HERMITE INTERPOLATION AND THE BOUNDARY VALUE PROBLEMS

by

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1. Introduction. In this paper, we investigate the approximate solution of the non-linear differential equation

$$(1.1) \qquad y'' = f(x, y(x); y'(x)) \quad (-1 \le x \le 1)$$

with the boundary values

$$(1.2) \qquad \begin{aligned} y(1) &= \alpha_0; & y'(1) &= \alpha_1 \\ y(-1) &= \beta_0; & y'(-1) &= \beta_1. \end{aligned}$$

The exact solution of this non-linear ordinary differential equation, if it exists, cannot always be found. Faced with this difficulty, mathematicians resorted to numerical methods for approximating the solutions e.g. Runge-Kutta, Euler, Galerkin, Adam.

Some mathematicians applied the interpolation in their methods approximating the solution. Adam applied the interpolation only in some of his methods. In his spline interpolation methods T. Fawzy [3] used the spline functions which are not polynomials over the whole given interval. K. Fanta and O. Kis applied the Hermite interpolation in their method for the solution approximation [2]. They used the Green's function which is not possible to calculate numerically.

In this paper, we apply the Hermite interpolation to obtain a method for approximating the solution of (1.1–2). Our method is economical in that we don't use the Green's function.

Suppose that $f(x, y(x) \cdot y'(x)) \in C^{(r-2)}([-1,1] \times R \times R)$ where $r \ge 2$ fixed integer, we instantly assume that $f(x, y(x), y'(x))$ satisfies the Lipschitz condition

$$(1.3) \qquad |f^{(q)}(x, y_1, y_1') - f^{(q)}(x, y_2, y_2')| \le L(|y_1 - y_2| + |y_1' - y_2'|)$$

for all $x \in [-1,1]$ and all y, y_1', y_2, y_2, in R. L is the Lipschitz constant and $q = 0, 1, 2, \ldots, r-2$.

Let $y(x)$ be the unique solution of (1.1 – 2) for all $x \in [-1,1]$. Suppose that we have the following triangular matrix

$$A(T):\{x_{\nu,n}\}_{\nu=0}^{n+1} \quad (n = 1,2,3,\ldots), \tag{1.4}$$

where

$$-1 = x_{n+1,n} < x_{n,n} < \ldots < x_{\nu+1,n} < x_{\nu,n} < \ldots < x_{1,n} < x_{0,n} = 1, \tag{1.5}$$

and

$$x_{\nu,n} = \cos\frac{2\nu-1}{2n}\pi \quad (1 \le \nu \le n) \tag{1.6}$$

are the roots of the Chebyshev polynomial of the first kind given by

$$T_n(x) = \cos(n \arccos x) \quad (n = 1, 2, 3, \ldots). \tag{1.7}$$

Our method for approximating the solution of (1.1 – 2) is carried into two processes, in the first process we use the method of step by step integration to obtain the approximate real values $\bar{y}_{\nu,n}$ where

$$\bar{y}_{\nu,n} \approx y_{\nu,n} = y(x_{\nu,n}) \quad (1 \le \nu \le n), \tag{1.8}$$

i.e. the approximate values of $y(x)$ at the nodal points.

In the second process we interpolate the real values in (1.8) at the nodal points which are given by (1.4 – 6) and obtain an interpolation process which we prove converges uniformly with the exact solution of (1.1 – 2) in the closed interval $[-1,1]$.

If there is no danger of misunderstanding we drop the second index in our notations whenever possible.

2. Preliminaries. In this section we state the definitions and formulae needed for the proofs in the rest of the paper.

We use the fundamental Lagrange interpolation polynomial

$$l_\nu(x) = \frac{T_n x}{T_n'(x_\nu)(x - x_\nu)} \quad (1 \le \nu \le n), \tag{2.1}$$

where $x_{\nu,n}$ and $T_n(x)$ are defined in (1.6 – 7). It is obvious that

$$T_n'(x_\nu) = \frac{n}{\sqrt{1 - x_\nu^2}}. \tag{2.2}$$

L. Fejér [4] proved that the following inequalities, for the Lebesgue function, are true for all $x \in [-1,1]$

$$\sum_{\nu=1}^{n} l_\nu^2(x) \le 2, \tag{2.3}$$

which implies

$$|l_\nu(x)| \le \sqrt{2} \tag{2.4}$$

and

$$\frac{\log n}{8\sqrt{\pi}} < \max_{-1\leq x\leq 1} \sum_{\nu=1}^{n} |l_\nu(x)| \leq \frac{2}{\pi}\log n. \tag{2.5}$$

