

AN ALGORITHM FOR THE BEST CHEBYSHEV APPROXIMATIONS

FERENC KÁLOVICS

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The first part of the paper gives two theorems for characterization of the best Chebyshev approximations (the basic idea of Theorem 1.1 can be found in [1]). In the second part we shall give possibilities of the numerical application of Theorem 1.2.

1. On the characteristic properties of the best approximation

The fixed symbols are as follows: $[a, b] \subset \mathbf{R}$, a closed and bounded interval; $f(x)$, a continuous function on $[a, b]$; n denotes a natural number; $g_1(x), g_2(x), \dots, g_n(x)$, a Chebyshev system of the continuous functions on $[a, b]$, i.e.

$$\begin{vmatrix} g_1(x_1) & \dots & g_1(x_n) \\ \vdots & & \vdots \\ g_n(x_1) & \dots & g_n(x_n) \end{vmatrix} > 0$$

if $a \leq x_1 < x_2 < \dots < x_n \leq b$;

$A_1^*, \dots, A_n^*$, the coefficients of the best Chebyshev approximation, i.e.

$$\begin{aligned} & \max_{x \in [a, b]} |A_1^* g_1(x) + \dots + A_n^* g_n(x) - f(x)| < \\ & < \max_{x \in [a, b]} |A_1 g_1(x) + \dots + A_n g_n(x) - f(x)|, \end{aligned}$$

where $A^* \equiv \{A_1^*, \dots, A_n^*\} \in \mathbf{R}^n$, $A \equiv \{A_1, \dots, A_n\} \in \mathbf{R}^n$ and $A^* \neq A$;

for all $a \leq x_1 < x_2 < \dots < x_{n+1} \leq b$. Now let $s \equiv \{s_1, s_2, \dots, s_{n+1}\}$ and $S \equiv \{S_1, S_2, \dots, S_{n+1}\} = \{s_1, -s_2, \dots, (-1)^n s_{n+1}\}$. If $s_1, \dots, s_{n+1}, x_1, \dots, x_{n+1}$ satisfies conditions of the problem, then

$$\begin{aligned} |s_1 f(x_1) - s_2 f(x_2) + \dots + (-1)^n s_{n+1} f(x_{n+1})| &= \left| \sum_{i=1}^{n+1} f(x_i) S_i \right| = \\ &= \left| \sum_{i=1}^{n+1} (A_1^* g_1(x_i) + \dots + A_n^* g_n(x_i) - f(x_i)) S_i \right| \leq \\ &\leq \max_i |A_1^* g_1(x_i) + \dots + A_n^* g_n(x_i) - f(x_i)| \sum_{i=1}^{n+1} |S_i| \leq \\ &\leq \max_{x \in [a, b]} |A_1^* g_1(x) + \dots + A_n^* g_n(x) - f(x)|. \end{aligned}$$

Equalities are in places of the inequalities if $x_1, \dots, x_n$ is a set of the external points. $\square$

Theorem 1.2. Assume that $f(x), g_1(x), \dots, g_n(x)$ are continuously differentiable on $[a, b]$ and $a, b \notin X^*$, where X^* is a set of the extremal points. Then the elements $x_1^* < x_2^* < \dots < x_{n+1}^*$ of X^* satisfy the system of equations

$$\begin{aligned} g'_1(x_i) D_1(x_1, \dots, x_{n+1}) + \dots + g'_n(x_i) D_n(x_1, \dots, x_{n+1}) &= \\ &= f'(x_i) D(x_1, \dots, x_{n+1}), \quad (i = 1, 2, \dots, n+1), \end{aligned}$$

where

$$\begin{aligned} D(x_1, \dots, x_{n+1}) &= \begin{vmatrix} 1 & g_1(x_1) & \dots & g_n(x_1) \\ -1 & g_1(x_2) & \dots & g_n(x_2) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ (-1)^n g_1(x_{n+1}) & \dots & g_n(x_{n+1}) \end{vmatrix}, \quad D_1(x_1, \dots, x_{n+1}) = \\ &= \begin{vmatrix} 1 & f(x_1) & g_2(x_1) & \dots & g_n(x_1) \\ -1 & f(x_2) & g_2(x_2) & \dots & g_n(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^n f(x_{n+1}) & g_2(x_{n+1}) & \dots & g_n(x_{n+1}) \end{vmatrix}, \quad \dots, \quad D_n(x_1, \dots, x_{n+1}) = \\ &= \begin{vmatrix} 1 & g_1(x_1) & \dots & g_{n-1}(x_1) & f(x_1) \\ -1 & g_1(x_2) & \dots & g_{n-1}(x_2) & f(x_2) \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ (-1)^n g_1(x_{n+1}) & \dots & g_{n-1}(x_{n+1}) & f(x_{n+1}) \end{vmatrix}. \end{aligned}$$

Proof. If we determine $s_1^*, \dots, s_{n+1}^*$ for the sequence $x_1^* < x_2^* < \dots < x_{n+1}^*$ from the former formulas, then we get a solution of the problem of Theorem 1.1.

