HERMITE INTERPOLATION AND THE TWO POINT BOUNDARY VALUE PROBLEMS. APPLICATION OF THE METHODS

By

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The Hermite interpolation methods established in [2] and [3] for the solution of non-linear ordinary differential equations are applied in this paper to solve the two boundary value problems. Two examples are given and programs are written for them in FORTRAN IV. The programs are run in the computer ODRA – 1304 (compatible with the ICL – 1900 series) giving approximately 11 digits of accuracy. Numerical results are obtained for comparison between the numerical exact solutions and the approximate ones.

```
      MASTER DIFFEQ
      INTEGER NN(2)
      REAL F(10)
      REAL X(21),Y(21),YD(21),YDD(21)
      DATA IDB/2/,NN/10,20/
      CALL GENFAKT(F,10)
      DO 1 II=1,IDB
      N=NN(II)+1
      CALL GENXNU(X,N)
      Y   (1)=0.0
      YD  (1)=-1.0+COT(1.0)
      YDD(1)=-1.0
      DO 2 I=2,N/2+1
      H=X(I)-X(I-1)
      H2=H*H/2.0
      H3=H2*H/3.0
      H4=H3*H/4.0
      H5=H4*H/5.0
      H6=H5*H/6.0
      H7=H6*H/7.0
      H8=H7*H/8.0
      H9=H8*H/9.0
      H10=H9*H/10.0
      YDD(I)=Y   (I-1)*(-1.0+H2-H4+H6)+
     1       YD  (I-1)*(-H+H3-H5+H7)+
     2       YDD(I-1)*H8+
     3       X   (I-1)*(H6-H4+H2)+
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     4                  (H7-H5+H3-H)-X(I-1)
      YD  (I)=Y    (I-1)*(-H+H3-H5+H7)+
     1          YD  (I-1)*(1.0-H2+H4-H6+H8)+
     2          YDD(I-1)*H9+
     3          X   (I-1)*(H7-H5+H3-H)+
     4                  (H8-H6+H4-H2)
      Y   (I)=Y    (I-1)*(1.0-H2+H4-H6+H8)+
     1          YD  (I-1)*(H-H3+H5-H7+H9)+
     2          YDD(I-1)*H10+
     3          X   (I-1)*(H8-H6+H4-H2)+
     4                  (H9-H7+H5-H3)
   2  CONTINUE
      WRITE (2,100) NN(II)
 100  FORMAT (1H1,25X, 'EXAMPLE 1.'/1H0,25X, 'N=',I3/1H0,5X,
     1'NU   X(NU)         EXACT SOLUTION          NUM. SOLUTION'/1H0)
      DO 3 I=1,N/2+1
      CAL EXACT1(X(I),E,ED,EDD)
      NU=I-2
      WRITE (2,101) NU,X(I),E,Y(I),ED,YD(I),EDD,YDD(I)
 101  FORMAT (1H0,I7,F13.7    ,4X,3HY  =,F15.10,3X,3HY  =,F15.10/
     1                  1X,24X.3HY '=,F15.10,3X,3HY '=,F15.10/
     2                  1X,24X,3HY"=,F15.10,3X,3HY"=,F15.10)
   3  CONTINUE
      DO 4 I=1,N/2
   4  Y(N+1-I)=-Y(I+1)
      WRITE(2,300) NN(II)
 300  FORMAT(1H1,25X,'EXAMPLE 1.'/1H0,25X,'N=',I3/1H0,8X,
     1'X                EXACT SOLUTION          NUM. SOLUTION'/H0)
      XX=-1.0
      DO 5 I=1,9
      SE=SIN(XX)/SIN(1.0)-XX
      S  =HEX1(XX,Y,X,N)
      WRITE(2,102) XX,SE,S
 102  FORMAT (1H0,F15.7,7X,F15.10,6X,F15.10)
   5  XX=XX+0.25
   1  CONTINUE
      DO 10 II=1,IDB
      N=NN(II)+1
      CALL GENXNU(X,N)
      Y   (1)=0.0
      YD  (1)=-(EXP(1.0)-EXP(-1.0))/(EXP(1.0)+EXP(-1.0))
      YDD(1)=-1.0
      DO 20 I=2,N/2+1
      H=X(I)-X(I-1)
