# APPLICATIONS OF THE GRADIENT METHOD TO THE APPROXIMATE SOLUTION OF BOUNDARY VALUE PROBLEMS INVOLVING A SELFADJOINT ORDINARY DIFFERENTIAL EQUATION

A. SHAMANDY

Mathematical Department, Faculty of Science, Mansoura, Egypt.

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**1. Introduction.** In the papers [1], [2] we have introduced an application of the gradient method to the solution of boundary value problems involving a self-adjoint ordinary linear differential equation.

The problem is the following:

$$Au = \sum_{k=0}^{N} (-1)^k \frac{d^k}{dx^k}\left(P_k(x) \cdot \frac{d^k u}{dx^k}\right) = f \tag{1.1}$$

$$u(a) = u'(a) = \ldots = u^{(N-1)}(a) = u(b) = u'(b) = \ldots = u^{(N-1)}(b) = 0 \tag{1.2}$$

where $f \in L_2(I)$ is a given function, $I = [a, b]$ and the functions $P_0, P_1, \ldots P_N$ satisfy the conditions

(1.3)

i) $P_k(x) \in C^{(k)}(I), \quad k = 0, 1, \ldots, N$

ii) $P_k(x) > 0 \quad$ for every $x \in I$, $k = 0, 1, \ldots, N-1$

iii) there exists a constant $m > 0$ such that for all $x \in I, P_N(x) \geq m$.

Let us choose an arbitrary function $u_0 \in H_2^{0(2N)}(I)$ and assume that we have obtained the $(n-1)^{\text{th}}$ approximation of the solution $U \in H_2^{0(2N)}(I)$ of the boundary value problem (1.1). Suppose we already have

$$u_1, u_2, \ldots, u_{n-1},$$

by introducing the notation

$$f_n = Au_{n-1} - f \tag{1.4}$$

and at each step we solve the boundary value problem

(1.5)

$$(-1)^N \frac{d^{2N} V}{dx^{2N}} = f_n,$$

$$V(a) = V'(a) = \ldots = V^{(N-1)}(a) = V(b) = V'(b) = \ldots = V^{(N-1)}(b) = 0.$$

Then we get the $n^{\text{th}}$ approximation of $u$

$$u_n = u_{n-1}+t_n V_n \tag{1.6}$$

where $t_n$ is defined by

$$t_n = -\frac{\int_a^b |V_n^{(N)}|^2}{\sum_{k=0}^{N} \int_a^b p_k |V_n^{(k)}|^2 dx}. \tag{1.7}$$

From the above algorithm we obtain that the sequence $(U_n) \in H_2^{0(2N)}(I)$ converges to the solution $u$ of the boundary value problem (1.1) in the norm

$$\|u\|_N'' = \sum_{k=0}^{N-1} \max_I |u^{(k)}| + \|u^{(N)}\|_{L_2}$$

and the error is estimated by

$$|u_n - u| \leq K_0 q^n, \quad (n = 0, 1. \dots) \tag{1.8}$$

$K_0 > 0$ is a constant, $0 < q < 1$ [2].

**2.1. Introducing a simple spline function to obtain some approximate results.** Practically it is not easy to use this method [2] to get the approximate solution for (1.1). So we are forced to use a simple spline function for this purpose [4], [5], [6], [7].

Our main purpose will be to study applications of simple spline functions to the numerical solution of (1.1). We develop a method which produces a smooth approximation to the solution $U$ in the form of piecewise polynomial functions of degree $< r$ which are joined at points called knots which have at least $m$ continuous derivatives. If $S$ is the spline function then it satisfies:

$$S \in C^{(m)}(I), \quad m < r. \tag{2.1.1}$$

(2.1.2) $S \in \pi_m$ in each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \dots, (n-1)$

where $\pi_m$ denotes the set of all polynomials of degree $< r$.
We define the knots by

$$\Delta : a = x_0 < x_1 < \dots < x_n = b, \tag{2.1.3}$$

and in our case we shall deal with equal subintervals and in this paper we denote

(2.1.4) $$h := x_{\nu+1} - x_\nu, \quad \nu = 0, 1, \ldots, (m-1),$$
$$h = \frac{b-a}{m},$$

(2.1.5) $$S_\nu(x_{\nu+1}, g_\nu) = S_{\nu+1}(x_{\nu+1}, g_\nu), \quad \nu = 0, 1, \ldots (m-1),$$

where $S_\nu$ is a simple spline function interpolated on the mesh (2.1.3) which gives the sets of points