Since s_i^* and x_i^* are inner points of the intervals $[0, 1]$ and $[a, b]$ ($i = 1, \dots, n+2$), respectively, therefore $s_1^*, \dots, s_{n+1}^*, x_1^*, \dots, x_{n+1}^*$ satisfy the Lagrange's conditions for

$$\begin{aligned} F(s_1, \dots, s_{n+1}, x_1, \dots, x_{n+1}) &= s_1 f(x_1) - \dots + (-1)^n s_{n+1} f(x_{n+1}) + \\ &+ C_1(s_1 + \dots s_{n+1} - 1) + C_2(s_1 g_1(x_1) - \dots + (-1)^n s_{n+1} g_1(x_{n+1})) + \\ &+ \dots + C_{n+1}(s_1 g_n(x_1) - \dots + (-1)^n s_{n+1} g_n(x_{n+1})), \text{ i.e.} \\ F'_{s_i} &= 0, \quad F'_{x_i} = 0 \quad (i = 1, \dots, n+1). \end{aligned}$$

Hence

$$\begin{aligned} f(x_1) + C_1 + C_2 g_1(x_1) + \dots + C_{n+1} g_n(x_1) &= 0 \\ -f(x_2) + C_1 - C_2 g_1(x_2) - \dots - C_{n+1} g_n(x_2) &= 0 \\ \vdots &\vdots \\ (-1)^n f(x_{n+1}) + C_1 + (-1)^n C_2 g_1(x_{n+1}) + \dots + (-1)^n C_{n+1} g_n(x_{n+1}) &= 0 \end{aligned}$$

and

$$\begin{aligned} s_1 f'(x_1) + s_1 C_2 g'_1(x_1) + \dots + s_1 C_{n+1} g'_n(x_1) &= 0 \\ -s_2 f'(x_2) - s_2 C_2 g'_1(x_2) - \dots - s_2 C_{n+1} g'_n(x_2) &= 0 \\ \vdots &\vdots \\ (-1)^n s_{n+1} f'(x_{n+1}) + (-1)^n s_{n+1} C_2 g'_1(x_{n+1}) + \dots + \\ &+ (-1)^n s_{n+1} C_{n+1} g'_n(x_{n+1}) = 0. \end{aligned}$$

If we determine $C_1, C_2, \dots, C_{n+1}$ from the first system of the equations and we use these values in the second system, then we get the system of equations of our theorem. (Since $s_i \neq 0$, we can divide the equations of the second system by s_i !) $\square$

Remark. If $a \in X^*$ or $b \in X^*$ or $a, b \in X_1^*$, then $x_1^* = a$ or $x_{n+1}^* = b$ or $x_1^* = a, x_{n+1}^* = b$, respectively.

Therefore from

$$\begin{aligned} F'_{s_i} &= 0 \quad (i = 1, \dots, n+1), \quad F'_{x_i} = 0 \quad (i = 2, \dots, n+1) \text{ or} \\ F'_{s_i} &= 0 \quad (i = 1, \dots, n+1), \quad F'_{x_i} = 0 \quad (i = 1, \dots, n) \text{ or} \\ F'_{s_i} &= 0 \quad (i = 1, \dots, n+1), \quad F'_{x_i} = 0 \quad (i = 2, \dots, n) \end{aligned}$$

we get a result similar to Theorem 1.2.

The results in case $g_1(x) \equiv 1, g_2(x) \equiv x$:

(1) If $a, b \notin X^*$, then

$$\begin{vmatrix} 1 & 1 & f(x_1^*) \\ -1 & 1 & f(x_2^*) \\ 1 & 1 & f(x_3^*) \end{vmatrix} = f'(x_i^*) \begin{vmatrix} 1 & 1 & x_1^* \\ -1 & 1 & x_2^* \\ 1 & 1 & x_3^* \end{vmatrix} \Leftrightarrow f'(x_i^*) = \frac{f(x_3^*) - f(x_1^*)}{x_3^* - x_1^*}, \quad (i = 1, 2, 3).$$

(2) If $a \in X^*$ and $b \notin X^*$, then

$$\begin{vmatrix} 1 & 1 & f(a) \\ -1 & 1 & f(x_2^*) \\ 1 & 1 & f(x_3^*) \end{vmatrix} = f'(x_i^*) \begin{vmatrix} 1 & 1 & a \\ -1 & 1 & x_2^* \\ 1 & 1 & x_3^* \end{vmatrix} \Leftrightarrow f'(x_i^*) = \frac{f(x_3^*) - f(a)}{x_3^* - a}, \quad (i = 2, 3).$$

(3) If $a \notin X^*$ and $b \in X^*$, then

$$f'(x_i^*) = \frac{f(b) - f(x_1^*)}{b - x_1^*}, \quad (i = 1, 2).$$

(4) If $a, b \in X^*$, then

$$f'(x_2^*) = \frac{f(b) - f(a)}{b - a}.$$

These results give a simple geometrical information about the extremal points.