      H2=H*H/2.0
      H3=H2*H/3.0
      H4=H3*H/4.0
      H5=H4*H/5.0
      H6=H5*H/6.0
      H7=H6*H/7.0
      H8=H7*H/8.0
      H9=H8*H/9.0
      H10=H9*H/10.0
      YDD(I)=Y    (I-1)*(1.0+H2+H4+H6)+
     1          YD  (I-1)*(H+H3+H5+H7)+
     2          YDD(I-1)*H8-
     3                  (1.0+H2+H4+H6)
      YD  (I)=Y    (I-1)*(H+H3+H5+H7)+
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     1            YD  (I-1)*(1.0+H2+H4+H6+H8)+
     2            YDD(I-1)*H9-
     3                       (H+H3+H5+H7)
      Y    (I)=Y    (I-1)*(1.0+H2+H4+H6+H8)+
     1            YD  (I-1)*(H+H3+H5+H7+H9)+
     2            YDD(I-1)*H10-
     3                       (H2+H4+H6+H8)
 20   CONTINUE
      WRITE (2,200) NN(II)
200   FORMAT (1H1,25X,'EXAMPLE 2.'/1H0,25X,'N=',I3/1H0,5X,
     1'NU         X(NU)          EXACT SOLUTION    NUM.SOLUTION'/1H0)
      DO 30 I=1,N/2+1
      CALL EXACT2(X(I),E,ED,EDD)
      NU=I-2
      WRITE (2,101) NU,X(I),E,Y(I),ED,YD(I),EDD,YDD(I)
 30   CONTINUE
      DO 40 I=1,N/2
 40   Y(N+1-I)=Y(I+1)
      WRITE(2,400) NN(II)
400   FORMAT (1H1,25X,'EXAMPLE 2.'/1H0,25X,'N=',I3/1H0,8X,
     1'X                    EXACT SOLUTION        NUM. SOLUTION'/1H0)
      XX=-1.0
      DO 50 I=1,9
      SE=1.0-(EXP(XX)+EXP(-XX))/(EXP(1.0)+EXP(-1.0))
      S=HEX2(XX,Y,X,N)
      WRITE(2,102) XX,SE,S
 50   XX=XX+0.25
 10   CONTINUE
      STOP
      END


      SUBROUTINE GENFAKT(FAKT,N)
      REAL FAKT (N)
      FAKT(1)=1.0
      DO 1 I=2,N
  1   FAKT(I)=FAKT(I-1)*I
      RETURN
      END


      SUBROUTINE GENXNU(X,N)
      REAL X(N)
      M=N+N-2
      X(1)=1.0
      PI=ACOS(-1.0)
      D0 1 I=2,N
  1   X(I)=COS((I+I-3)*PI/M)
      RETURN
      END


      SUBROUTINE EXACT1(X,Y,YD,YDD)
      S=SIN(1.0)
      Y   =SIN(X)/S-X
      YD  =COS(X)/S-1.0
      YDD=-SIN(X)/S
      RETURN
      END
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```
      SUBROUTINE EXACT2(X,Y,YD,YDD)
      E=EXP(X)
      F=EXP(-X)
      C=1.0/(EXP(1.0)+EXP(   -1.0))
      Y   =1.0-(E+F)*C
      YD  =-C*(E-F)
      YDD=-C*(E+F)
      RETURN
      END

      FUNCTION HEX1(X,Y,XNU,N)
      REAL Y(N),XNU(N)
      C=COS((N-1)*ACOS(X))
      R0 =((X-1.0)+0.5*(X-1.0)**2)*0.5*(1.0+X)*C
      RN=((X+1.0)-0.5*(X+1.0)**2)*0.5*(1.0-X)*C
      HEX1=(-1.0+COT(1.0))*(R0+RN)
      S=0.0
      DO 1 I=2,N
    1 S=S+Y(I)*(-1.0)**I/((X-XNU(I))*(1.0-XNU(I)**2)**1.5)
      HEX1=HEX1+(1.0-X*X)**2*C/(N-1)*S
      RETURN
      END

      FUNCTION HEX2(X,Y,XNU,N)
      REAL Y(N),XNU(N)
      C=COS((N-1)*ACOS(X))
      R0 =((X-1.0)+0.5*(X-1.0)**2)*0.5*(1.0+X)*C
      RN=((X+1.0)-0.5*(X+1.0)**2)*0.5*(1.0-X)*C
      E=EXP(1.0)
      E=E*E
      HEX2=(1.0-E)/(1.0+E)*(R0-RN)
      S=0.0
      DO 1 I=2,N
    1 S=S+Y(I)*(-1.0)**I/((X-XNU(I))*(1.0-XNU(I)**2)**1.5)
      HEX2=HEX2+(1.0-X*X)**2*C/(N-1)*S
      RETURN
      END
```

