n \in \mathbb{N}).$$

Proposition. For any $k \in \mathbf{N}$ there exists $N(k) \in \mathbf{N}$ so that the sequence $\{\lambda_n^{(N(k))}\} \in \mathcal{A}$ is interval filling of order k .

Proof. In view of Theorem 2.1 it suffices to show that for $k \in \mathbf{N}$ fixed

$$\frac{k}{(N+n)^2} \leq \sum_{i=n+1}^{\infty} \frac{1}{(N+i)^2}$$

holds for any $n \in \mathbf{N}$, if only N is large enough. An investigation of the function $x \rightarrow \frac{x^2}{x+1}$ shows that for any $k \in \mathbf{N}$ there exists such an x_0 that $x > x_0$ implies $\frac{x^2}{x+1} > k$. Choosing $N = N(k) > x_0$ we now infer that

$$\frac{(N+n)^2}{N+n+1} > k$$

i.e.

$$\frac{k}{(N+n)^2} < \frac{1}{N+n+1}$$

holds for any $n \in \mathbf{N}$. From this

$$\begin{aligned} \frac{k}{(N+n)^2} &< \frac{1}{N+n+1} - \frac{1}{N+n+2} + \frac{1}{N+n+2} - \frac{1}{N+n+3} + \dots = \\ &= \sum_{i=n+1}^{\infty} \frac{1}{(N+i)(N+i+1)} \leq \sum_{i=n+1}^{\infty} \frac{1}{(N+i)^2} \end{aligned}$$

i.e. $\{\lambda_n^{(N)}\}$ is interval filling of order k . \square

3. Let $\{\lambda_n\} \in \mathcal{A}$ be an interval filling sequence.

Definition 3.1. The function $F: [0, L] \rightarrow \mathbf{R}$ is said to be *completely additive* if for any sequence $\varepsilon_n \in \{0, 1\}$ ($n \in \mathbf{N}$) the equality

$$(3.1) \quad F\left(\sum_{n=1}^{\infty} \varepsilon_n \lambda_n\right) = \sum_{n=1}^{\infty} \varepsilon_n F(\lambda_n)$$

holds.

We have investigated completely additive functions in the papers [3] and [4]. A more general form of the following theorem can be found in [4], but the method of proof is quite different.

Theorem 3.1. If $\{\lambda_n\} \in \mathcal{A}$ is an interval filling sequence of second order and $F: [0, L] \rightarrow \mathbf{R}$ is completely additive, then there exists $c \in \mathbf{R}$ so that $F(x) = cx$ for any $x \in [0, L]$.

Proof. (i) Let $P := \{n \mid F(\lambda_n) > 0\}$ and $M := \mathbf{N} - P$. Putting $\xi := \sum_{n \in P} \lambda_n$ we see by (3.1) that

$$\sum_{n=1}^{\infty} |F(\lambda_n)| = \sum_{n \in P} F(\lambda_n) - \sum_{n \in M} F(\lambda_n) = F(\xi) - F(L - \xi) < \infty,$$

hence F is bounded in $[0, L]$.

(ii) Let $x, y \geq 0$ and $x + y \leq L$ be arbitrary. Then there exist sequences $\varepsilon_n, \delta_n \in \{0, 1\}$ such that $\varepsilon_n + \delta_n \in \{0, 1\}$ ($n \in \mathbf{N}$) and

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n, \quad y = \sum_{n=1}^{\infty} \delta_n \lambda_n.$$

By the complete additivity

$$\begin{aligned} F(x + y) &= F\left[\sum_{n=1}^{\infty} (\varepsilon_n + \delta_n) \lambda_n\right] = \sum_{n=1}^{\infty} (\varepsilon_n + \delta_n) F(\lambda_n) = \\ &= \sum_{n=1}^{\infty} \varepsilon_n F(\lambda_n) + \sum_{n=1}^{\infty} \delta_n F(\lambda_n) = F\left(\sum_{n=1}^{\infty} \varepsilon_n \lambda_n\right) + \\ &\quad + F\left(\sum_{n=1}^{\infty} \delta_n \lambda_n\right) = F(x) + F(y). \end{aligned}$$

Now, as is known from [2], there exists an additive function $A : \mathbf{R} \rightarrow \mathbf{R}$ (i. e. $A(u + v) = A(u) + A(v)$ for any $u, v \in \mathbf{R}$) such that $A(x) = F(x)$ for $x \in [0, L]$. On the other hand, the boundedness of F implies that A is bounded in $[0, L]$, and from this $A(x) = cx$ follows (see e.g. [1]), i.e. we have $F(x) = cx$. \square

4. The following definitions have also been introduced in [4].

Definition 4.1. Let $\{\lambda_n\} \in \mathcal{A}$ be an interval filling sequence. The number $x \in [0, L]$ is said to be *uniquely determined*, if there exists a unique sequence $\varepsilon_n \in \{0, 1\}$ ($n \in \mathbf{N}$) for which (1.1) is satisfied.

