GENERAL NECESSARY CONDITION FOR SMOOTH-CONVEX PROBLEMS

by

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In [1] a general necessary condition for smooth-convex problems was proved. In the present paper we consider a more general problem in which an infinite number of inequality constraints is given. The proof of the corresponding necessary condition needs essential modifications at several points while some parts can be adapted from [1]. For the reader's convenience we include a complete proof [2, 3].

Let X and Y be real Banach spaces, let U be an arbitrary set, let $f: X \times U \to l^\infty, f = (f_0, f_1, \ldots, f_n, \ldots)$ and $F: X \times U \to Y$ be arbitrary functions.

We consider the problem:

$$f_0(x, u) \to \inf, \tag{1}$$

$$(x, u) \in X \times U, \tag{2}$$

$$F(x, u) = 0, \tag{3}$$

$$f_i(x, u) \leq 0, \quad (i \in \mathbf{N}). \tag{4}$$

Problem (1)–(4) will be called *smooth-convex problem* at a point $(x_*, u_*) \in X \times U$, if

A) the point (x_*, u_*) satisfies conditions (2)–(4),

B) for every $u \in U$, the functions $x \mapsto F(x, u)$ $(x \in X)$ and $x \mapsto f(x, u)$ $(x \in X)$ belong to the class C_1 at the point $x_* \in X$: and exists a $K > 0$ such that for every $i \in \mathbf{N}_0$, $u \in U$ we have

$$|f_i(x_*, u)| < K \quad \text{and} \quad \|\partial_1 f_i(x_*, u_*)\| < K,$$

C) there exists a neighbourhood $k(x_*) \subset X$ of $x_* \in X$ such that for every $x \in k(x_*)$ the functions $u \mapsto F(x, u)$ $(u \in U)$ and $u \mapsto f(x, u)$ $(u \in U)$ satisfy the following convexity condition: for every u^1, $u^2 \in U$ and arbitrary $\alpha \in [0, 1]$ there exists a $\bar{u} \in U$ such that

$$F(x, \bar{u}) = \alpha F(x, u^1) + (1-\alpha) F(x, u^2),$$

$$f_i(x, \bar{u}) \leq \alpha f_i(x, u^1) + (1-\alpha) f_i(x, u^2), \quad (i \in \mathbf{N}_0).$$

We shall be concerned with necessary conditions for a local minimum in the problem (1)–(4) in the following sense:

We shall say that a point $(x_*, u_*)\in X\times U$ is a *local minimum point* of the problem (1)–(4) if there exists neighbourhood $\tilde{k}(x_*)\subset X$ such that for every $(x, u)\in\tilde{k}(x_*)\times U$ which satisfies constraints (2)–(4), the inequality

$$f_0(x_*, u_*)\leq f_0(x, u)$$

holds.

We consider the Lagrange function of the problem (1)–(4):

$$\mathcal{L}: X\times U\times l_\infty^*\times Y^*\to\mathbf{R},$$

$$\mathcal{L}(x, u, \lambda, y^*):=\langle\lambda, f(x, u)\rangle+\langle y^*, F(x, u)\rangle.$$

Theorem *(necessary conditions for smooth-convex extremum problems). Suppose that*

a) (1)–(4) *is a smooth-convex problem at the point* $(x_*, u_*)\in X\times U$,

b) the range of the operator $\partial_1 F(x_*, u_*): X\to Y$ *is finite codimensional and closed in* Y,

c) (x_*, u_*) *is a local minimum point of the problem* (1)–(4), *then there exists a finitely additive nonnegative measure* $\lambda\in l_\infty^*$ *and a* $y^*\in Y^*$ *(Lagrange multipliers), for which* $0\neq(\lambda, y^*)$ *on the following statements hold:*

(5) $$\partial_1\mathcal{L}(x_*, u_*, \lambda, y^*) = \partial_1 f^*(x_*, u_*)\lambda+\partial_1 F^*(x_*, u_*)y^* = 0,$$

(6) $$\mathcal{L}(x_*, u_*, \lambda, y^*) = \min_{u\in U}\mathcal{L}(x_*, u, \lambda, y^*),$$

and for any set $A\subset\mathbf{N}$

(7) $$\int_A f(x_*, u_*)d\lambda = 0.$$

(In particular: for $i\in\mathbf{N}$ $\lambda(\{i\})\, f_i(x_*, u_*) = 0$.) □

Proof. We shall use the following notations:

$$L_0 := R_{\partial_1 F(x_*, u_*)}\subset Y,$$
$$B := L_0+F(x_*, U)\subset Y,$$
$$L := \operatorname{cl}\operatorname{lin} B.$$

Then 1. $L\neq Y$, or

2. *a)* $L = Y$ and $0\notin\operatorname{int} B$, or

b) $L = Y$ and $0\in\operatorname{int} B$.