(2.1.6) $$\{(g_0, g_1, \ldots, g_\nu \ldots, g_n)\}, \ \{\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_\nu, \ldots, \bar{g}_m\}.$$

**2.2. Some notations.** *a)* in (1.7) assume that

(2.2.1) $$(V_n^{(N)}(x))^2 = g_n(x), \ (V_n^{(N)}(x))^2_{x=x_\nu} = g_n(x_\nu),$$
$$\nu = 0, 1, \ldots, (m-1), m = g_{n,\nu}.$$

Also

(2.2.2) $$\sum_{k=0}^{N} p_k(x)(V_n^{(k)}(x))^2 = \bar{g}_n(x),$$
$$\sum_{k=0}^{N} p_k(x_\nu)(V_n^{(k)}(x))^2_{x=x_\nu} = \bar{g}_n(x_\nu) = \bar{g}_{n,\nu},$$

(2.2.3) $$S_{h,n} = g_{n,\nu} + \frac{g_{n,\nu+1} - g_{N,\nu}}{h}(x - x_\nu) = S_{\nu,n}(x, g).$$

Also we have

$$\bar{S}_{h,n}(x, \bar{g}) = \bar{g}_{\nu,n} + \frac{\bar{g}_{\nu+1,n} - \bar{g}_{\nu,n}}{h}(x - x_\nu) = \bar{S}_{\nu,n}(x, \bar{g}), \ \nu = 0, 1, \ldots, \ldots, m-1.$$

(2.2.4)

*b)* Let $w(h, g)$ and $w(h, \bar{g})$ be the modulis of continuity of the functions $g$ and $\bar{g}$ respectively.

**Lemma.** *The inequalitities*

$$|g(x)S_\nu(x, g)| \le 2w(h, g), \quad = 0, 1, \ldots, m-1,$$
$$\left| \int_a^b g_n(x)dx - \int_a^b s_{h,n}(x, g_n)dx \right| < 2w(h, g_n)(b-a)$$

*are true.* □

**Proof.**

$$|g(x) - S_\nu(x, g)| = \left| g(x) - g(x_\nu) - \frac{g(x_{\nu+1}) - g(x_\nu)}{h}(x - x_\nu) \right| \le$$

(2.2.5) $$\le |g(x) - g(x_\nu)| + \frac{|g(x_{\nu+1}) - g(x_\nu)|}{h}, \quad \nu = 0, 1, \ldots, m-1.$$

$$|g(x) - S_\nu(x, g)| \le 2w(h, g),$$

where $w(h, g) \to 0$ as $m \to \infty$.

Also we have

$$\left|\int_a^b g_n(x)dx - \int_a^b s_{h,n}(x, g_n)dx\right| =$$

$$\leq \left|\int_a^b |g_n(x) - s_{h,n}(x, g_n)|dx\right| =$$

$$\leq \sum_{\nu=0}^{m-1} \int_{x_\nu}^{x_{\nu+1}} |g_n(x) - s_{h,n}(x, g)dx| ,$$

from (2.2.4) if follows

$$\leq \sum_{\nu=0}^{m-1} \int_{x}^{x_{\nu+1}} |g_n(x) - s_{\nu,n}(x, g_n)|dx \leq$$

$$\leq \sum_{\nu=0}^{m-1} \int_{x_\nu}^{x_{\nu+1}} |g_n(x) - s_{\nu,n}(x, g_n)|dx < 2w(h, g_n) \sum_{\nu=0}^{m-1} \int_{x_\nu}^{x_{\nu+1}} dx.$$

Then we get

$$\left|\int_a^b g_n(x)dx - \int_a^b s_{h,n}(x, g_n)dx\right| < 2w(h, g_n) \cdot (b-a). \tag{2.2.6}$$

From lemma (1) we can calculate the value of the integral

$$\int_a^b (V_n^{)N)}(x))^2dx$$

in equation (1.7). From (2.2.1) we know that

$$\int_a^b g_n(x)dx = \int_a^b (V_n^{(N)}(x))^2dx \rightarrow \int_a^b s_{h,n}(x, g_n)dx.$$