Definition 4.2. Let $\{\lambda_n\} \in \mathcal{A}$ be an interval filling sequence. The number $x \in]0, L[$ is said to be *finite* if there exist $N \in \mathbf{N}$ and $\varepsilon_n \in \{0, 1\}$ ($n = 1, 2, \dots, N - 1$) such that

$$(4.1) \quad x = \sum_{n=1}^{N-1} \varepsilon_n \lambda_n + \lambda_N.$$

Remarks. (i) Clearly, 0 and L are always uniquely determined.

(ii) If $x \in [0, L]$ is finite, then it is not uniquely determined. Indeed, by Theorem 1.1, $\mu_n := \lambda_{N+n}$ ($n \in \mathbf{N}$) is in this case interval filling and $\lambda_N \leq \sum_{i=N+1}^{\infty} \lambda_i = \sum_{n=1}^{\infty} \mu_n$. Thus there exists a sequence $\delta_n \in \{0, 1\}$ such that

$$\lambda_N = \sum_{n=1}^{\infty} \delta_n \mu_n = \sum_{i=N+1}^{\infty} \delta_{i-N} \lambda_i,$$

hence

$$x = \sum_{n=1}^{N-1} \varepsilon_n \lambda_n + \lambda_N = \sum_{n=1}^{N-1} \varepsilon_n \lambda_n + \sum_{i=N+1}^{\infty} \delta_{i-N} \lambda_i,$$

i.e. x is not uniquely determined.

Theorem 4.1. *Let $\{\lambda_n\} \in \mathcal{A}$ be an interval filling sequence for which it is true that any non-finite number from $]0, L[$ is uniquely determined. Then*

$$\lambda_n = \frac{2\lambda_1}{2^n} \quad (n \in \mathbb{N}).$$

Proof. (i) Suppose that there exists an $n \in \mathbb{N}$ for which

$$\lambda_n < \sum_{i=n+1}^{\infty} \lambda_i.$$

Let us show that in this case there exists a number in $]0, L[$ which is neither finite nor uniquely determined. The finite numbers form a countable set in $]0, L[$, so there exists $\eta > 0$ such that $\lambda_n + \eta$ is non-finite and

$$\lambda_n + \eta < \sum_{i=n+1}^{\infty} \lambda_i.$$

Now the non-finite number $x = \lambda_n + \eta$ has a representation in which λ_n occurs, and also such a one in which λ_n does not occur. Thus x is neither finite nor uniquely determined. From this, by Theorem 1.1, we obtain that

$$\lambda_n = \sum_{i=n+1}^{\infty} \lambda_i$$

for any $n \in \mathbb{N}$, hence

$$\lambda_n = \sum_{i=n+1}^{\infty} \lambda_i = \lambda_{n+1} + \sum_{i=n+2}^{\infty} \lambda_i = 2\lambda_{n+1}.$$

By induction we get (4.2).

(ii) On the other hand it is known that the sequence (4.2) satisfies the hypothesis of the theorem (see e.g. [5]). \square

Remark. Theorem 4.1. can be regarded as a characterization of the sequence $\frac{c}{2^n}$ ($c > 0$).

5. Let $1 < q \leq 2$ and $\lambda_n := \frac{1}{q^n}$ ($n \in \mathbb{N}$). Then $\left\{ \frac{1}{q^n} \right\} \in \mathcal{A}$ is an interval filling sequence, and we have shown in [4] that for $q \leq \frac{\sqrt{5}+1}{2}$ every number $x \in]0, L[$ $\left(L = \frac{1}{q-1} \right)$ is non uniquely determined. By Theorem 4.1 we know

that there always exists an $x \in]0, L[$ which is neither finite nor uniquely determined. Also, we have proved in [4] that for $\frac{\sqrt{5}+1}{2} < q \leq 2$ there exist uniquely determined numbers.

Now we are going to investigate the following question: given a sequence $\varepsilon_n \in \{0, 1\}$ ($n \in \mathbb{N}$) i.e. given an element $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) \in \{0, 1\}^{\mathbb{N}}$, for which values $1 < q \leq 2$ will the number

$$(5.1) \quad x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q^n} \in \left[0, \frac{1}{q-1}\right].$$

be uniquely determined?

