APPROXIMATE SOLUTION OF THE INITIAL VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

By

THARWAT FAWZY

Math. Dept., Faculty of Science
Suez Canal Univ., Ismailia, EGYPT

(Received November 25, 1983)

1. Assumptions and procedures of the method

Consider the initial value problem

(1) $$y' = f(x, y), \quad 0 = x_0 \leq x \leq 1 \quad \text{and} \quad y(0) = y_0.$$

We assume that $f: (x_0-\alpha,\ x_0+\alpha) \times \mathbf{R} \to \mathbf{R}$ is defined and continuous with its first r derivatives, where $\alpha > 1$.

For $q = 0, 1, \ldots, r$ we define $f^{[q]}$ by the algorithm:

$$f^{[0]} = f,$$

$$f^{[q]} = \frac{\partial}{\partial x} f^{[q-1]} + f \frac{\partial}{\partial y} f^{[q-1]}.$$

We remark that the derivatives of y, the solution of (1), can be expressed by the help of $f^{[q]}$ as follows:

$$y^{(q)}(x) = f^{[q-1]} \quad (x,\ y(x)).$$

We also assume for $|x-x_0| < \alpha$ and $y, y_1, y_2 \in \mathbf{R}$

(2) $$|f^{[q]}(x, y)| \leq M, \quad q = 0, 1, \ldots, r$$

and the Lipschitz condition

(3) $$|f^{[q]}(x, y_1) - f^{[q]}(x, y_2)| \leq L|y_1 - y_2|, \quad q = 0, 1, \ldots, r$$

where L and M are some constants.

Consider the interval $0 \leq x \leq 1$ and define the mesh Δ by

$$\Delta : 0 = x_0 < x_1 < x_2 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

and let

$$x_{k+1} - x_k = h, \quad k = 0, 1, \ldots, n-1.$$

The modulus of continuity of $y^{(r+1)}$ is denoted by ω_r.

Choosing the required positive integer m (the order of the error to be achieved), we define the function approximating the solution of (1) as $S_\Delta(x)$ given by:

$$S_\Delta(x) = S_k^{(m)}(x) = S_{k-1}^{(m)}(x_k) + \int_{x_k}^{x} f(t, S_k^{(m-1)}(t))dt \tag{4}$$

where $x_k \le x < x_{k+1}$, $k = 0, 1, \ldots, n-1$ and $S_{-1}^{(m)}(x_0) = y_0$.

The following m iteration processes are considered in the above formula (4) for $x_k \le x < x_{k+1}$ and $k = 0, 1, \ldots, n-1$:

$$S_k^{(0)}(x) = S_{k-1}^{(m)}(x_k) + \sum_{q=0}^{r} \frac{f^{[q]}(x_k, S_{k-1}^{(m)}(x_k))}{(q+1)!}(x-x_k)^{q+1}, \tag{5}$$

$$S_k^{(j)}(x) = S_{k-1}^{(m)}(x_k) + \int_{x_k}^{x} f(t, S_k^{(j-1)}(t))dt, \quad j = 1, 2, \ldots, m. \tag{6}$$

2. Error estimations and convergence

The exact solution of (1) can be written in the following forms, for $x_k \le x \le x_{k+1}$ and $k = 0, 1, \ldots, n-1$:

$$y(x) = y(x_k) + \sum_{q=0}^{r-1} \frac{f^{[q]}(x_k, y_k)}{(q+1)!}(x-x_k)^{q+1} + \frac{f^{[r]}(\xi_k, y(\xi_k))}{(r+1)!}(x-x_k)^{r+1}, \tag{7}$$

where $y_k = y(x_k)$ and $x_k < \xi_k < x_{k+1}$,

$$y(x) = y_k + \int_{x_k}^{x} f[t, y(t)]dt. \tag{8}$$

If we denote for $x_k \le x < x_{k+1}$

$$|S_k^{(m)}(x) - y(x)| = e(x) \quad \text{and} \quad |S_k^{(m)}(x_k) - y(x_k)| = e_k,$$

then (6) and (8) together with the Lipschitz condition give

$$e(x) \le e_k + L\int_{x_k}^{x} |S_k^{(m-1)}(t_1) - y(t_1)|dt_1 \le e_k + L\int_{x_k}^{x} \{|S_{k-1}^{(m)}(x_k) - y_k| +$$

$$+ L\int_{x_k}^{t_1} |f[t_2, S_k^{(m-2)}(t_2)] - f[t_2, y(t_2)]|dt_2\}dt_1 \le \ldots \le$$

$$\le e_k \sum_{j=0}^{m-1} \frac{L^j h^j}{j!} + L^m \int_{x_k}^{x}\int_{x_k}^{t_1} \ldots \int_{x_k}^{t_{m-1}} |S_k^{(0)}(t_m) - y(t_m)|dt_m \ldots dt_1.$$