Case 1. Assume that $L\neq Y$. Then, by the Hahn-Banach theorem, there exists a non-zero functional $y^*\in Y^*$ which belongs to the annihilator of the space L. Since $B\subset L$ and for all $x\in X$, $u\in U$

$$\partial_1 F(x_*, u_*)x+F(x_*, u)\in B,$$

hence

(8) $$\langle y^*, \partial F(x_*, u_*)\,x+F(x_*, u)\rangle = 0.$$

In particular, it follows for $u:=u_*$ (since $F(x_*, u_*)=0$) that $\langle y_*, \partial_1 F(x_*, u_*)x\rangle = 0$ for all $x\in X$, i.e.,

$$\partial_1 F^*(x_*, u_*)y^* = 0. \tag{9}$$

On the other hand, for $x=0$ and all $u\in U$ from (8) we get

$$\langle y^*, F(x_*, u)\rangle = \langle y^*, F(x_*, u_*)\rangle = 0. \tag{10}$$

Setting $\lambda:=0\in l_\infty^*$, from (9) and (10) we obtain the relations (5)–(7). Thus, the assertion of the Theorem holds in this case.

Case 2. a) $L=Y$. We shall show that then int $B\neq\emptyset$. Indeed, since codim $L_0<\infty$, the factor space Y/L_0 is finite-dimensional. We denote by $\pi: Y\to Y/L_0$ the canonical mapping. Since the closed linear hull of the set $\pi(B)$ coincides with Y, the linear hull of the set $\pi(B)$ coincides with Y/L_0. Obviously $F(x_*, U)\subset Y$ is a convex set; L_0 is a subspace, so their algebraic sum (the set B) is convex. Therefore, the set $\pi(B)$ is also convex.

Since $F(x_*, u_*)=0$, we have $0\in\pi(B)$. Thus aff $\pi(B)$ = lin $\pi(B)$ = $= Y/L_0$. Hence int $\pi(B)\neq\emptyset$.

Since π is a continuous mapping, and

$$\pi^{-1}(\pi(B)) = B,$$

we obtain that int $B\neq\emptyset$.

Assume now that $0\notin$ int B. By the separation theorem, there exists a non-zero functional $y^*\in Y^*$, which separates the set $B\subset Y$ and the point $0\in Y$, i.e.

$$\langle y^*, y\rangle\geq 0$$

for all $y\in B$. This means that

$$\langle y^*, \partial_1 F(x_*, u_*)x + F(x_*, u)\rangle\geq 0$$

for all $x\in X$ and $u\in U$. Setting in this inequality $u:=u_*$, we obtain

$$\langle y^*, \partial_1 F(x_*, u_*)x\rangle\geq 0$$

for all $x\in X$, from which

$$\partial_1 F^*(x_*, u_*)\, y^* = 0.$$

Put $x:=0$, then

$$\langle y^*, F(x_*, u)\rangle\geq 0 = \langle y^*, F(x_*, u_*)\rangle$$

holds for all $u\in U$. Thus, in this case, as in the preceding one, $\lambda:=0\in l_\infty^*$, and y^* are the required Lagrange multipliers.

b) Finally, we assume that

$$L = Y \quad\text{and}\quad 0\in\text{int } B.$$

Define

$$S:=\{i\in\mathbf{N}\,|\,f_i(x_*, u_*)=0\}.$$

If S is finite, the proof coincides with that of [1]. Assume that S is infinite. Further let $\bar{S}:=S\cup\{0\}$.

We consider the set $C \subset l_\infty(\bar{S}) \times Y$ of all vectors $(\mu, y) \in l_\infty(\bar{S}) \times Y$ for each of which there exists a $x \in X$ and a $u \in U$ such that

$$\mu_i > \partial_1 f_i(x_*, u_*)x + f_i(x_*, u) - f_i(x_*, u_*) \quad (i \in \bar{S}),$$
$$y = \partial_1 F(x_*, u_*)x + F(x_*, u) - F(x_*, u),$$

where $\mu := (\mu_i)_{i \in \bar{S}}$.