The program

Example 1

Consider the following differential equation (see [4], p. 212. for the problem itself, but the numerical method here is different and the results are better)

(1) $$y''+y+x = 0 \qquad -1 \leqq x \leqq 1,$$

with the boundary conditions

(2) $$y(-1) = y(1) = 0$$ $$y'(-1) = y'(1) = -1+\operatorname{ctg} 1.$$

The exact solution is obviously

(3) $$y = \frac{\sin x}{\sin 1} - x.$$

Equations (1.13–14) and (3.10–11) presented in [2] give the approximate values $\bar{y}_{\nu+1}$ of the exact values $y_{\nu+1}$ at the points

(4) $$x_{\nu+1} = \cos \frac{2\nu+1}{2n} \pi \quad \left(0 \leqq \nu \leqq \left[\frac{n}{2}\right] - 1\right).$$

In this example, we have from the asymmetry

(5) $$y_{n-\nu} = -\bar{y}_{\nu+1} \quad \left(\left[\frac{n}{2}\right] \leqq \nu \leqq n-1\right).$$

From (3.9–14) and (3.8) in [2], we obtain $\bar{y}_{-1} = y(-1) = 0$, $\bar{y}'_{-1} = -1 + \operatorname{ctg} 1$ and

(6) $$\bar{y}_{\nu+1} = A_{11}\bar{y}_\nu + B_{11}\bar{y}'_\nu + \frac{(x_{\nu+1}-x_\nu)^8}{8!}\bar{y}''_\nu + C_{11}x_\nu + D_{11}$$
$$\left(-1 \leqq \nu \leqq \left[\frac{n}{2}\right] - 1,\ x_{-1} = -1\right)$$

where

(7) $$A_{11} = -1 + \frac{(x_{\nu+1}-x)^2}{2!} - \frac{(x_{\nu+1}-x_\nu)^4}{4!} + \frac{(x_{\nu+1}-x_\nu)^6}{6!},$$
$$B_{11} = -\frac{x_{\nu+1}-x_\nu}{1!} + \frac{(x_{\nu+1}-x_\nu)^3}{3!} - \frac{(x_{\nu+1}-x_\nu)^5}{5!} + \frac{(x_{\nu+1}-x_\nu)^7}{7!},$$
$$C_{11} = A_1 + 1,$$
$$D_{11} = B_{11} - x_\nu.$$

(8) $$\bar{y}'_{\nu+1} = A_{12}\bar{y}_\nu + B_{12}\bar{y}'_\nu + \bar{y}''_\nu\frac{(x_{\nu+1}-x_\nu)^9}{9!} + A_{12}x_\nu + D_{12},$$

where

(9) $$A_{12} = B_{11},$$
$$B_{12} = -A_{11} + \frac{(x_{\nu+1}-x_\nu)^8}{8!},$$
$$D_{12} = B_{12} - 1.$$