But

$$\int_a^b s_{h,n}(x, g_n)dx \sum_{\nu=0}^{m-1} \int_{x_\nu}^{x_{\nu+1}} s_{h,n}(x, g_n)dx = \sum_{\nu=0}^{m-1} \int_{x_\nu}^{x_{\nu+1}} S_{\nu,n}(x, g_n)dx =$$

$$= \sum_{\nu=0}^{m-1} \int_{x_\nu}^{x_{\nu+1}} \left[g_{\nu,n} + \frac{g_{\nu+1,n} - g_{\nu,n}}{h}(x - x_\nu)\right]dx =$$

$$= \sum_{\nu=0}^{m-1} \left\{g_{\nu,n}\, h + \frac{g_{\nu+1,n} - g_{\nu,n}}{2} h\right\} = \tag{2.2.7}$$

$$= \frac{h}{2} \sum_{\nu=0}^{m-1} |g_{\nu+1,n} + g_{\nu,n}| = \frac{h}{2} \sum_{\nu=0}^{m-1} \{(V_n^{(N)}(x))^2_{x=x_{\nu+1}} + (V_n^{(N)}(x))^2_{x=x_\nu}\}.$$

Then we have from (2.2.6)

$$\left|\int_a^b (V_n^{(N)}(x))^2dx - \frac{h}{2}\sum_{\nu=0}^{m-1}\{(V_n^{(N)}(x))^2_{x=x_{\nu+1}} + (V_n^{(N)}(x))^2_{x=x_\nu}\}\right| \le 2(b-a)w(h, V_n^{(N)}(x)). \quad \square \tag{2.2.8}$$

**Lemma 2.** *The following inequalities*

$$|\bar{g}(x) - \bar{s}_\nu(x, \bar{g})| \le 2w(h, \bar{g}),$$

$$\left|\int_a^b \bar{g}(x)dx = \int_a^b \bar{s}_{h,n}(x, \bar{g}_n)dx\right| \le 2\cdot(b-a)\cdot w(h, \bar{g}_n)$$

*are true.* □

**Proof.** The same as in Lemma 1. □
We can prove that

$$\left|\sum_{k=0}^{N}\int_a^b p_k(x)(V_n^{(k)})^2dx - \frac{h}{2}\sum_{\nu=0}^{m-1}(\bar{g}_{\nu+1,n} + \bar{g}_{\nu,n})\right| \le 2(b-a)w(h, \bar{g}_n). \tag{2.2.9}$$

From (1.7), (2.2.7), (2.2.8), (2.2.9) we can define $t_n^*$ as

$$t_n^*(m) = -\frac{\int_a^b s_{h,n}(x, g_n)dx}{\int_a^b \bar{s}_{h,n}(x, \bar{g}_n)dx}, \tag{2.2.10}$$

and we can prove that

$$t_n^*(m) = -\frac{\sum_{\nu=0}^{m-1}\{g_{n,\nu} + g_{n,\nu+1}\}}{\sum_{\nu=0}^{m-1}\{\bar{g}_{n,\nu} + \bar{g}_{n,\nu+1}\}}, \quad n = 1, 2, \ldots. \quad \square \tag{2.2.11}$$

**Lemma 3.** *The inequality*

$$|t_n^*(m) - t_n| \le K_1 \max\{w(h, \bar{g}_n), w(h, g_n)\} \tag{2.2.12}$$

*is true, where $K_1$ is a constant.* □

**Proof.** Assume that

$$t_n = \frac{\gamma}{\delta}, \ \gamma = \int_a^b (V_n^{(N)})^2dx, \ \delta = \sum_{k=0}^{N}\int_a^b p_k(x)(V_n^{(k)}(x))^2dx,$$

$$t_n^*(m) = \frac{\gamma(m)}{\delta(m)}, \ \gamma(m) = \int_a^b s_{h,n}(x, g_n)dx, \ \delta(m) = \int_a^b \bar{s}_{h,n}(x, \bar{g}_n)dx,$$

then

$$|t_n - t_n^*(m)| = \left|\frac{\gamma}{\delta} - \frac{\gamma(m)}{\delta(m)}\right| = \frac{|\gamma \cdot \delta(m) - \delta \cdot \gamma(m)|}{|\delta \cdot \delta(m)|} \leq$$

$$\leq \frac{|\delta(m)| \cdot |\delta - \delta(m)| + |\delta(m)| \cdot |\delta(m) - \delta|}{|\delta \cdot \delta(m)|}.$$

From lemmas (1) and (2) we get

$$|t_n - t_n^*(m)| \leq \frac{\delta\omega(h, g_n) + \gamma\omega(h, \bar{g}_n)}{\delta^2} \to 0 \text{ as } m \to \infty.$$