For the proof of the assertion of the theorem it is enough to verify that

(11′) the set C is convex,

(11″) $\operatorname{int} C \neq \emptyset$,

(11‴) $0 \notin \operatorname{int} C$.

Indeed, by the separation theorem there exists a non-zero functional $(\lambda, y^*) \in l_\infty^*(\bar{S}) \times Y^*$ such that for all $(\mu, y) \in C$

(12) $$\langle \lambda, \mu \rangle + \langle y^*, y \rangle \geq 0.$$

The $\lambda \in l_\infty^*(\bar{S})$ can not be negative finitelly additive measure. Indeed, if there is a set $A \subset \bar{S}$, $\lambda(A) < 0$, then let $(\mu^0, y^0) \in C$ be such that $\mu_i^0 > 0$ $(i \in \bar{S})$. Further, for every $\alpha \in]1, +\infty[$ define

$$\mu_i^\alpha := \begin{cases} \alpha \mu_i^0, & \text{if} \quad i \in A, \\ \mu_i^0, & \text{if} \quad i \in \bar{S} \setminus A. \end{cases}$$

Obviously $(\mu^\alpha, y^0) \in C$ and

$$\langle \lambda, \mu^\alpha \rangle + \langle y^*, y^0 \rangle = \int_A \mu^\alpha d\lambda + \int_{\bar{S} \setminus A} \mu^\alpha d\lambda + \langle y^*, y^0 \rangle =$$
$$= \alpha \int_A \mu^0 d\lambda + \int_{\bar{S} \setminus A} \mu^0 d\lambda + \langle y^*, y^0 \rangle.$$

Since $\int_A \mu^0 d\lambda < 0$, there is an $\alpha > 1$ such that

$$\alpha \int_A \mu^0 d\lambda + \int_{S \setminus A} \mu^0 d\lambda + \langle y^*, y^0 \rangle < 0,$$

which contradicts inequality (12).

For arbitrary $x \in X$ and $u \in U$ define the sequences

$$(\mu_i^k)_{k \in \mathbf{N}} := \left(\partial_1 f_i(x_*, u_*)x + f_i(x_*, u) - f_i(x_*, u_*) + \frac{1}{k} \right)_{k \in \mathbf{N}}, \quad (i \in \bar{S}).$$

Obviously, for $k \in \mathbf{N}$ with $\mu^k := (\mu_i^k)_{i \in \bar{S}} \in l_\infty(\bar{S})$, and $y := \partial_1 F(x_*, u_*)x + F(x_*, u) - F(x_*, u_*)$

$$(\mu^k, y) \in C,$$

therefore by (12)

$$\langle \lambda, \mu^k \rangle + \langle y^*, y \rangle \geq 0, \quad (k \in \mathbf{N}).$$

But $\lim (\mu^k)_{k\in\mathbf{N}} = (\partial_1 f_i(x_*, u_*)x + f_i(x_*, u) - f_i(x_*, u_*))_{i\in \bar{S}}$ and the $(\lambda, y^*) \in (l_\infty \times Y)^*$ continuous, therefore (12) is true for the limit of the sequence $(\mu^k, y)_{k\in\mathbf{N}}$ i.e.

$$\int_{\bar{S}} (\partial_1 f_i(x_*, u_*)x + f_i(x_*, u) - f_i(x_*, u_*))_{i\in\bar{S}} d\lambda + \tag{12'}$$
$$+ \langle y^*, \partial_1 F(x_*, u_*)x + F(x_*, u) - F(x_*, u_*) \rangle \geq 0.$$

Let $\lambda \in l_\infty^*(\mathbf{N}_0)$ be an extension of the functional $\lambda \in l_\infty^*(\bar{S})$ such that for an arbitrary $A \subset \mathbf{N}_0 \setminus \bar{S}$ $\lambda(A) = 0$.

In terms of the Lagrange function, the inequality (12′) means the following:

$$\partial_1 \mathcal{L}(x_*, u_*, \lambda, y^*)x + \mathcal{L}(x_*, u, \lambda, y^*) - \mathcal{L}(x_*, u_*, \lambda, y^*) \geq 0. \tag{12''}$$

If $x \in X$ then $-x \in X$, therefore (12″) satisfies only if

$$\partial_1 \mathcal{L}(x_*, u_*, \lambda, y^*) = 0.$$

This is assertion (5) of the theorem.