And

(10) $$\bar{y}_{\nu+1} = A_{13}\bar{y}_\nu + B_{13}y' + \bar{y}''_\nu\frac{(x_{\nu+1}-x_\nu)^{10}}{10!} + C_{13}x_\nu + D_{13},$$

where

(11) $$A_{13} = -A_{11} + \frac{(x_{\nu+1}-x_\nu)^8}{8!},$$
$$B_{13} = -B_{11} + \frac{(x_{\nu+1}-x_\nu)^9}{9!},$$
$$C_{13} = D_{12},$$
$$D_{13} = \frac{(x_{\nu+1}-x_\nu)^9}{9!} - \frac{(x_{\nu+1}-x_\nu)^7}{7!} + \frac{(x_{\nu+1}-x_\nu)^5}{5!} - \frac{(x_{\nu+1}-x_\nu)^3}{3!}.$$

The approximate solution of (1 – 2) is thus given as follows

(12)
$$\bar{H}(x) = (-1+\operatorname{ctg} 1)\{r_{0,1}(x)+r_{n+1,1}(x)\}+$$
$$+\frac{1}{n}\sum_{\nu=0}^{n-1}\bar{y}_{\nu+1}\frac{(1-x^2)^2[\cos(n\arccos x)](-1)^\nu}{(x-x_{\nu+1})(1-x_{\nu+1}^2)^{3/2}}$$

where

(13)
$$r_{0,1}(x) = \left\{(x-1)+\frac{1}{2}(x-1)^2\right\}\left\{\frac{1+x}{2}\right\}\cos(n\arccos x),$$
$$r_{n+1,1}(x) = \left\{(x+1)-\frac{1}{2}(x+1)^2\right\}\left\{\frac{1-x}{2}\right\}\cos(n\arccos x).$$

In the equations (4 – 12), we let $n = 10$ and the program 1 is set for them. This program is run and the numerical results obtained are set out in tables 1.1 – 2.

To see the effect of n we repeated the procedure for $n = 20$ and the numerical results obtained are given in tables 1.3 – 4.

EXAMPLE 1.
N=10

X	EXACT SOLUTION	NUM. SOLUTION
–1.0000000	0.0000000000	0.0000000000
–0.7500000	–0.0600561663	–0.0600561664
–0.5000000	–0.0697469637	–0.0697469634
–0,2500000	–0.0440136543	–0.0440136546
0.0000000	0.0000000000	0.0000000003
0.2500000	0.0440136543	0.0440136542
0.5000000	0.0697469637	0.0697469638
0.7500000	0.0600561663	0.0600561663
1.0000000	0.0000000000	0.0000000000

Table 1.1.

EXAMPLE 1.
N=10

NU	X(NU)	EXACT SOLUTION	NUM. SOLUTION
–1	1.0000000	Y = 0.0000000000 Y' = –0.3579073840 Y'' = –1.0000000000	Y = 0.0000000000 Y' = –0.3579073841 Y'' = –1.0000000000

0	0.9876883	Y = 0.0043308460 Y' = −0.3456446982 Y" = −0.9920191866	Y = 0.0043308460 Y' = −0.3456446983 Y" = −0.9920191866
1	0.8910065	Y = 0.0332141397 Y' = −0.2529396999 Y" = −0.9242206639	Y = 0.0332141397 Y' = −0.2529397000 Y" = −0.9242206639
2	0.7071068	Y = 0.0649185778 Y' = −0.0965290416 Y" = −0.7720253589	Y = 0.0649185777 Y' = −0.0965290416 Y" = −0.7720253589
3	0.4539905	Y = 0.0671864668 Y' = 0.0680156820 Y" = −0.5211769666	Y = 0.0671864668 Y' = 0.0680156820 Y" = −0.5211769666
4	0.1564345	Y = 0.0287141744 Y' = 0.1738836862 Y" = −0.1851486395	Y = 0.0287141744 Y' = 0.1738836862 Y" = −0.1851486396

Table 1.2.

EXAMPLE 1.