For any polynomial $g_k(x)$ of real coefficients and of degree k, S. Bernstein and Markov have proved that the following inequalitites are true respectively

$$|g_h^{(q)}(x)| = 0(1)\frac{n^q}{(1-x^2)^{q/2}} \max_{-1\leq x\leq 1} |g_k(x)|, \tag{2.6}$$

$$|g_k^{(q)}(x)| = 0(1)n^{2q} \max_{-1\leq x\leq 1} |g_k(x)|. \tag{2.7}$$

Theorem 2.1 *(I. E. Gopengaus)* [5]. *Let $f(x)$ be an arbitrary real function for which $f\in C^{(q)}([-1,1])$. Then there exists a polynomial $G_m(x; f)$ of degree at most m $(m\geq 4q+5)$ and such that*

$$|G_m^{(i)}(x; f) - f^{(i)}(x)| = 0(1)\omega\left(\frac{1}{m}; f^{(q)}\right)\left(\frac{\sqrt{1-x^2}}{m}\right)^{q-i} \qquad 0\leq i\leq q \tag{2.8}$$

is true for all $q\geq 0$, fixed integer and for all $x\in[-1,1]$. $\omega(\delta; f^{(q)})$ is the modulus of continuity of $f^{(q)}(x)$. □

In the following, we describe the two processes for obtaining the method of solution approximation.

3. First approximation process. Considering the non-linear differential equation (1.1), we assume that $f(x; y(x); y'(x))\in C^{(r-2)}([-1,1]\times R\times R)$ and satisfies the Lipschitz condition (1.3). Suppose that the boundary conditions are as given in (1.2). Let us write them as follows

$$\begin{aligned} y(1) &= y_0 = \alpha_0; & y'(1) &= y \;\; = \alpha_1, \\ y(-1) &= y_{n+1} = \beta_0; & y'(-1) &= y_{n+1} = \beta_1. \end{aligned} \tag{3.1}$$

Thus we can integrate (1.1) between x_ν and x where $x\in[x_{\nu+1}, x_\nu]$ $\left(0\leq \nu\leq\left[\frac{n}{2}\right]-2\right)$ and obtain

$$y'(x) = \bar{y}'_\nu + \int_{x_\nu}^{x} f(t; y(t); y'(t))dt \tag{3.2}$$

and

$$\begin{aligned} y(x) = y_\nu + y'_\nu(x-x_\nu) + \int_{x_\nu}^{x}\int_{x_\nu}^{t} f[u; y(u); y'_\nu + \\ + \int_{x_\nu}^{\omega} f(v; y(v); y'(v))dv]dudt. \end{aligned} \tag{3.3}$$

It is obvious that $y(x)$ and $y'(x)$ can have the following Taylor's expansions respectively

$$y(x) = \sum_{j=0}^{r-1} \frac{y_\nu^{(j)}}{j!}(x-x_\nu)^j + \frac{y^{(r)}(t_\nu)}{r!}(x-x_\nu)^r, \tag{3.4}$$

$$y'(x) = \sum_{j=0}^{r-2} \frac{y_\nu^{(j+1)}}{j!}(x-x_\nu)^j + \frac{y^{(r)}(\bar{t}_\nu)}{(r-1)!}(x-x_\nu)^{r-1}, \tag{3.5}$$

where

$$x_{\nu+1} < x < t_\nu, \quad \bar{t}_\nu < x_\nu \quad \left(0 \le \nu \le \left[\frac{n}{2}\right] - 2\right).$$

Given

$$y_0 = \alpha_0, \quad y_0' = \alpha_1, \quad y_{n+1} = \beta_0, \quad y_{n+1}' = \beta_1 \tag{3.6}$$

we can obtain simply from (1.1) the following values

$$y_0^{(s)} = \alpha_s; \quad y_{n+1}^{(s)} = \beta_s \quad (s = 2, 3, \ldots, r). \tag{3.7}$$

Thus we can adopt the following definitions

$$\begin{aligned} \bar{y}_0^{(s)} &\stackrel{\text{def}}{=} y_0^{(s)} = \alpha_s \\ \bar{y}_{n+1}^{(s)} &\stackrel{\text{def}}{=} y_{n+1}^{(s)} = \beta_s \end{aligned} \qquad (s = 0, 1, 2, \ldots, r) \tag{3.8}$$

and

$$y''(x) \stackrel{\text{def}}{=} f(x;\, y_\nu^*(x);\, \bar{y}_\nu' + \int_{x_\nu}^{x} f(t;\, y_\nu^*(t);\, y_\nu^*(t)dt), \tag{3.9}$$

where $y_\nu^*(x)$ and $y_\nu^{*\prime}(x)$ are the approximated functions given by using the approximate values $\bar{y}_\nu^{(j)}$ instead of y_ν in the two series (2.4–5) $(j = 0, 1, 2, \ldots \ldots, r)$ i.e.