Hence, by (12″) for every $u \in U$

$$\mathcal{L}(x_*, u, \lambda, y^*) - \mathcal{L}(x_*, u_*, \lambda, y^*) \geq 0.$$

This means that inequality (6) is satisfied in the Theorem.

By the definition of $\lambda \in l_\infty^*(\mathbf{N}_0)$ the requirement (7) is fulfilled.

Hence in order to prove the theorem it is sufficient to verify (11′), (11″) and (11‴).

The set C is, obviously, convex.

Now we shall prove that int $C \neq \emptyset$. Since $0 \in \text{int } B$, therefore $0 \in \text{int } \pi(B)$. The space Y/L_0 is finite-dimensional. Thus there exists a finite number of points $z_1, z_2, \ldots, z_m$ of $\pi(B)$ whose linear hull coincides with Y/L_0 such that $z_1 + z_2 + \ldots + z_m = 0$.

Since $z_j \in \pi(B)$ $(j = 1, 2, \ldots, m)$, there exists a $u_j \in U$ such that $\pi(F(x^*, u_j)) = z_j$, and by the linearity of π,

$$\pi\left(\sum_{j=1}^{m} F(x_*, u_j)\right) = 0.$$

Set

$$c_0 := \sup_{\substack{1 \leq j \leq m \\ i \in S}} \{f_i(x_*, u_j) - f_i(x_*, u_*) + \|\partial_1 f_i(x_*, u_*)\|\},$$

$$U_0 := \{u \in U \mid \exists \alpha_j \geq 0, \quad 1 \leq j \leq m, \quad \sum_{j=1}^{m} \alpha_j = 1 \quad \text{such that } F(x_*, u) =$$
$$= \sum_{j=1}^{m} \alpha_j F(x_*, u_j), \ f_j(x_*, u) \leq \sum_{j=1}^{m} \alpha_j f_i(x_*, u_j), \quad i \in \bar{S}\}.$$

(From requirements $B)$ and $C)$ of the smooth-convex problem it follows that $c_0 < +\infty$ and $U_0 \neq \emptyset$.)

In addition put

$B_0 := \{y \in Y \mid \exists x \in k_1(0)$ and $u \in U_0$ such that $y = \partial_1 F(x_*, u_*)x + F(x_*, u)\}$.
It is easy to prove that B_0 is convex.

Since $z_j \in \pi(F(x_*, U_0))$ $(j = 1, \ldots, m)$, we have

$$\operatorname{lin} \pi(F(x_*, U_0)) = Y/L_0.$$

It is not hard to see that

$$0 \in \pi(F(x_*, U_0)),$$

thus

$$\operatorname{aff} \pi(F(x_*, U_0)) = \operatorname{lin} \pi(F(x_* U_0)) = Y/L_0.$$

Therefore

$$\operatorname{int} \pi(F(x_*, U_0)) \neq \emptyset$$

moreover

$$\operatorname{int} (L_0 + F(x_*, U_0)) \neq \emptyset.$$

On the other hand $\partial_1 F(x_*, U_0) k_1(0)$ is open in L_0. Hence we obtain $\operatorname{int} B_0 \neq \emptyset$.

We set the half-line $E_{c_0} := \{\mu \in \mathbf{R} \mid \mu > c_0\}$ and let

$$C_0 := \Big(\mathop{\times}_{i \in \bar{S}} E_{c_0} \Big) \times B_0.$$

The set $\mathop{\times}_{i \in \bar{S}} E_{c_0} \subset l_\infty(\bar{S})$ is open and $\operatorname{int} B_0 \neq \emptyset$, therefore

$$\operatorname{int} C_0 \neq \emptyset.$$

We shall prove that $C_0 \subset C$. Indeed, let $(\bar{\mu}, \bar{y}) \in C_0$. This means that for every $i \in \bar{S}$ $\bar{\mu}_i > c_0$ and $\bar{y} \in B_0$. Since $\bar{y} \in B_0$, there exist $\bar{x} \in k_1(0)$ and $\bar{u} \in U_0$ such that $\bar{y} = \partial_1 F(x_*, u_*)\bar{x} + F(x_*, \bar{u})$. But $k_1(0) \subset X$ and $U_0 \subset U$, therefore $\bar{y}$ can be a second component of an element of C. Hence, it is sufficient to verify that

$$\bar{\mu}_i > \partial_1 f_i(x_*, u_*)\bar{x} + f_i(x_*, \bar{u}) - f_i(x_*, u_*) \quad (i \in \bar{S}).$$