2. On numerical applications of Theorem 1.2

Assume that $a, b \notin X^*$ (the other cases are similar). We shall use Theorem 1.2 in two versions:

(1) If $f(x), g_1(x), \dots, g_n(x)$ are twice-continuously differentiable and we have a suitable approach of sequence $x_1^* < x_2^* < \dots < x_{n+1}^*$, then we can compute $x_1^*, \dots, x_{n+1}^*$ by Newton's method ($A_1^*, \dots, A_n^*$ is easily computed from $x_1^*, \dots, x_{n+1}^*$). Here we apply the Newton's method to $g_1(x) \equiv 1, g_2(x) \equiv x$, as follows:

$$f'(x_i) \begin{vmatrix} 1 & 1 & x_1 \\ -1 & 1 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & f(x_1) \\ -1 & 1 & f(x_2) \\ 1 & 1 & f(x_3) \end{vmatrix} = 0, \quad (i = 1, 2, 3)$$

and the elements of the Jacobi matrix:

$$\begin{aligned} J_{11} &= f''(x_1) \begin{vmatrix} 1 & 1 & x_1 \\ -1 & 1 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} + f'(x_1) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} - \begin{vmatrix} 0 & 0 & f'(x_1) \\ -1 & 1 & f(x_2) \\ 1 & 1 & f(x_3) \end{vmatrix} = \\ &= 2(x_3 - x_1)f''(x_1), \end{aligned}$$

$$J_{12} = f'(x_1) \begin{vmatrix} 1 & 1 & x_1 \\ 0 & 0 & 1 \\ 1 & 1 & x_3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & f(x_1) \\ 0 & 0 & f'(x_2) \\ 1 & 1 & f(x_3) \end{vmatrix} = 0,$$

$$J_{13} = 2(f'(x_1) - f'(x_3)), \quad J_{21} = 2(f'(x_1) - f'(x_2)),$$

$$J_{22} = 2(x_3 - x_1)f''(x_2), \quad J_{23} = 2(f'(x_2) - f'(x_3)),$$

$$J_{31} = 2(f'(x_1) - f'(x_3)), \quad J_{32} = 0,$$

$$J_{33} = 2(x_3 - x_1)f''(x_3).$$

Hence

$$\begin{bmatrix} (x_3 - x_1)f''(x_1) & 0 & f'(x_1) - f'(x_3) \\ f'(x_1) - f'(x_2) & (x_3 - x_1)f''(x_2) & f'(x_2) - f'(x_3) \\ f'(x_1) - f'(x_3) & 0 & (x_3 - x_1)f''(x_3) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 - x_1 \\ \tilde{x}_2 - x_2 \\ \tilde{x}_3 - x_3 \end{bmatrix} =$$

$$= \begin{bmatrix} f(x_3) - f(x_1) - (x_3 - x_1)f'(x_1) \\ f(x_3) - f(x_1) - (x_3 - x_1)f'(x_2) \\ f(x_3) - f(x_1) - (x_3 - x_1)f'(x_3) \end{bmatrix},$$

where x_1, x_2, x_3 and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are "old and new approach" of x_1^*, x_2^*, x_3^* , respectively.

(2) If $f(x), g_1(x), \dots, g_n(x)$ are thrice-continuously differentiable, then we can look for the solution (solutions) of our system of equations in some intervals $a_1 \leq x_1 \leq b_1, \dots, a_{n+1} \leq x_{n+1} \leq b_{n+1}$ ($a < a_1 < b_1 < a_2 < \dots < a_{n+1} < b_{n+1} < b$) of $\mathbf{R}^{n+1}$ by method of [2]. (This method does not require an initial approach of x_1^*, x_2^*, x_3^* .)

Numerical example. Let $[a, b] = [-1.5; 2]$,

$$f(x) = 3x^4 - 4x^3 - 12x^2, \quad g_1(x) \equiv 1, \quad g_2(x) \equiv x.$$

From the graph of $f(x)$ we can see that $a, b \notin X^*$ ($f(x)$ has local minima at (-1) and 2 , local maximum at 0 ; furthermore $f(-1) = -5, f(0) = 0, f(2) = -32$). If we use the Newton's method with initial approach $x_1 = -1, x_2 = 0, x_3 = 2$, then for the first approach of the unique solution:

$$\begin{bmatrix} 108 & 0 & 0 \\ 0 & -72 & 0 \\ 0 & 0 & 216 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 + 1 \\ \tilde{x}_2 \\ \tilde{x}_3 - 2 \end{bmatrix} = \begin{bmatrix} -27 \\ -27 \\ -27 \end{bmatrix} \text{ and } \begin{matrix} \tilde{x}_1 = -1,25, \\ \tilde{x}_2 = 0,375, \\ \tilde{x}_3 = 1,875. \end{matrix}$$

We can get from Kantorovich's theorem (used for the second approach) that $\|\tilde{x}^{(3)} - x^*\|_\infty < 0,0007$, where $\tilde{x}^{(3)} \approx \{-1,2018; 0,3325; 1,8609\}$ is the third approach of the exact solution $x^* = (x_1^*, x_2^*, x_3^*)$.

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