Definition 5.1. Let $\varepsilon \in \{0, 1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Denote by $E_n^+(\varepsilon)$ ($E_n^-(\varepsilon)$) the set of all those real numbers $1 < q \leq 2$ for which there does not exist a $\delta \in \{0, 1\}^{\mathbb{N}}$ such that $\delta_i = \varepsilon_i$ for $i < n$, $\delta_n > \varepsilon_n$ ($\delta_n < \varepsilon_n$) and

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.$$

Let moreover $E_n(\varepsilon) := E_n^+(\varepsilon) \cap E_n^-(\varepsilon)$,

$$E^+(\varepsilon) := \bigcap_{n=1}^{\infty} E_n^+(\varepsilon), \quad E^-(\varepsilon) := \bigcap_{n=1}^{\infty} E_n^-(\varepsilon)$$

and

$$E(\varepsilon) := E^+(\varepsilon) \cap E^-(\varepsilon).$$

Remark. (i) It follows from the definition that for $q \in E(\varepsilon)$ the number (5.1) is uniquely determined. The set $E(\varepsilon)$ will accordingly be called the *uniqueness set* of the sequence $\varepsilon \in \{0, 1\}^{\mathbb{N}}$.

(ii) We have $E_n^+(\varepsilon) = E_n^-(1-\varepsilon)$ for any $n \in \mathbb{N}$, where $1 := (1, 1, 1, \dots) \in \{0, 1\}^{\mathbb{N}}$. Thus it often suffices to consider the set $E_n^+(\varepsilon)$.

Theorem 5.1. *The relation $q \in E_n^+(\varepsilon)$ holds if and only if either*

$$(1) \quad \varepsilon_n = 1$$

or else

$$(2) \quad \varepsilon_n = 0 \quad \text{and} \quad \sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} < \frac{1}{q^n}.$$

Proof. (i) Suppose that $q \notin E_n^+(\varepsilon)$ and $\varepsilon_n = 0$. Then there exists a $\delta \in \{0, 1\}^{\mathbb{N}}$ such that

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}$$

moreover $\varepsilon_i = \delta_i$ for $i < n$, $\delta_n = 1$, $\varepsilon_n = 0$ and so

$$\sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} = \frac{1}{q^n} + \sum_{i=n+1}^{\infty} \frac{\delta_i}{q^i} \geq \frac{1}{q^n}.$$

(ii) Conversely, if

$$\sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} \cong \frac{1}{q^n}$$

then

$$0 \leq \sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} - \frac{1}{q^n} < \sum_{i=n+1}^{\infty} \frac{1}{q^i}$$

and so by the interval filling property there exists a sequence $\delta_i \in \{0, 1\}$ ($i \geq n+1$) such that

$$0 \leq \sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} - \frac{1}{q^n} = \sum_{i=n+1}^{\infty} \frac{\delta_i}{q^i}.$$

Putting $\delta_i = \varepsilon_i$ for $i < n$, $\delta_n = 1$ we now get

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}. \quad \square$$

Theorem 5.2. *Let $n \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}^{\mathbb{N}}$ be fixed. Then there exists $q_0 = q_0(n; \varepsilon) \in [1, 2]$ such that*

$$E_n^+(\varepsilon) = [q_0, 2].$$

Proof. The condition

$$(5.2) \quad \sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} < \frac{1}{q^n}$$

occurring in Theorem 5.1 is equivalent to

$$f(q) := \sum_{i=1}^{\infty} \varepsilon_{i+n} \left(\frac{1}{q} \right)^i < 1.$$

As sum function of a power series f is continuous, and the coefficients being nonnegative it is a monotone increasing function of $\frac{1}{q}$. Thus there exists $q_0 \in [1, 2]$ such that for $q < q_0$ (5.2) is satisfied. \square

Theorem 5.1 enables us determine the unicity set $E(\varepsilon)$ for certain sequences $\varepsilon \in \{0, 1\}^{\mathbb{N}}$.

Theorem 5.3. *Let $\varepsilon \in \{0, 1\}^{\mathbb{N}}$ be defined by*

$$(5.3) \quad \varepsilon_n = \begin{cases} 1 & \text{if } n \in \{1, k+1, 2k+1, 3k+1, \dots\} \\ 0 & \text{if } n \notin \{1, k+1, 2k+1, 3k+1, \dots\} \end{cases}$$

with $k \geq 2$ a fixed natural number. Then

$$(5.4) \quad E(\varepsilon) = \{q \in [1, 2] \mid q^k - q^{k-1} - q^{k-2} - \dots - q - 1 > 0\}.$$

Proof. For $1 < q \leq 2$ we have

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = \frac{1}{q} \left(1 + \frac{1}{q^k} + \frac{1}{q^{2k}} + \dots \right) = \frac{q^{k-1}}{q^k - 1}.$$