Since $\bar{u} \in U_0$, there are $\bar{\alpha}_j \geq 0$, $j = 1, 2, \ldots, m$, such that $\sum_{j=1}^{m} \bar{\alpha}_j = 1$ and

$$f_i(x_*, \bar{u}) \leq \sum_{j=1}^{m} \bar{\alpha}_j f_i(x_*, u_j) \quad (i \in \bar{S}).$$

Then

$$\partial_1 f_i(x_*, u_*)x + f_i(x_*, \bar{u}) - f_i(x_*, u_*) \leq \|\partial_1 f_i(x_*, u_*)\| +$$
$$+ \max_{1 \leq j \leq m} \{f_i(x_*, u_j) - f_i(x_*, u_*)\} \leq \sup_{\substack{1 \leq j \leq m \\ i \in \bar{S}}} \{\|\partial_1 f_i(x_*, u_*)\| +$$
$$+ f_i(x_*, u_j) - f_i(x_*, u_*)\} = c_0 < \bar{\mu}_i.$$

Hence $(\bar{\mu}, \bar{y}) \in C$, i.e. $C_0 \subset C$. Summing up, int $C_0 \neq \emptyset$ and $C_0 \subset C$, therefore int $C \neq \emptyset$, i.e. proposition (11″) is proved.

Finally, we shall prove $0 \notin \operatorname{int} C$. Let us assume that this is false, namely there exists a neighbourhood $k_{2\delta}(0) \subset l_\infty(\bar{S}) \times Y$ with $k_{2\delta}(0) \subset C$. Let $\mu_i := -\delta \ (i \in \bar{S})$ and $\mu := (\mu_i)_{i \in \bar{S}}$. Obviously $(\mu, 0) \in C$. From the last relation it follows that there exist $x_0 \in X$ and $u_0 \in U$ such that

(13) $$-\delta > \partial_1 f_i(x_*, u_*)x_0 + f_i(x_*, u_0) - f_i(x_*, u_*) \quad (i \in \bar{S})$$

and

(14) $$0 = \partial_1 F(x_*, u_*)x_0 + F(x_*, u_0) - F(x_*, u_*).$$

Let us fix $\varepsilon > 0$ and consider the following function:

$$\mathcal{F} : X \times \mathbf{R}^{m+1} \to Y$$

$$\mathcal{F}(x, \alpha_0, \alpha_1, \ldots, \alpha_m) := F(x_* + x, u_*) + \alpha_0\big(F(x_* + x, u_0) - F(x_* + x, u_*)\big) + \\ + \varepsilon \sum_{j=1}^{m} \alpha_j\big(F(x_* + x, u_j) - F(x_* + x, u_*)\big).$$

By the smoothness conditions there exists a neighbourhood $k(x_*) \subset X$ such that at every point $(x, \alpha_0, \alpha_1, \ldots, \alpha_m) \in (k(x_*) - x_*) \times \mathbf{R}^{m+1}$ the function $\mathcal{F}$ is differentiable, $\mathcal{F}'$ is continuous at the point $(0, 0, \ldots, 0) \in X \times \mathbf{R}^{m+1}$ and

(15) $$\mathcal{F}'(\mathbf{0}, 0, \ldots, 0)(x, \alpha_0, \alpha_1, \ldots, \alpha_m) = \partial_1 F(x_*, u_*)x + \\ + \alpha_0\big(F(x_*, u_0) - F(x_*, u_*)\big) + \\ + \varepsilon \sum_{j=1}^{m} \alpha_j\big(F(x_*, u_j) - F(x_*, u_*)\big).$$