$$y_\nu^*(x) = \sum_{j=0}^{r} \frac{y_\nu^{(j)}}{j!}(x-x_\nu)^j \tag{3.10}$$

$$y_\nu^{*\prime}(x) = \sum_{j=0}^{r-1} \frac{\bar{y}_\nu^{(j+1)}}{j!}(x-x_\nu)^j \quad \left(0 \le \nu \le \left[\frac{n}{2}\right] - 2\right). \tag{3.11}$$

Integrating (3.9) between x_ν and $x_{\nu+1}$ $\left(0 \le \nu \le \left[\frac{n}{2}\right] - 2\right)$ we get

$$\bar{y}_{\nu+1}' = \bar{y}_\nu' + \int_{x_\nu}^{x_{\nu+1}} f(t;\, y_\nu^*(t);\, y_\nu^{**\prime}(t))dt, \tag{3.12}$$

and

$$\bar{y}_{\nu+1} = \bar{y}_\nu + \bar{y}_\nu'(x_{\nu+1} - x_\nu) + \int_{x_\nu}^{x_{\nu+1}} \int_{x_\nu}^{t} f(u;\, y_\nu^*(u);\, y_\nu^{**\prime}(u))du dt \tag{3.13}$$

where

$$(3.14) \qquad y_\nu^{**\prime}(x) = \bar{y}'_\nu + \int_{x_\nu}^{x} f(z, y_\nu^*(z); y_\nu^{*\prime}(z))dz.$$

On the other hand, we consider the sub-interval $[x_{n-\nu+1}, x_{n-\nu}]$ and have the following approximated Taylor's expansions

$$(3.15) \qquad y_{n-\nu+1}^*(x) = \sum_{j=0}^{r} \frac{\bar{y}_{n-\nu+1}^{(j)}}{j!}(x-x_{n-\nu+1})^j$$

$$(3.16) \qquad y_{n-\nu+1}^{*\prime}(x) = \sum_{j=0}^{r-1} \frac{\bar{y}_{n-\nu+1}^{(j+1)}}{j!}(x-x_{n-\nu+1})^j.$$

Define

$$(3.17) \qquad y''(x) \overset{\text{def}}{=} f(x; y_{n-\nu+1}^*(x); y_{n-\nu+1}^{**\prime}(x))$$

where

$$(3.18) \qquad y_{n-\nu+1}^{**\prime}(x) = \bar{y}'_{n-\nu+1} + \int_{x_{n-\nu+1}}^{x_{n-\nu}} f(t; y_{n-\nu+1}^*(t); y_{n-\nu+1}^{*\prime}(t))dt$$

and

$$0 \le \nu \le \left[\frac{n}{2}\right].$$

Integrating (3.17) between $x_{n-\nu+1}$ and $x_{n-\nu}$ $\left(0 \le \nu \le \left[\frac{n}{2}\right]\right.$ we obtain the following approximate values

$$(3.19) \qquad \bar{y}'_{n-\nu} = \bar{y}'_{n-\nu+1} + \int_{x_{n-\nu+1}}^{x_{n-\nu}} f(t; y_{n-\nu+1}^*(t); y_{n-\nu+1}^{**\prime}(t))dt,$$

and

$$(3.20) \qquad \begin{aligned} \bar{y}_{n-\nu} &= y_{n-\nu+1} + \bar{y}'_{n-\nu+1}(x_{n-\nu} - x_{n-\nu+1}) + \\ &+ \int_{x_{n-\nu+1}}^{x_{n-\nu}} \int_{x_{n-\nu+1}}^{t} f(u; y_{n-\nu+1}^*(u); y_{n-\nu+1}^{**\prime}(u))dudt, \end{aligned}$$

To estimate the error between the exact and the approximate values of the solution of (1.1) at the non-equidistant nodal points $\{x_\nu\}_{\nu=0}^{n+1}$, we follow the same method of proof carried out by T. Fawzy in [3] in detail. The only difference is that we have here $n = |x_{\nu+1} - x_\nu|$ increasing from the two boundaries towards the origin. For x_ν defined in (1.6)

$$(3.21) \qquad |x_{\nu+1} - x_\nu| = 0(1)\left(\frac{\nu+1}{n^2}\right) \quad \left(0 \le \nu \le \left[\frac{n}{2}\right] - 2\right)$$

and from the symmetry we have

$$|x_{n-\nu}-x_{n-\nu+1}| = 0(1)\left(\frac{\nu+2}{n^2}\right) \quad \left(0\le\nu\le\left[\frac{n}{2}\right]-1\right). \tag{3.22}$$

Thus we can easily attain the following results.