N = 20

X	EXACT SOLUTION	NUM. SOLUTION
−1.0000000	0.0000000000	0.0000000000
−0.7500000	−0.0600561663	−0.0600561665
−0.5000000	−0.0697469637	−0.0697469637
−0.2500000	−0.0440136543	−0.0440136542
0.0000000	0.0000000000	0.0000000000
0.2500000	0.0440136543	0.0440136545
0.5000000	0.0697469637	0.0697469634
0.7500000	0.0600561663	0.0600561662
1.0000000	0.0000000000	0.0000000000

Table 1.3.

EXAMPLE 1.

N = 20

NU	X(NU)	EXACT SOLUTION	NUM. SOLUTION
−1	1.0000000	Y = 0.0000000000 Y' = −0.3579073840 Y" = −1.0000000000	Y = 0.0000000000 Y' = −0.3579073841 Y" = −1.0000000000
0	0.9969173	Y = 0.0010985608 Y' = −0.3548277735 Y" = −0.9980158945	Y = 0.0010985607 Y' = −0.3548277735 Y" = −0.9980158945

1	0.9723699	Y =	0.0095095804	Y =	0.0095095804
		Y' =	−0.3305258978	Y' =	−0.3305258979
		Y" =	−0.9818795008	Y" =	−0.9818795008
2	0.9238795	Y =	0.0243955005	Y =	0.0243955005
		Y' =	−0.2837197551	Y' =	−0.2837197552
		Y" =	−0.9482750330	Y" =	−0.9482750330
3	0.8526402	Y =	0.0422454138	Y =	0.0422454138
		Y' =	−0.2180391764	Y' =	−0.2180391765
		Y" =	−0.8948855781	Y" =	−0.8948855781
4	0.7604060	Y =	0.0586545359	Y =	0.0586545359
		Y' =	−0.1389408718	Y' =	−0.1389408718
		Y" =	−0.8190605015	Y" =	−0.8190605015
5	0.6494480	Y =	0.0692302243	Y =	0.0692302243
		Y' =	−0.0535410902	Y' =	−0.0535410902
		Y" =	−0.7186782726	Y" =	−0.7186782726
6	0.5224986	Y =	0.0705663134	Y =	0.0705663134
		Y' =	0.0298334710	Y' =	0.0298334709
		Y" =	−0.5930648781	Y" =	−0.5930648781
7	0.3826834	Y =	0.0610765385	Y =	0.0610765385
		Y' =	0.1024336786	Y' =	0.1024336786
		Y" =	−0.4437599709	Y" =	−0.4437599709
8	0.2334454	Y =	0.0414670253	Y =	0.0414670253
		Y' =	0.1561600693	Y' =	0.1561600692
		Y" =	−0.2749123892	Y" =	−0.2749123892
9	0.0784591	Y =	0.0146856771	Y =	0.0146856771
		Y' =	0.1847392029	Y' =	0.1847392028
		Y" =	−0.0931447728	Y" =	−0.0931447728

Table 1.4.

Example 2

Consider the differential equation (see [4] for the problem, but a different method is used and the numerical results are here considerably better).

(14) $$y''-y+1=0$$

with the boundary conditions

(15) $$y(-1)=y(1)=0$$
$$y'(-1)=\frac{e^2-1}{1+e^2},\quad y'(1)=\frac{1-e^2}{1+e^2},$$

The exact solution is obviously

(16) $$y=1-\frac{\operatorname{ch} x}{\operatorname{ch} 1}.$$

The approximate values $\bar{y}_{\nu+1}$ are evaluated at

$$x_{\nu+1} = \cos\frac{2\nu+1}{2n}\pi \quad \left(0 \leqq \nu \leqq \left[\frac{n}{2}\right]-1\right). \tag{17}$$

In this example, we have

$$\bar{y}_{n-\nu} = \bar{y}_{\nu+1} \quad \left(\left[\frac{n}{2}\right] \leqq \nu \leqq n-1\right). \tag{18}$$