The function $\mathcal{F}$ has the following properties

1. $\mathcal{F}(\mathbf{0}, 0, \ldots, 0) = 0$, indeed,

   $\mathcal{F}(\mathbf{0}, 0, \ldots, 0) = F(x_*, u_*) = 0.$

2. $\mathcal{R}_{\mathcal{F}'(\mathbf{0}, 0, \ldots, 0)} = Y.$

First, we shall prove $\mathcal{R}_{\mathcal{F}'(\mathbf{0}, 0, \ldots, 0)} \subset B_0$. For all $y \in B_0$ there exist $\bar{x} \in k_1(0) \subset X$ and $\bar{u} \in U_0$ such that

$$y = \partial_1 F(x_*, u_*)\bar{x} + F(x_*, \bar{u}).$$

Since $\bar{u} \in U_0$, there exist numbers $\bar{\alpha}_j \geq 0$ $(j = 1, \ldots, m)$, $\sum_{j=1}^{m} \bar{\alpha}_j = 1$, such that

$$F(x_*, \bar{u}) = \sum_{j=1}^{m} \bar{\alpha}_j F(x_*, u_j).$$

Then, for $x:=\bar{x}$, $\alpha_0:=0$, $\alpha_j:=\frac{\bar{\alpha}_j}{\varepsilon}$ $(j=1,\ldots,m)$

$$\mathcal{F}'(\mathbf{0},0,\ldots,0)(x,\alpha_0,\alpha_1,\ldots,\alpha_m)=\partial_1 F(x_*,u_*)\bar{x}+$$

$$+0\cdot\big(F(x_*,u_0)-F(x_*,u_*)\big)+\varepsilon\sum_{j=1}^{m}\frac{\bar{\alpha}_j}{\varepsilon}\big(F(x_*,u_j)-F(x_*,u_*)\big)=y,$$

that is $y\in\mathcal{R}_{\mathcal{F}'(\mathbf{0},0,\ldots,0)}$. Since $\operatorname{int} B_0\neq\emptyset$,

$$\mathcal{R}_{\mathcal{F}'(\mathbf{0},0,\ldots,0)}=Y.$$

3. There exists an $x'\in X$ such that

$$\mathcal{F}'(\mathbf{0},0,\ldots,0)(x_0+\varepsilon x',1,1,\ldots,1)=0.$$

Indeed, by (14)

$$\sum_{j=1}^{m}F(x_*,u_j)\in L_0=\mathcal{R}_{\partial_1 F(x_*,u_*)},$$

therefore there exists a $(-x')\in X$ such that

$$\sum_{j=1}^{m}F(x_*,u_j)=-\partial_1 F(x_*,u_*)x'. \tag{16}$$

By (15)

$$\mathcal{F}'(\mathbf{0},0,\ldots,0)(x_0+\varepsilon x',1,\ldots,1)=\partial_1 F(x_*,u_*)(x_0+\varepsilon x')+$$

$$+F(x_*,u_0)-F(x_*,u_*)+\varepsilon\sum_{j=1}^{m}\big(F(x_*,u_j)-F(x_*,u_*)\big)=$$

$$=\partial_1 F(x_*,u_*)x_0+F(x_*,u_0)+\varepsilon\left(\partial_1 F(x_*,u_*)x'+\sum_{j=1}^{m}F(x_*,u_j)\right).$$

The sum ot the first two terms is equal to zero by (14) and the sum in the brackets also equals to zero by (16). Hence the vector $(x_0+\varepsilon x',1,\ldots,1)\in X\times\times\mathbf{R}^{m+1}$ belongs to the kernel of the operator $\mathcal{F}'(\mathbf{0},0,\ldots,0)$.

The function $\mathcal{F}$ satisfies the conditions of Ljusternik theorem at the point $(\mathbf{0},0,\ldots,0)\in X\times\mathbf{R}^{m+1}$, therefore the tangent space of the set

$$M:=\{(x,\alpha_0,\alpha_1,\ldots,\alpha_m)\in X\times\mathbf{R}^{m+1}\mid\mathcal{F}(x,\alpha_0,\alpha_1,\ldots,\alpha_m)=0\}$$

is equal to the kernel of the operator $\mathcal{F}'(\mathbf{0},0,\ldots,0)$, i.e.

$$TM(\mathbf{0},0,\ldots,0)=\operatorname{Ker}\mathcal{F}'(\mathbf{0},0,\ldots,0).$$