Using (5) and (7) and $y^{(q)}(x) = f^{[q-1]}(x, y(x))$, it is easy to prove that

$$|S_k^{(0)}(t_m) - y(t_m)| \leq e_k + Le_k \sum_{q=0}^{r-1} \frac{(t_m - x_k)^{q+1}}{(q+1)!} + \frac{\omega_r(h)}{(r-1)!}(t_m - x_k)^{r+1}.$$

Thus, for $c_0 = Le^L$ if $L > 1$ and $c_0 = e$ if $L \leq 1$, it is easy to get the following inequalities for $x_k \leq x \leq x_{k+1}$:

$$e(x) = |S_k^{(m)}(x) - y(x)| \leq e_k(1 + c_0 h) + \frac{L^m}{(m+r+1)!}\omega_r(h)h^{m+r+1},$$

and consequently

$$e_k(1 + c_0 h) \leq e_{k-1}(1 + c_0 h)^2 + \frac{L^m}{(m+r+1)!}\omega_r(h)h^{m+r+1}(1 + c_0 h),$$

$$e_1(1 + c_0 h)^k \leq e_0(1 + c_0 h)^{k+1} + \frac{L^m}{(m+r+1)!}\omega_r(h)h^{m+r+1}(1 + c_0 h)^k$$

and $e_0 = 0$ gives

(9) $$|S_k(x) - y(x)| \leq \frac{L^m}{(m+r+1)!}\omega_r(h)h^{m+r+1} \sum_{j=0}^{k} (1 + c_0 h)^j$$

and thus we have proved the following theorem.

Theorem 1. *Let* $y(x)$ *be the exact solution of* (1). *If* $S_\Delta(x)$ *given in* (4) *is the approximate solution, then the inequality*

$$|S_\Delta(x) - y(x)| \leq c_1 \omega_1(h)\, h^{m+r}$$

holds for all $x \in [0, 1]$, *where* $c_1 = \dfrac{L^m(e^{c_0} - 1)}{c_0(m+r+1)!}$ *and* $c_0 = Le^L$ *if* $L < 1$, *and* $c_0 = e$, *if* $L \leq 1$. □

Theorem 2. *The inequalities*

$$\left|\frac{d^{q+1}}{dx^{q+1}}S_\Delta(x) - \frac{d^{q+1}}{dx^{q+1}}y(x)\right| \leq c_2\omega_r(h)h^{m+r+1} \quad \text{if } m > 1,$$

$$\left|\frac{d^{q+1}}{dx^{q+1}}S_\Delta(x) - \frac{d^{q+1}}{dx^{q+1}}y(x)\right| \leq c_2\omega_r(h)h^{r+1} \quad \text{if } m = 1$$

hold for all $x \in [0, 1]$ *and all* $q = 0, 1, \ldots, r$, *where* $c_2 = (m+r+1)c_1$. □

Proof. From (4) and (8), using the Lipschitz condition we get

$$\left|\frac{d^{q+1}}{dx^{q+1}}S_\Delta(x) - \frac{d^{q+1}}{dx^{q+1}}y\right| \leq L\left|S_k^{(m-1)}(t_1) - y^{(m-1)}(t_1)\right|$$

and applying theorem 1, taking into consideration $m-1$ instead of m, it is easy to complete the proof. □

Theorem 3. *The error by which $S_\Delta(x)$ fails to satisfy the differential equation* (1) *is given by the inequality*

$$|S'(x)-f(x, S_\Delta(x))| \leq c_3\omega_r(h)h^{m+r-1}$$

which holds for all $x\in[0,1]$ *where* $c_3 = Lc_1+c_2\cdot$ □

Proof. Adding and substracting $y'(x)$ and using Theorems 1 and 2, it is easy to prove Theorem 3. □

REFERENCES

Fawzy T. and *Al-Mutib A.:* Spline Functions and Cauchy Problems, XII. Error of an arbitrary order for the approximate solution of the D. E. $y' = f(x, y)$ with spline functions. Proc. of the 1st Int. Conf. (Trinity College, Dublin, June, 1980) Bool Press Lim., Dublin.