From (3.8) and (3.9 – 14) presented in [2], we obtain

$$\bar{y}_{-1} = y(-1) = 0, \ \bar{y}'_{-1} = y'(-1) = \frac{e^2-1}{1+e^2}$$

and

$$\bar{y}_{\nu+1} = A_{21}\bar{y}_\nu + B_{21}\bar{y}'_\nu + \bar{y}''_\nu \frac{(x_{\nu+1}-x_\nu)^8}{8!} - A_{21}, \tag{19}$$

$$\left(-1 \leqq \nu \leqq \left[\frac{n}{2}\right]-1, \ x_{-1} = -1\right)$$

where

$$\begin{aligned} A_{21} &= 1 + \frac{(x_{\nu+1}-x_\nu)^2}{2!} + \frac{(x_{\nu+1}-x_\nu)^4}{4!} + \frac{(x_{\nu+1}-x_\nu)^6}{6!}, \\ B_{21} &= \frac{(x_{\nu+1}-x_\nu)}{1!} + \frac{(x_{\nu+1}-x_\nu)^3}{3!} + \frac{(x_{\nu+1}-x_\nu)^5}{5!} + \frac{(x_{\nu+1}-x_\nu)^7}{7!}. \end{aligned} \tag{20}$$

$$\bar{y}'_{\nu+1} = A_{22}\bar{y}_\nu + B_{22}\bar{y}'_\nu + \bar{y}''_\nu \frac{(x_{\nu+1}-x_\nu)^9}{9!} - A_{22} \tag{21}$$

where

$$\begin{aligned} A_{22} &= B_{21} \\ B_{22} &= A_{21} + \frac{(x_{\nu+1}-x_\nu)^8}{8!}. \end{aligned} \tag{22}$$

And

$$\bar{y}''_{\nu+1} = A_{23}\bar{y}_\nu + B_{23}\bar{y}'_\nu + \bar{y}''_\nu \frac{(x_{\nu+1}-x_\nu)^{10}}{10!} + C_{23} \tag{23}$$

where

$$\begin{aligned} A_{23} &= B_{22} \\ B_{23} &= A_{22} + \frac{(x_{\nu+1}-x_\nu)^9}{9!} \\ C_{23} &= -B_{22} + 1. \end{aligned} \tag{24}$$

The approximate solution of (14 – 15) is thus given as follows

(25)
$$\bar{H}(x) = \frac{1-e^2}{1+e^2}\{r_{0,1}(x) - r_{n+1,1}(x)\} + \\ + \frac{1}{n}\sum_{\nu=0}^{n-1} \bar{y}_{\nu+1} \frac{(1-x^2)^2(\cos(n \arccos x))(-1)^\nu}{(x-x_{\nu+1})(1-x_{\nu+1}^2)^{3/2}},$$

where

(26)
$$r_{0,1}(x) = \left\{(x-1) + \frac{1}{2}(x-1)^2\right\}\left(\frac{1+x}{2}\right)(\cos(n \arccos x)),$$
$$r_{n+1,1}(x) = \left\{(x+1) - \frac{1}{2}(x+1)^2\right\}\left(\frac{1-x}{2}\right)(\cos(n \arccos x)).$$

In the equations (17 – 26) we let $n = 10$ and we set for them program 2. The program is run and the numerical results obtained are given in tables 2.1 – 2. To see the effect of n, we repeated the procedure for $n = 20$ and the numerical results obtained are given in tables 2.3 – 4.

EXAMPLE 2.

N = 10

X	EXACT SOLUTION	NUM. SOLUTION
–1.0000000	0.0000000000	0.0000000000
–0.7500000	0.1609749643	0.1609749644
–0.5000000	0.2692371742	0.2692371732
–0.2500000	0.3315883327	0.3315883332
0.0000000	0.3519457263	0.3519457256
0.2500000	0.3315883327	0.3315883333
0.5000000	0.2692371742	0.2692371742
0.7500000	0.1609749643	0.1609749644
1.0000000	0.0000000000	0.0000000000

Table 2.1.

EXAMPLE 2.