Then there exists an $\varepsilon_0>0$ (generally speaking, depends on ε) and a function

$$[-\varepsilon_0,\varepsilon_0]\ni t\mapsto\big(\tilde{x}(t),\tilde{\alpha}_0(t),\tilde{\alpha}(t),\ldots,\tilde{\alpha}_m(t)\big)\in X\times\mathbf{R}^{m+1},$$

$$\lim_0\left(t\mapsto\|\tilde{x}(t)\|+\sum_{j=1}^{m}|\tilde{\alpha}_j(t)|\right)=0,$$

for which at all point $t\in[-\varepsilon_0, \varepsilon_0]$ we have

$$(\mathbf{0}, 0, \dots, 0)+t(x_0+\varepsilon x', 1, \dots, 1)+t(\tilde{x}(t), \tilde{\alpha}_0(t), \dots, \tilde{\alpha}_m(t))\in M,$$

that is

$$\mathcal{F}\big(t(x_0+\varepsilon x'+\tilde{x}(t)), t(1+\tilde{\alpha}_0(t)), \dots, t(1+\tilde{\alpha}_m(t))\big) = 0.$$

This result holds for all $\varepsilon>0$.

We now choose an $\varepsilon>0$ such that for every $i\in\bar{S}$

$$\varepsilon\left|\partial_1 f_i(x_*, u_*)x'+\sum_{j=1}^{m}[f_i(x_*, u_j)-f_i(x_*, u_*)]\right|<\frac{\delta}{2}$$

holds.

Now, we consider the functions for all $i\in\bar{S}$

$$g_i : X\times\mathbf{R}^{m+1}\to\mathbf{R}$$

$$g_i(x, \alpha_0, \alpha_1, \dots, \alpha_m) := f_i(x_*+x, u_*)+$$

$$+\alpha_0(f_i(x_*+x, u_0)-f_i(x_*+x, u_*))+\varepsilon\sum_{j=1}^{m}\alpha_j[f_i(x_*+x, u_j)-f_i(x_*+x, u_*)].$$

Obviously,

$$g_i(\mathbf{0}, 0, \dots, 0)=\begin{cases} f_0(x_*, u_*), & \text{if} \quad i=0 \\ 0, & \text{if} \quad i\in S,\end{cases}$$

the functions g_i $(i\in\bar{S})$ are continuously differentiable at the point $(\mathbf{0}, 0, \dots, 0)$ and

$$g_i'(\mathbf{0}, 0, \dots, 0)(x, \alpha_0, \alpha_1, \dots, \alpha_m) = \partial_1 f_i(x_*, u_*)x+$$

$$+\alpha_0(f_i(x_*, u_0)-f_i(x_*, u_*))+\varepsilon\sum_{j=1}^{m}\alpha_j[f_i(x_*, u_j)-f_i(x_*, u_*)].$$

By the choice of ε, from (13) and (16) it follows

$$g_i'(\mathbf{0}, 0, \dots, 0)(x_0+\varepsilon x', 1, \dots, 1) = \partial_1 f_i(x_*, u_*)x_0+f_i(x_*, u_0)-f_i(x_*, u_*)+$$

(17)
$$+\varepsilon\left(\partial_1 f_i(x_*, u_*)x'+\sum_{j=1}^{m}[f_i(x_*, u_j)-f_i(x_*, u_*)]\right)<$$

$$<-\delta+\delta/2 = -\delta/2 \qquad (i\in\bar{S}).$$

By the Lagrange mean-value theorem there exists a vector $(\xi^i, \vartheta_0^i, \vartheta_1^i, \dots$ $\dots, \vartheta_m^i)\in X\times\mathbf{R}^{m+1}$ (depending on t), for which

$$g_i\big(t(x_0+\varepsilon x'+\tilde{x}(t)), t(1+\tilde{\alpha}_0(t)), \dots, t(1+\tilde{\alpha}_m(t))\big) =$$

$$= g_i(\mathbf{0}, 0, \dots, 0)+tg_i'(\xi^i, \vartheta_0^i, \vartheta_1^i, \dots, \vartheta_m^i)(x_0+\varepsilon x'+\tilde{x}(t), 1+\tilde{\alpha}_0(t), \dots, 1+\tilde{\alpha}_m(t))$$

for all $t\in[-\varepsilon_0, \varepsilon_0]$ and $i\in\bar{S}$.