Theorem 3.1. *The error of the approximated values $\bar{y}_\nu^{(s)}$ to the exact values $y_\nu^{(s)}$ corresponding to the nodal points* (1.4 – 6) *is given as follows*

$$|\bar{y}_\nu^{(s)}-y_\nu^{(s)}| = 0(1)\omega\left(\frac{1}{n};\, y^{(r)}\right)\left(\frac{\nu+1}{n^2}\right)^r \tag{3.23}$$

for all $1\le\nu\le\left[\frac{n}{2}\right]-1$,

and

$$|\bar{y}_{n-\nu}^{(s)}-y_{n-\nu}^{(s)}| = 0(1)\omega\left(\frac{1}{n};\, y^{(r)}\right)\left(\frac{\nu+2}{n^2}\right)^r \tag{3.24}$$

for all $0\le\nu\le\left[\frac{n}{2}\right]$, $s = 0, 1, 2, \dots, r$ *and* $r\ge2$ *fixed integer.* □

4. The first interpolation. Suppose that we have the triangular matrix (1.4) whose elements are defined by (1.5 – 6). Suppose that $f(x;y(x);y'(x))$ in (1.1) satisfies the Lipschitz condition (1.3) and $f(x;y(x);y'(x))\in C^{(r-2)}([-1,1]\times\times R\times R)$. Hence there exists a unique solution for (1.1 – 2). Let this solution be $y(x)$ $(-1\le x\le 1)$.

Corresponding to the triangular matrix (1.4), suppose we have the following matrices

$$M : \{m_\nu\}_{\nu=0}^{n+1}, \tag{4.1}$$

where

$$\begin{aligned} m_\nu &= 1 \qquad (1\le\nu\le n)\\ m_0 &= m_{n+1} = 2 \end{aligned} \tag{4.2}$$

and

$$Y : \{y_\nu^{(s)}\}_{\nu=0}^{n+1} \quad (0\le s\le m_\nu-1;\; 0\le\nu\le n+1), \tag{4.3}$$

where

$$y^{(s)}(x_\nu) = y_\nu^{(s)}.$$

Hence there exists a unique Hermite interpolation polynomial $H_{n+3}(x;\,Y;\,A)$ of degree at most $n+3$ [7] which satisfies the following

$$\begin{aligned} H_{n+3}(x_\nu;\,Y;\,A) &= y_\nu \qquad\qquad (1\le\nu\le n),\\ H_{n+3}^{(s)}(1;\,Y;\,A) &= y^{(s)}(1) = y_0^{(s)} - \alpha_s,\\ H_{n+3}^{(s)}(-1;\,Y;\,A) &= y^{(s)}(-1) = y_{n+1}^{(s)} = \beta_s. \end{aligned} \qquad (s = 0, 1) \tag{4.4}$$

It is easy to show that the explicit form of this interpolation process is as follows

$$H_{n+3}(x, Y; A) = \alpha_0 r_0(x) + \alpha_1 r_{0,1}(x) + \beta_0 r_0(x) + \beta_1 r_{0,1}(x) + \tag{4.5}$$
$$+ \sum_{\nu=1}^{n} y_\nu \left(\frac{1-x^2}{1-x_\nu^2} \right)^2 l_\nu(x),$$

where

$$r_0(x) = \left\{ 1 - \left(\frac{1}{2} + n^2 \right)(x-1) - \frac{1}{2}(1+n^2)(x-1)^2 \right\} \frac{1+x}{2} T_n(x), \tag{4.6}$$

$$r_{n+1}(x) = \left\{ 1 + \left(\frac{1}{2} + n^2 \right)(x+1) - \frac{1}{2}(1+n^2)(x+1)^2 \right\} \frac{1-x}{2} \frac{T_n(x)}{T_n(-1)}, \tag{4.7}$$

$$r_{0,1}(x) = \left\{ (x-1) + \frac{1}{2}(x-1)^2 \right\} \frac{1-x}{2} T_n(x), \tag{4.8}$$

$$r_{n+1,1}(x) = \left\{ (x+1) - \frac{1}{2}(x+1)^2 \right\} \frac{1-x}{2} \frac{T_n(x)}{T_n(-1)}, \tag{4.9}$$

and

$$l_\nu(x) = \frac{T_n(x)}{T_n'(x_\nu)(x-x_\nu)} \qquad (1 \le \nu \le n;\ n = 1, 2, 3, \dots) \tag{4.10}$$

the well known Lagrange fundamental polynomial of interpolation.

In the following lemma we obtain a useful result for the Lebesque function.