By Theorem 5.1

$$(5.5) \quad E_n^+(\varepsilon) =]1, 2] \text{ for } n \in \{1, k+1, 2k+1, \dots\}.$$

If $n \in \{2, 3, \dots, k\}$ and $q \in E_n^+(\varepsilon)$ (in case $\varepsilon_n = 0$) then by Theorem 5.1 it is necessary and sufficient that

$$\frac{1}{q^n} \geq \frac{1}{q^k} > \sum_{i=k+1}^{\infty} \frac{\varepsilon_i}{q^i} = \frac{1}{q^{k+1}} + \frac{1}{q^{2k+1}} + \dots = \frac{1}{q^k} \frac{q^{k-1}}{q^k - 1}$$

should hold. By the periodicity of the sequence ε we get

$$(5.6) \quad E_n^+(\varepsilon) \supseteq \{q \in]1, 2] \mid q^k - q^{k-1} - 1 > 0\}$$

for $n \notin \{1, k+1, 2k+1, \dots\}$; if $n = lk$ ($l \in \mathbb{N}$) then equality holds in (5.6). From equations (5.5) and (5.6) we obtain

$$(5.7) \quad E^+(\varepsilon) = \{q \in]1, 2] \mid q^k - q^{k-1} - 1 > 0\}.$$

By the definition and by Theorem 5.1

$$(5.8) \quad E_n^-(\varepsilon) = E_n^+(1 - \varepsilon) =]1, 2] \text{ for } n \notin \{1, k+1, 2k+1, \dots\}.$$

If $n = 1$ (in case $1 - \varepsilon_n = 0$) and $q \in E_1^-(\varepsilon) = E_1^+(1 - \varepsilon)$, then by Theorem 5.1 it is necessary and sufficient that

$$\frac{1}{q} > \sum_{i=2}^{\infty} \frac{1 - \varepsilon_i}{q^i} = \frac{1}{q-1} - \frac{q^{k-1}}{q^k - 1}.$$

should hold, and this in turn is equivalent to the inequality

$$(5.9) \quad q^k - q^{k-1} - q^{k-2} - \dots - q - 1 > 0.$$

By the periodicity of the sequence $1 - \varepsilon$ and by (5.9) we get

$$(5.10) \quad E_n^-(\varepsilon) = E_n^+(1 - \varepsilon) = \{q \in [1, 2] \mid q^k - q^{k-1} - \dots - q - 1 > 0\}.$$

for $n \in \{1, k+1, 2k+1, \dots\}$. Thus by (5.8) and (5.10)

$$(5.11) \quad E^-(\varepsilon) = \{q \in [1, 2] \mid q^k - q^{k-1} - \dots - q - 1 > 0\}.$$

By the definition (5.7) and (5.11) imply

$$E(\varepsilon) = E^+(\varepsilon) \cap E^-(\varepsilon) = \{q \in [1, 2] \mid q^k - q^{k-1} - \dots - q - 1 > 0\}$$

i.e. (5.4) is valid. \square

Remark. From Theorem 5.3 we get as a special case ($k = 2$) that if $\varepsilon = (101010\dots) \in \{0, 1\}^{\mathbb{N}}$ then

$$E(\varepsilon) = \{q \in [1, 2] \mid q^2 - q - 1 > 0\} = \left] \frac{\sqrt{5} + 1}{2}, 2 \right].$$

This will say that in case $\frac{\sqrt[5]{5}+1}{2} < q \leq 2$ the number

$$x = \sum_{i=1}^{\infty} \frac{1}{q^{2i-1}} = \frac{q}{q^2-1} \in \left] 0, \frac{1}{q-1} \right[$$

is always uniquely determined with respect to the interval filling sequence $\left\{ \frac{1}{q^n} \right\} \in \mathcal{A}$ (see [4]).

REFERENCES

- [1] *Aczél J.*: Lectures on Functional Equations and Their Applications. Academic Press, New York, 1966.
- [2] *Daróczy Z. and Losonczi L.*: Über die Erweiterung der auf einer Punktmenge additiven Funktionen. *Publ. Math. Debrecen* **14** (1967), 239–245.
- [3] *Daróczy Z., Járαι A. and Kátai I.*: On functions defined by digits of real numbers. *Acta Math. Acad. Sci. Hung.* **47** (1–2) (1986), 73–80.
- [4] *Daróczy Z., Járαι A. and Kátai I.*: Intervallfüllende Folgen und volladditive Funktionen. *Acta Sci. Math. Szeged* **50** (1986), 337–350.
- [5] *Galambos J.*: Representation of real numbers by infinite series. Lecture Notes in Mathematics **502**. Springer Verlag, Berlin, 1976.

