SOME REVERSE MAXIMAL INEQUALITIES FOR SUPERMARTINGALES

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1. Let $\varphi(t)$ be a nonnegative increasing function, continuous on the left, such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. For $x \geq 0$ define the function

$$\Phi(x) = \int_0^x \varphi(t)dt.$$

Then Φ is increasing, continuous and convex. Φ is called a Young-function.

We define the conjugate Young-function as follows: for $t > 0$ put $\psi(t) = \sup(x > 0 : \varphi(x) < t)$ and $\psi(0) = 0$. It is easily shown that ψ satisfies all the properties imposed on φ. It is also true that

$$\psi(\varphi(x)) \leq x \leq \psi(\varphi(x)+0).$$

The Young-function

$$\Psi(x) = \int_0^x \psi(t)dt$$

is said to be conjugate to Φ.

The pair (Φ, Ψ) of conjugate Young-functions satisfies the following inequality of Young: $xy \leq \Phi(x) + \Psi(y)$, for all $x \geq 0$, $y \geq 0$. Equality holds if and only if $x \in [\psi(y), \psi(y+0)]$ or $y \in [(x), \varphi(x+0)]$.

We say that Φ satisfies the growth condition if there exist constants $a > 1$ and $A > 0$ such that for all $x \geq 0$, the inequality

$$\Phi(ax) \leq A\Phi(x) \tag{1.1}$$

holds. The growth condition (1.1) is equivalent to

$$\sup_{x>0} \frac{x\varphi(x)}{\Phi(x)} = p < \infty. \tag{1.2}$$

p is called the power of Φ. The power q of Ψ is defined similarly.

2. Let $(X_n, \mathfrak{F}_n)$, $n \geq 0$, be a nonnegative supermartingale. Denote by X_n^* the corresponding maximal function defined as follows

$$X_n^* = \max_{0 \leq k \leq n} X_k.$$

Recall the following inequality for nonnegative supermartingales (e.g. [7]): For any nonnegative supermartingale $(X_n, \mathfrak{F}_n)$, $n \geq 0$, the inequality

$$xP(X_n^* \geq x) \leq EX_0 \wedge x$$

holds for all $x > 0$. We shall reverse this inequality under Gundy's condition.

Lemma 2. 1. *Let $(X_n, \mathfrak{F}_n)$ be any nonnegative supermartingale satisfying Gundy's condition*

$$X_n \leq cX_{n-1} \text{ a.e.},$$

for all $n \geq 1$ and for some positive constant c. Then

$$EX_n\chi(X_n^* \geq x) \leq EX_0\chi(X_0 \geq x) + cxE\chi(X_n^* \geq x) \tag{2.1}$$

for all $x > 0$.

Proof. Calculate $EX_n\chi(X_n^* \geq x)$.

$$EX_n\chi(X_n^* \geq x) = E\sum_{i=0}^{n} X_n[\chi(X_i^* \geq x) - \chi(X_{i-1}^* \geq x)]$$

with $X_{-1} \equiv X_{-1}^* \equiv 0$. It is clear that the random variable $\chi(X_i^* \geq x) - \chi(X_{i-1}^* \geq x)$ is nonnegative (since the function $\chi(y \geq x)$ increases in y) and $\mathfrak{F}_i$-measurable. The supermartingale property and the conditional expectation property together with the above facts imply

$$EX_n\chi(X_n^* \geq x) \leq \sum_{i=1}^{n} EX_i[\chi(X_i^* \geq x) - \chi(X_{i-1}^* \geq x)].$$

By Gundy's condition it ensues that

$$EX_n\chi(X_n^* \geq x) \leq EX_0\chi(X_0 \geq x) + c\sum_{i=1}^{n} EX_{i-1}[\chi(X_i^* \geq x) - \chi(X_{i-1}^* \geq x)] \leq$$

$$\leq EX_0\chi(X_0 \geq x) + c\sum_{i=1}^{n} EX_{i-1}^*[\chi(X_i^* \geq x) - \chi(X_{i-1}^* \geq x)] \leq$$

$$\leq EX_0\chi(X_0 \geq x) + cE\int_{X_0}^{X^*_n} yd(\chi(y \geq x)) =$$

$$= EX_0\chi(X_0 \geq x) + cxE[\chi(X_n^* \geq x) - \chi(X_0 \geq x)] \leq$$

$$\leq EX_0\chi(X_0 \geq x) + cxE\chi(X_n^* \geq x),$$

and this was to be proved. □

Consider an arbitrary pair of conjugate Young-functions (Φ, Ψ) defined on $[0, \infty)$, and introduce the Laplace transform

$$I(t) = \int_0^\infty e^{-t\lambda} d\Psi(\lambda),$$

for all $0<t<1$.

Theorem 2.2. *Consider a potential $(X_n, \mathfrak{F}_n)$, generated by an adapted increasing process A_n, $n \geq 0$ i.e.*

$$X_n = E(A_\infty \mid \mathfrak{F}_n) - A_n.$$

If $I(t) < \infty$, then

$$\text{(2.2)} \qquad EX_n^* \leq E\Big(\max_{1 \leq i \leq n} E(A_\infty \mid \mathcal{F}_n)\Big) \leq E\Phi(A_\infty) + I(t)(1-t)^{-1}.$$

The first inequality on the left-hand side is trivial. The second one is the consequence of the following statement ([5]): For every nonnegative submartingale $(X_n, \mathfrak{F}_n)$ we have

$$\text{(2.3)} \qquad EX_n^* \leq E\Phi(X_n) + I(^1)(1-t)^{-1},$$

provided that $I(t) < \infty$.