Lemma 4.4. *Let $l_\nu(x)$ be the polynomial defined in* (4.10). *Then for all $x \in [-1,1]$ the inequality*

$$\sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu^2} |l_\nu(x)| = 0(1) \log n, \tag{4.11}$$

holds true. □

Proof. Let $x \in [x_{j+1}, x_j]$ $(0 \le j \le n)$. Hence it is obvious from (1.6) that

$$1 - x^2 < 1 - x_j^2 = 0(1) \frac{j^2}{n^2}, \tag{4.12}$$

$$\frac{1-x^2}{1-x_\nu^2} = 0(1) \quad \text{for} \quad \nu = j, j+1, \tag{4.13}$$

and

$$|x - x_\nu| = \begin{cases} 0(1) \dfrac{|j-\nu|\ |j+\nu-1|}{n^2} & \text{for} \quad 1 \le \nu \le j-1, \\ 0(1) \dfrac{|j-\nu+1|\ |j+\nu|}{n^2} & \text{for} \quad j+2 \le \nu \le \left[\dfrac{n}{2} \right], \\ 0(1) \dfrac{|j-(n-\nu)+1|\ |j+(n-\nu)|}{n^2} & \text{for} \quad \left[\dfrac{n}{2} \right] + 1 \le \nu \le n. \end{cases} \tag{4.14}$$

Now, we have

$$(4.15)\qquad \sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu^2}|l_\nu(x)| = \sum_{\nu=1}^{j-1} + \sum_{\nu=j}^{j+1} + \sum_{\nu=j+2}^{\left[\frac{n}{2}\right]} + \sum_{\nu=\left[\frac{n}{2}\right]+1}^{n} .$$

Thus, using (1.6–7), (2.2) and (4.10) into (4.15), we get the following

$$(4.16)\qquad \sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu^2}|l_\nu(x)| \leq \sum_{\nu=1}^{n} \frac{1-x^2}{n\sqrt{1-x_\nu^2}|x-x_\nu|} .$$

It is obvious when (4.12) and (4.14) are eliminated in (4.16) we get

$$(4.17)\qquad \begin{aligned} \sum_{\nu=1}^{j-1} \frac{1-x^2}{1-x_\nu^2}|l_\nu(x)| &= 0(1)\sum_{\nu=1}^{j-1} \frac{j^2}{(2\nu-1)|j-\nu|\,|j+\nu-1|} = \\ &= 0(1)\sum_{\nu=1}^{j-1} \frac{j}{(2\nu-1)|j-\nu|} . \end{aligned}$$

If $\nu \geq \frac{j}{2}$; then (4.17) reduces to

$$(4.18)\qquad \sum_{\nu=1}^{j-1} \frac{1-x^2}{1-x_\nu^2}|l_\nu(x)| = 0(1)\sum_{\nu=1}^{j-1} \frac{1}{|j-\nu|} = 0(1)\log n.$$

If $\nu < \frac{j}{2}$, then (4.17) reduces to

$$(4.19)\qquad \sum_{\nu=1}^{j-1} \frac{1-x^2}{1-x_\nu^2}|l_\nu(x)| = 0(1)\sum_{\nu=1}^{j-1} \frac{1}{2\nu-1} = 0(1)\log n.$$

Similarly if we use (4.12) and (4.14) in (4.16), we get

$$(4.20)\qquad \begin{aligned} \sum_{\nu=j+2}^{\left[\frac{n}{2}\right]} \frac{1-x^2}{1-x_\nu^2}|l_\nu(x)| &= 0(1)\sum_{\nu=j+2}^{\left[\frac{n}{2}\right]} \frac{j^2}{(2\nu-1)|j-\nu+1|\,|j+\nu|} = \\ &= 0(1)\sum_{\nu=1}^{\left[\frac{n}{2}\right]} \frac{j}{(2\nu-1)|j-\nu+1|} = 0(1)\log n, \end{aligned}$$

and

$$(4.21)\qquad \begin{aligned} &\sum_{\nu=\left[\frac{n}{2}\right]+1}^{n} \frac{1-x^2}{1-x_\nu^2}|l_\nu(x)| = \\ &\sum_{\nu=\left[\frac{n}{2}\right]+1}^{n} \frac{j^2}{(2n-2\nu+1)|j-(n-\nu)+1|\,|j+(n-\nu)|} = 0(1)\log n. \end{aligned}$$

It is easy to show from (4.13) and (2.4) that

$$\sum_{\nu=j}^{j+1} \frac{1-x^2}{1-x_\nu^2} |l_\nu(x)| = 0(1). \tag{4.22}$$

Thus the inequalities (4.18–22) when added give the proof of the lemma. □

Considering the Hermite interpolation process given by (4.5–10), we prove the following.