N = 10

NU	X(NU)	EXACT SOLUTION		NUM. SOLUTION	
–1	1.0000000	Y =	0.0000000000	Y =	0.0000000000
		Y' =	–0.7615941560	Y' =	–0.7615941559
		Y" =	–1.0000000000	Y" =	–1.0000000000
0	0.9876883	Y =	0.0093009353	Y =	0.0093009353
		Y' =	–0.7493399063	Y' =	–0.7493399063
		Y" =	–0.9906990647	Y" =	–0.9906990647

1	0.8910065	Y = 0.0772275720 Y' = −0.6569129411 Y" = −0.9227724280	Y = 0.0772275720 Y' = −0.6569129411 Y" = −0.9227724280
2	0.7071068	Y = 0.1830680730 Y' = −0.4973966543 Y" = −0.8169319270	Y = 0.1830680730 Y' = −0.4973966543 Y" = −0.8169319270
3	0.4539905	Y = 0.2840063716 Y' = −0.3044216390 Y" = −0.7159936284	Y = 0.2840063716 Y' = −0.3044216389 Y" = −0.7159936284
4	0.1564345	Y = 0.3440000340 Y' = −0.1017920127 Y" = −0.6559999660	Y = 0.3440000339 Y' = −0.1017920127 Y" = −0.6559999661

Table 2.2.

EXAMPLE 2.

N = 20

X	EXACT SOLUTION	NUM. SOLUTION
−1.0000000	0.0000000000	0.0000000000
−0.7500000	0.1609749643	0.1609749648
−0.5000000	0.2692371742	0.2692371743
−0.2500000	0.3315883327	0.3315883317
0.0000000	0.3519457263	0.3519457274
0.2500000	0.3315883327	0.3315883343
0.5000000	0.2692371742	0.2692371731
0.7500000	0.1609749643	0.1609749640
1.0000000	0.0000000000	0.0000000000

Table 2.3.

EXAMPLE 2.

N = 20

NU	X(NU)	EXACT SOLUTION	NUM. SOLUTION
−1	1.0000000	Y = 0.0000000000 Y' = −0,7615941560 Y" = −1.0000000000	Y = 0.0000000000 Y' = −0.7615941559 Y" = −1.0000000000
0	0.9969173	Y = 0.0023429929 Y' = −0.7585151035 Y" = −0.9976570071	Y = 0.0023429929 Y' = −0.7585151035 Y" = −0.9976570071
1	0.9723699	Y = 0.0206638498 Y' = −0.7342512877 Y" = −0.9793361502	Y = 0.0206638498 Y' = −0.7342512877 Y" = −0.9793361502

2	0.9238795	Y =	0.0551303431	Y =	0.0551303431
		Y' =	−0.6876076839	Y' =	−0.6876076839
		Y" =	−0.9448696569	Y" =	−0.9448696569
3	0.8526402	Y =	0.1017578804	Y =	0.1017578804
		Y' =	−0.6219843758	Y' =	−0.6219843758
		Y" =	−0.8982421196	Y" =	−0.8982421196
4	0.7604060	Y =	0.1553840371	Y =	0.1553840371
		Y' =	−0.5416657486	Y' =	−0.5416657486
		Y" =	−0.8446159629	Y" =	−0.8446159629
5	0.6494480	Y =	0.2104048939	Y =	0.2104048939
		Y' =	−0.4510943249	Y' =	−0.4510943249
		Y" =	−0.7895951061	Y" =	−0.7895951061
6	0.5224986	Y =	0.2614538534	Y =	0.2614538534
		Y' =	−0.3542260141	Y' =	−0.3542260140
		Y" =	−0.7385461466	Y" =	−0.7385461466
7	0.3826834	Y =	0.3039111095	Y =	0.3039111095
		Y' =	−0.2540972292	Y' =	−0.2540972292
		Y" =	−0.6960888905	Y" =	−0.6960888905
8	0.2334454	Y =	0.3342069647	Y =	0.3342069647
		Y' =	−0.1526631070	Y' =	−0.1526631070
		Y" =	−0.6657930353	Y" =	−0.6657930353
9	0.0784591	Y =	0.3499500470	Y =	0.3499500470
		Y' =	−0.0508979347	Y' =	−0.0508979347
		Y" =	−0.6500499530	Y" =	−0.6500499530

Table 2.4.

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