Then for some constant $K_1$ we have

$$|t_n - t_n^*(m)| \leq K_1 \cdot \max\{(w(h, \bar{g}_n), w(h, g))\}.$$

**3.1. Application of the Gradient method to the approximate solution of a boundary value problem of a self-adjoint ordinary differential equation.** We can apply the gradient method given in [2] to obtian a numerical solution of (1.1) by using (1. 4), (1.5), (2.2.8), (2.2.9), (2.2.11) and the boundary condition

$$(3.1.1) \quad \left(\frac{d^{(j)}V_n}{dx^j}\right)_{x=0} = \left(\frac{d^{(j)}(V_n)}{dx^j}\right)_{x=1} = 0, \quad j = 0, 1, 2, \ldots 2N-1.$$

We can summarise the algorithm as follows:
$f(x)$ is a given function in (1.1). Consider the interval $I = [0, 1]$.
Assume that $u_0 = 0$, from (1.4), (1.5) we have,

$$f_1(x) = Au_0 - f(x) = -f(x),$$

$$(-1)^{(N)} \frac{d^{2N}V_1}{dx^{2N}} = f_1(x) = -f(x).$$

We can prove that

$$(3.1.2) \quad V_1(x) = (-1)^N \int_0^x \int_0^{\xi_{2N-1}} \ldots \int_0^{\xi_1} f_1\{(R)dR\}d\xi, \ldots d\xi_{2N-2}\xi_{2N-1}.$$

From (2.2.10) we can prove that

$$t_1^* = -\frac{\sum_{\nu=0}^{m-1} \{g_{1,\nu} + g_{1,\nu+1}\}}{\sum_{\nu=0}^{m-1} \{\bar{g}_{1,\nu} + \bar{g}_{1,\nu+1}\}},$$

and then by (1.6)

$$u_1^* = u_0^* + t_1^* V_1^*, \; u_0 = 0, \; V_1 = V_1^*.$$

By the same way we can calculate

$$(3.1.3) \quad u_1^*, u_2^*, \ldots, u_{n-1}^*.$$

Then the $n$-th approximation of the $u$ solution of (1.1) is

$$u_n^* = u_{n-1}^* + t_n^* V_n, \quad V_n = V_n^*, \tag{3.1.4}$$

where $t_n^*$ is given by (2.2.11).

**3.2. The convergence of the sequence** $(u_n^*)$. In [2] we showed that the sequence of approximations $(u_n)$ converges to the solution of (1.1) and the error is estimated by (1.8).
For the sequence $(u_n^*)$ we have [2]

$$|u(x) - u_n^*| \leq |u(x) - u_n(x)| + |u_n(x) - u_n^*(x)| <$$
$$< K_0 q^n + |u_n(x) - u_n^*(x)|.$$

We can prove that

$$|u_n(x) - u_n^*(x)| \leq \left| \sum_{\nu=1}^{n} \{t_\nu v_\nu - t_\nu^* v_\nu^*\} \right| \leq$$
$$\leq K_2 \max\{w(h, \bar{g}_n), \ w(h, g_n)\}$$

$$|u(x) - u_n^*| < K_0 q^n + K_2 \max\{w(h, g_n), w(h, \bar{g}_n)\}. \tag{3.1.5}$$

Then

$$u(x) \to u_n^* \text{ as } n \to \infty, \ 0 < q < 1, \ K_2 \text{ is constant.}$$

The above results can be formulated in the following assertion.

**Theorem.** *Consider the boundary value problem* (1.1), (1.2). *Let* $S$, $\bar{S}$ *be simple spline functions* (2.2.3), (2.2.4) *interpolated on the mesh* $\Delta : a = x_0 < x_1 < \ldots < x_m = b$, $(x_{\nu+1} - x_\nu = h)$ *to give the sets of points* $\{g_0, g_1, \ldots, g_n\}$, $\{\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_n\}$. *Suppose we have obtained the* $(n-1)^{\text{th}}$ *approximationof the solution* $u$ *of* (1.1), *as in* [2],

$u_1^*, u_2^*, \ldots u_{n-1}^*$ *by solving for each step the boundary value problem* (1.5). *Then the n-th approximation of* $u$ *is* $u_n^* = u_{n-1}^* + t_n^* V_n$, *where* $t_n^*$ *is defined by* (2.2.11).

*The sequence* $(u_n^*)$ *converges to the solution* $u$ *and the error is estimated by* (3.1.5). □

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