Theorem 4.1. *Let $y(x) \in C^{(r)}([-1,1])$; then for all $x \in [-1,1]$*

$$|H_{n+3}^{(s)}(x; Y; A) - y^{(s)}(x)| = 0(1)\omega\left(\frac{1}{n+3}; y^{(r)}\right)\frac{\log n}{n^{r-s}}, \tag{4.23}$$

holds true, where $n \geq 10$, $r \geq 2$, fixed integer $s = 0, 1, 2$ and $\omega(\delta; y^{(r)})$ is the modulus of continuity of $y^{(r)}(x)$. □

Proof. Let $G_{n+3}(x; y)$ be the Gopengaus polynomial for the function $y(x)$ and of degree $n+3$ at most. It is known from (4.5) that

$$G_{n+3}(x; y(x)) \equiv H_{n+3}(x; Y; A) \tag{4.24}$$

where

$$Y: \{G_{n+3}(x_\nu); G'_{n+3}(1); G'_{n+3}(-1)\}. \tag{4.25}$$

Therefore using (4.5) and (4.24), we obtain

$$|H_{n+3}^{(s)}(x; Y; A) - G_{n+3}^{(s)}(x; y(x))| \equiv |\{(1-x^2)g_{n+1}(x)\}^{(s)}| \tag{4.26}$$

where

$$g_{n+1}(x) = \sum_{\nu=1}^{n} [G_{n+1}(x_\nu) - y_\nu] \frac{1-x^2}{(1-x_\nu^2)^2} l_\nu(x) \tag{4.27}$$

and $s = 0, 1, 2$.

Using Bernstein inequality (2.6) we can show simply that

$$|\{(1-x^2)g_{n+1}(x)\}^{(s)}| = 0(1)n^s \max_{-1 \leq x \leq 1} |g_{n+1}(x). \tag{4.28}$$

Using (2.8) into (4.27), we can easily obtain

$$\begin{aligned} |g_{n+1}(x)| &= 0(1)\frac{\omega\left(\frac{1}{n+3}; y^{(r)}\right)}{n^r} \sum_{\nu=1}^{n} (1-x_\nu^2)^{\frac{r-2}{2}} \frac{1-x^2}{1-x_\nu^2} |l_\nu(x) = \\ &= 0(1)\frac{\omega\left(\frac{1}{n+3}; y^{(r)}\right)}{n^r} \sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu^2} |l_\nu(x)| \end{aligned} \tag{4.29}$$

for all $n \geq 10$, $r \geq 2$.

Using Lemma 4.1 into (4.29), we achieve

$$|g_{n+1}(x)| = 0(1)\frac{\omega\left(\frac{1}{n+3};\, y^{(r)}\right)}{n^r}\log n. \tag{4.30}$$

Substituting (4.30) into (4.28) and the result into (4.26), we obtain

$$\left|H^{(s)}_{n+3}(x;Y;A)-G^{(s)}_{n+3}(x;y(x))\right| = 0(1)\frac{\omega\left(\frac{1}{n+3};\, y^{(r)}\right)}{n^{r-s}}\log n \tag{4.31}$$

for all $n\geq 10$, $r\geq 2$ fixed integer and $s = 0, 1, 2$.

Using (2.8) together with (4.31) into the R.H.S. of the triangular inequality

$$\left|H^{(s)}_{n+3}(x;Y;A)-y^{(s)}(x)\right|\leq$$

$$\leq\left|H^{(s)}_{n+3}(x;Y;A)-G^{(s)}_{n+3}(x;y(x))\right|+\left|G^{(s)}_{n+3}(x)-y^{(s)}(x)\right|,$$

we obtain

$$\left|H^{(s)}_{n+3}(x;Y;A)-y^{(s)}(x)\right| = 0(1)\frac{\omega\left(\frac{1}{n+3};\, y^{(r)}\right)}{n^{r-s}}\log n \tag{4.32}$$

for all $n\geq 10$, $r\geq 2$ fixed integer and $s = 0, 1, 2$. □

5. Second interpolation – The second approximation process. Using the method of approximation given in Section 3, in (3.13) and (3.20) together with the definition (3.8), we obtain

$$\overline{Y}:\{\bar{y}_\nu, y_0', y_{n+1}'\}_{\nu=0}^{n+1} \quad (n = 1,2,3,\ldots), \tag{5.1}$$

which corresponds to the matrix defined by (1.4–7). Hence we can obtain a Hermite interpolation polynomial $\overline{H}_{n+3}(x;\overline{Y},A)$ of degree $n+3$ at most which satisfies

$$\begin{aligned}\overline{H}_{n+3}(x_\nu;\overline{Y};A) &= \bar{y}_\nu\approx y_\nu \quad (1\leq\nu\leq n),\\ \overline{H}_{n+3}(1;\overline{Y};A) &= \bar{y}_0^{(s)} \overset{\text{def}}{=\!=} y_0^{(s)} = \alpha_s \quad (s = 0, 1),\\ \overline{H}_{n+3}(-1;\overline{Y};A) &= \bar{y}_{n+1}^{(s)} \overset{\text{def}}{=\!=} y_{n+1}^{(s)} = \beta_s \quad (s = 0, 1).\end{aligned} \tag{5.2}$$