We are now in the position to reverse inequality (2.2).

Theorem 2.3. *Let $(X_n, \mathfrak{F}_n)$ be a potential. Suppose that*

$$\Psi(\varphi(x)) = x\varphi(x) - \Phi(x) = O(x), \quad as \quad x \to \infty.$$

Then under Gundy's condition, we have

$$\text{(2.4)} \qquad E\Phi(X_n^*) \leq cx_0\varphi(x_0) + EX_0\varphi(X_0) + cK_\Phi EX_n^*,$$

where $x_0 > 0$ is a suitable canstant.

Proof. Integrate inequality (2.1) on the interval $(0, \infty)$ with respect to the measure $d(\varphi(x))$. By Fubini's theorem, we have

$$EX_n\varphi(X_n^*) \leq EX_0\varphi(X_0) + cE\int_0^{X^*n} xd(\varphi(x)).$$

Since φ is an increasing function, it is obvious that $EX_n\varphi(X_n) \leq EX_n\varphi(X_n^*)$. Consequently,

$$EX_n\varphi(X_n) \leq EX_0\varphi(X_0) + cE\int_0^{X^*n} xd(\varphi(x)).$$

One can easily verify that

$$\int_0^x td(\varphi(t)) = \Psi(\varphi(x)).$$

which implies

$$EX_n\varphi(X_n) \leq EX_0\varphi(X_0) + cE\Psi(\varphi(X_n^*)).$$

Further, the assumption of the theorem ensures the existence of a positive constant K_Φ, depending only on Φ, such that

$$\Psi(\varphi(x)) \leq K_\Phi \cdot x, \quad x \geq x_0.$$

Hence

$$EX_n\varphi(X_n) \leq EX_0\varphi(X_0) + c \int_{(X^*_n \leq x_0)} \Psi(\varphi(X_n^*))dP +$$

$$+ c \int_{(X^*_n \geq x_0)} \Psi(\varphi(X_n^*))dP \leq EX_0\varphi(X_0) + c\Psi(\varphi(x_0)) + cK_\Phi \int_{(X^*_n \geq x_0)} X_n^* dP \leq$$

$$\leq c\Psi(\varphi(x_0)) + EX_0\varphi(X_0) + cK_\Phi EX_n^*.$$

Using the fact that $x\varphi(x) \geq \Psi(\varphi(x))$ for all $x \geq 0$, we can conclude on the validity of the desired inequality, finishing the proof of the theorem. □

We shall give an example of a potential satisfying Gundy's condition.

Let (x_n), $n \geq 1$, be any real sequence such that $x_n \uparrow x$, for an arbitrary positive constant x. Consider the probability space $(\Omega, \mathfrak{A}, P)$, with $\Omega = \mathbf{N}$ and $\mathfrak{A}$ being the σ-field of all the subsets of $\mathbf{N}$. The probability measure will be defined on $(\Omega, \mathfrak{A})$ by the formula,

$$P(\{n\}) = \frac{1}{n} - \frac{1}{n+1}.$$

Define for all $n \geq 1$ the random variables

$$X_n(\omega) = x - x_n \chi(\omega \leq n),$$

and let $\mathfrak{F}_n = \sigma(\{1\}, \ldots, \{n\}, \{n+1, n+2, \ldots\})$ be the minimal σ-field generated by the measurable partition given in the brackets.

Then it is easily checked that $(X_n, \mathfrak{F}_n)$, $n \geq 1$, is a potential on the probability space $(\Omega, \mathfrak{A}, P)$, satisfying Gundy's condition with $c = 1$, since $-x_{n+1}\chi(\omega \leq n+1) \leq -x_n\chi(\omega \leq n)$. Consequently, $x - x_{n+1}\chi(\omega \leq n+1) \leq x - x_n\chi(\omega \leq n)$, i.e. $X_{n+1} \leq X_n$.

REFERENCES

[1] *Garsia A. M.:* Martingale Inequalities. Benjamin, Reading, Massachusetts, 1973.

[2] *Gundy R. F.:* On the class L log L martingales and singular integrals, *Studia Math.* **23** (1969), 109–118.

[3] *Krasnosel'ski M. A.* and *Rutickii B. Ya.:* Convex Functions and Orlicz-spaces. (Transl. from Russian by L. F. Boron) Noordhoff, Groningen, 1961.

[4] *Meyer P. A.:* Martingales and Stochastic Integrals I. *Lecture Notes in Mathematics* **284** Springer, Berlin, 1972.

[5] *Mogyoródi J.:* On some problems for predictable r.v.'s. In: Stat. and Probability: Proc. of the 3rd Pannonian Symp. on Math. Stat. (Visegrád, 1982), 1984, 221–230.

[6] *Mogyoródi J.* and *Móri T. F.:* Necessary and sufficient condition for the maximal inequality of convex Young-functions. *Acta Sci. Math.* **45** (1983), 325–332.

[7] *Neveu J.:* Martingales à temps discrets. Masson et Cie, Paris, 1972.