It is easy to see from (17) that there exists a $\gamma>0$ such that for every $t\in]0,\gamma[$

$$g_i'(\xi^i, \vartheta_0^i, \vartheta_1^i, \dots, \vartheta_m^i)(x_0+\varepsilon x'+\tilde{x}(t), 1+\tilde{\alpha}_0(t), \dots, 1+\tilde{\alpha}_m(t)) < -\frac{\delta}{2}$$

also holds. From this we obtain

$$\begin{aligned} &g_i\big(t(x_0+\varepsilon x'+\tilde{x}(t)), t(1+\tilde{\alpha}_0(t)), \dots, t(1+\tilde{\alpha}_m(t))\big)< \\ (18)\qquad &< g_i(\mathbf{0}, 0, \dots, 0) - t\frac{\delta}{2}, \quad t\in]0,\gamma[, \quad i\in\overline{S}. \end{aligned}$$

Define the function

$$]0,\gamma[\ni t\mapsto x(t):=x_*+t(x_0+\varepsilon x'+\tilde{x}(t))\in X.$$

Obviously, $\lim\limits_{0}(t\mapsto x(t))=0$. We can assume that $\gamma>0$ is sufficiently small. Then all $t\in]0,\gamma[$ satisfy

$$t\left[1+\tilde{\alpha}_0(t)+\varepsilon\sum_{j=1}^{m}(1+\tilde{\alpha}_j(t))\right]\leq 1,$$

$$1+\tilde{\alpha}_0(t)\geq 0,\ 1+\tilde{\alpha}(t)\geq 0, \dots, 1+\tilde{\alpha}_m(t)\geq 0.$$

For such $t\in]0,\gamma[$, according to the condition *C)* of the smooth-convex problem, there exists an element $u(t)\in U$ such that the following relations hold:

$$F(x(t),u(t)) = \left[1-t(1+\tilde{\alpha}_0(t))-\varepsilon t\sum_{j=1}^{m}(1+\tilde{\alpha}_j(t))\right]F(x(t),u_*)+$$

$$+t(1+\tilde{\alpha}_0(t))F(x(t),u_0)+\varepsilon t\sum_{j=1}^{m}(1+\tilde{\alpha}_j(t))F(x(t),u_j)=$$

$$=\mathcal{F}\big(t(x_0+\varepsilon x'+\tilde{x}(t)), t(1+\tilde{\alpha}_0(t)), t(1+\tilde{\alpha}_1(t)), \dots, t(1+\tilde{\alpha}_m(t))\big)=0,$$

$$f_i(x(t),u(t))\leq\left[1-t(1+\tilde{\alpha}_0(t))-\varepsilon t\sum_{j=1}^{m}(1+\tilde{\alpha}_j(t))\right]f_i(x(t),u_*)+$$

$$+t(1+\tilde{\alpha}_0(t))f_i(x(t),u_0)+\varepsilon t\sum_{j=1}^{m}(1+\tilde{\alpha}_j(t))f_i(x(t),u_j)=$$

$$=g_i\big(t(x_0+\varepsilon x'+\tilde{x}(t)), t(1+\tilde{\alpha}_0(t)), t(1+\tilde{\alpha}_1(t)), \dots, t(1+\tilde{\alpha}_m(t))\big) \quad (i\in\overline{S}).$$

From these relations and from (18) it follows that, for every $t\in]0,\gamma[$

$$f_0(x(t),u(t))<f_0(x_*,u_*),$$
$$F(x(t),u(t))=0$$
$$f_i(x(t),u(t))<0, \quad i\in S.$$

Finally, it is obvious that

$$\limsup_{0} \left(t \mapsto f_i(x(t), u(t))\right) \leq f_i(x_*, u_*) < 0$$

for $i \in \mathbf{N} \setminus S$. Thus, if a $t \in]0, \gamma[$, then $(x(t), u(t)) \in X \times U$ is an admissible element of our problem, but $f_0(x(t), u(t)) < f_0(x_*, u_*)$. This means that the point (x_*, u_*) can not be a local minimum point. It is a contradiction. Therefore, the assumption $0 \in \operatorname{int} C$ was false, and the relation (11‴) holds.

The theorem has been proved. □

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REFERENCES

[1] *Ioffe A. D.*, and *Tichomirov V. M.:* Theory of Extremal Problems, North-Holland Publishing Company, Amsterdam, 1979.

[2] *Mezei I.:* Végtelen idejű diszkrét optimális folyamatok vizsgálata, Egyetemi doktori értekezés, Budapest, ELTE, 1980.

[3] *Mezei I.:* A necessary condition for extremum problems with infinite dimensional contraints, Szelvények az ELTE TTK Analízis II. tanszék tudományos munkáiból, 1981/3 (preprint).