It is not difficult to show the explicit form of this polynomial to be

$$\begin{aligned}\overline{H}_{n+3}(x;\overline{Y};A) = {}&\alpha_0 r_0(x)+\alpha_1 r_{0,1}(x)+\beta_0 r_{n+1}(x)+\beta_1 r_{n+1,1}(x)+\\ &+\sum_{\nu=1}^{n}\bar{y}_\nu\left(\frac{1-x^2}{1-x_\nu^2}\right)^2 l_\nu(x),\end{aligned} \tag{5.3}$$

where $r_0(x)$, $r_{n+1}(x)$, $r_{0,1}(x)$, $r_{n+1,1}(x)$ and $l_\nu(x)$ respectively are given by (4.6–10).

Considering (5.3) we prove the convergence of the interpolation polynomial $\overline{H}_{n+3}(x; \overline{Y}, A)$ to the exact solution of (1.1–2). An estimate order for the approximation is given too.

Theorem 5.1. *Let $y(x)$ be the exact solution of the differential equation given in* (1.1–2). *Let $f(x, y, y')\in C^{(r-2)}([-1,1]\times R\times R)$ which satisfies the Lipschitz condition* (1.3). *Then for all $x\in[-1,1]$ and all $n\geq 10$ the following inequality*

$$(5.4)\qquad \left|\overline{H}^{(s)}_{n+3}(x; \overline{Y}; A)-y^{(s)}(x)\right| = 0(1)\omega\left(\frac{1}{n}; y^{(r)}\right)\frac{\log n}{n^{r-s}}$$

holds true. The $r\geq 2$ fixed integer, $n\geq 10$ and $s = 0, 1, 2$.

If $y''(x)\in \mathrm{Lip}\,\mu \quad (0\leq\mu\leq 1)$, then

$$(5.5)\qquad \left|\overline{H}''_{n+3}(x; \overline{Y}; A)-y''(x)\right| = 0(1)\frac{\log n}{n^{\mu}},$$

for all $y(x)\in C^{(2)}([-1,1])$ and all $x\in[-1,1]$. □

Remark. *It is obvious from* (5.3) *that*

$$\left|\overline{H}^{(s)}_{n+1}(\pm 1; \overline{Y}; A)-y^{(s)}(x)\right| = 0 \quad (s = 0, 1).$$

Proof. It is obvious from (4.10) and (5.3) that

$$(5.6)\qquad \left|\overline{H}^{(s)}_{n+3}(x; \overline{Y}; A)-H^{(s)}_{n+3}(x; Y; A)\right| = \left|\{(1-x^2)g_{n+1}(x)\}^{(s)}\right|,$$

where $s = 0, 1, 2$ and

$$(5.7)\qquad g_{n+1}(x) = \sum_{\nu=1}^{n}(\overline{y}_\nu-y_\nu)\frac{1-x^2}{(1-x_\nu^2)^2}l_\nu(x).$$

Using (3.23–24) together with (1.6), we simply reduce (5.7) to give

$$(5.8)\qquad |g_{n+1}(x)| = 0(1)\frac{\omega\left(\frac{1}{n}; y^{(r)}\right)}{n^{2r-2}}\sum_{\nu=1}^{n}\nu^{r-2}\frac{1-x^2}{1-x_\nu^2}|l_\nu(x)|.$$

It is obvious when we use (4.11) into (5.8) we obtain for all $r\geq 2$ the following

$$(5.9)\qquad |g_{n+1}(x)| = 0(1)\frac{\omega\left(\frac{1}{n}; y^{(r)}\right)}{n^{r}}\log n.$$

Differentiating the R.H.S. of (5.6) and then using Bernstein inequality (2.6), we simply obtain

$$(5.10)\qquad \left|\overline{H}^{(s)}_{n+3}(x; \overline{Y}; A)-H^{(s)}_{n+3}(x; Y; A)\right| = 0(1)n^{s}\max_{-1\leq x\leq 1}|g_{n+1}(x)|,$$

where $s = 0, 1, 2$.

Substituting (5.9) into (5.10), we obtain

$$(5.11)\qquad \left|\overline{H}^{(s)}_{n+3}(x;\overline{Y};A)-H^{(s)}_{n+3}(x;Y;A)\right| = 0(1)\omega\left(\frac{1}{n};y^{(r)}\right)\frac{\log n}{n^{r-s}}$$

for all $r \geq 2$, fixed integer and $s = 0, 1, 2$.

Using (4.23) and (5.11) into the triangular inequality

$$(5.12)\qquad \begin{aligned}\left|\overline{H}^{(s)}_{n+3}(x;\overline{Y};A)-y^{(s)}(x)\right| &\leq \left|\overline{H}^{(s)}_{n+3}(x;\overline{Y};A)-H^{(s)}_{n+3}(x;Y;A)\right| + \\ &+\left|H^{(s)}_{n+3}(x,Y;A)-y^{(s)}(x)\right| \quad (s = 0, 1, 2),\end{aligned}$$

we obtain the proof of (5.4).

If $y''(x)\in \mathrm{Lip}\,\mu$ $(0\leq\mu\leq 1)$, then we have

$$(5.13)\qquad \omega\left(\frac{1}{n};y''\right) = 0(1)\frac{1}{n^{\mu}}.$$

Hence (5.4) and (5.13) prove (5.5). □

In the following theorem, we prove that the approximate solution (5.3) satisfies the differential equation (1.1).

Theorem 5.2. *Let $f\in C^{(r-2)}([-1,1]\times R\times R)$ where $r\geq 2$, a fixed integer and let it satisfy the Lipschitz condition* (1.3). *Then for all $x\in[-1,1]$, $n\geq 10$*

$$(5.14)\qquad \left|\overline{H}''_{n+3}(x;\overline{Y};A)-f''(x;\overline{H}_{n+1}(x);\overline{H}'_{n+1}(x)\right| = 0(1)\omega\left(\frac{1}{n};y^{(r)}\right)\frac{\log n}{n^{r-2}}$$

holds true.

If $y''(x)\in \mathrm{Lip}\,\mu$ $(0<\mu\leq 1)$, then for $f\in C([-1,1]\times R\times R)$ and all $x\in[-1,1]$ the inequality

$$(5.15)\qquad \left|\overline{H}''_{n+3}(x;Y;A)-f(x;\overline{H}_{n+3}(x);\overline{H}'_{n+3}(x)\right| = 0(1)\frac{\log n}{n^{\mu}},$$

holds true. □

Proof. Let us define

$$(5.16)\qquad \gamma_n(x) = f(x;\overline{H}_{n+3}(x);\overline{H}'_{n+3}(x)).$$

It is obvious that

$$(5.17)\quad \left|\overline{H}''_{n+3}(x;\overline{Y};A)-\gamma_n(x)\right|\leq\left|\overline{H}''_{n+3}(x;\overline{Y};A)-y''(x)\right|+\left|y''(x)-\gamma_n(x)\right|.$$

It is obvious from (1.1), (1.3) and (5.14) that

$$(5.18)\qquad \begin{aligned}\left|y''(x)-\gamma_n(x)\right| \leq L\{&\left|\overline{H}_{n+3}(x;\overline{Y};A)-y(x)\right| + \\ &\left|\overline{H}'_{n+3}(x;\overline{Y};A)-y'(x)\right|\}.\end{aligned}$$

Using (5.4) into (5.18), we obtain for $f\in C^{(r-2)}([-1,1]\times R\times R)$

$$(5.19)\qquad \left|y''(x)-\gamma_n(x)\right| = 0(1)\omega\left(\frac{1}{n};y^{(r)}\right)\frac{\log n}{n^{r-1}}$$

for all $n\geq 10$ and $r\geq 2$.

Substituting (5.19) and (5.4) into (5.17), we obtain the proof of (5.14).

The inequalitites (5.13) and (5.14) simply give the proof of (5.15) for $f \in C([-1,1] \times R \times R)$ and $y'' \in \mathrm{Lip}\,\mu$ $(0<\mu\leq 1)$. □

Remark 1. *We can use the roots of the second kind Chebyshev polynomial*

$$x_\nu = \cos\frac{\nu}{n+1}\pi \quad (1\leq\nu\leq n) \tag{5.20}$$

as the nodal points for our interpolation. The results which we will abtain will be the same as those given in (5.4–5) *and* (5.15–16).

Remark 2. *Any method of integration which gives* y_ν *such that*

$$|\bar{y}_\nu - y_\nu| = 0(1)\omega\left(\frac{1}{n}; y^{(r)}\right)\left(\frac{\nu}{n^2}\right)^r \quad (1\leq\nu\leq n) \tag{5.21}$$

can be used in the Hermite interpolation process that approximates the solution of the non-linear differential equation (1.1–2).

Remark 3. *If we impose a change on the calculated values* $\bar{y}_\nu$ *to the arbitrary real values* $y_\nu^*(1\leq\nu\leq n)$ *in such a way that*

$$|\bar{y}_\nu - y_\nu^*| = 0(1)\omega\left(\frac{1}{n}; y^{(r)}\right)\left(\frac{\nu}{n^2}\right)^r, \tag{5.22}$$

then our method of approximation given by (5.3) *shows that it is stable for all* $r>2$. *In the case when* $r=2$ *the method is stable only if condition* (4.22) *is satisfied and* $y''(x)\in\mathrm{Lip}\,\mu$ $(0<\mu\leq 1)$.

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