

THE APPROXIMATE SOLUTION OF NON-LINEAR FUNCTIONAL EQUATIONS BY A STEFFENSEN-TYPE METHOD

By

Á. VÁRHELYI

Dept. of Numerical analysis and computer science, Eötvös University,
1088 Budapest, Múzeum krt. 6–8.

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The classic secant method can be extended for solving a non-linear operator equation defined in the conditions of Banach spaces, by using the generalized divided differences treated in the last chapter of the monography [3]. In their paper [2] M. Balázs and G. Goldner proposed the

$$(1) \quad x_{n+1} = x_n - \frac{F(x_n)}{F_{x_n x_{n-1}}(y_n)} y_n \quad (n = 0, 1, \dots)$$

method for the computation of the solution of the equation

$$(2) \quad F(x) = 0,$$

where $F: X \rightarrow \mathbf{R}$, (X being a Banach space) is a non-linear functional, and the auxiliary vector series was chosen in the following form

$$(3) \quad y_n := \frac{x_n - x_{n-1}}{\|x_n - x_{n-1}\|}, \quad (n = 0, 1, \dots).$$

In this paper we are going to solve the functional equation (2) using a Steffensen-type method instead of the above secant method. Let's define a non-linear operator $\Phi: X \rightarrow X$ in the X Banach space using the above mentioned functional F

$$(4) \quad \Phi(x) := x - F(x) \frac{x}{\|x\|}$$

and let x_n, u_n be the nodal points of our method, where x_n is the n -th approximate solution of (2), but u_n is computed by the following form

$$(5) \quad u_n := \Phi(x_n).$$

We define step by step a scalar L_n and a vector y_n in the following forms

$$(6) \quad \left. \begin{aligned} L_n &:= \frac{1}{F_{x_n u_n}(y_n)}, \\ y_n &:= \frac{x_n - u_n}{\|x_n - u_n\|}, \end{aligned} \right\} \quad (n = 0, 1, \dots)$$

and suggest the following method

$$(7) \quad x_{n+1} = x_n - L_n F(x_n) y_n \quad (n = 0, 1, \dots)$$

for solving the equation (2). According to the essential feature of divided differences $F_{x' x''}(x' - x'') = F(x') - F(x'')$ the following relation

$$(8) \quad \|F_{x_n u_n}\| = |F_{x_n u_n}(y_n)|, \quad (n = 0, 1, \dots),$$

can be proved easily. Further on we assume the functional F as being continuous in the sphere $S(x_0, r)$ specified later on.

Theorem. *We suppose that the following conditions are satisfied*

1°. *For the divided difference of the functional F in the points x_0, u_0 we have*

$$|L_0| = \frac{1}{\|F_{x_0 u_0}\|} \leq B_0, \quad B_0 \geq 1.$$

2°. *For the approximation x_0*

$$|F(x_0)| \leq \eta_0.$$

3°. *The divided difference of the operator Φ and the divided difference of the second order of the functional F satisfy the following relations*

$$\|\Phi_{x' x''}\| \leq M; \quad \|F_{x' x''} - F_{x'' x''}\| \leq K \|x' - x''\|$$

if $x', x'', x''' \in S(x_0, r)$, where

$$r := \max \left\{ \frac{B_0 \eta_0 (1 - \alpha)}{1 - 2\alpha}; \frac{MB_0 \eta_0 (1 - \alpha)}{1 - 2\alpha} + \eta_0 \right\}.$$

4°. *The constant numbers B_0, η_0, K, M satisfy the inequality*

$$h_0 := 2K B_0^2 \eta_0 (M + 1) < \alpha < \frac{3 - \sqrt{5}}{2}.$$

Then equation (2) has at least a solution $x^* \in S(x_0, r)$ which is the limit of the approximations (7) and the rapidity of the convergence is characterized by the inequality

$$(9) \quad \|x_n - x^*\| \leq (1 - \alpha)^n \left(\frac{\alpha}{(1 - \alpha)^2} \right)^{2^{n-1}} \cdot \delta,$$

where

$$\delta := B_0 \eta_0 \sum_{k=1}^{\infty} (1-\alpha)^{k-1} \left(\frac{\alpha}{(1-\alpha)^2} \right)^{2(2^{k-1}-1)}.$$

Proof. First of all let's take the following estimates, which are given directly from the formulas (5), (7) and the assumptions of Theorem:

$$\begin{aligned} (10) \quad & \|x_1 - x_0\| = \|L_0 F(x_0) y_0\| \leq B_0 \eta_0, \\ & \|u_1 - u_0\| = \|\Phi(x_1) - \Phi(x_0)\| \leq MB_0 \eta_0, \\ & \|x_1 - u_0\| \leq \|x_0 - u_0\| + \|L_0 F(x_0) y_0\| \leq \eta_0 + B_0 \eta_0 \leq 2B_0 \eta_0, \\ & \|u_1 - x_0\| \leq MB_0 \eta_0 + \eta_0. \end{aligned}$$

According to formula (7) it results immediately

$$F_{x_0 u_0}(x_1 - x_0) = -F_{x_0 u_0}(L_0 F(x_0) y_0),$$

whence using the formulas (6) and the definition of the divided differences we obtain the identity

$$(11) \quad F_{x_0 u_0}(x_1 - x_0) = -F(x_0).$$

Substituting x_0 for x_1 from the conditions of this theorem, we have

$$\|F_{x_1 u_1}\| \geq \|F_{x_0 u_0}\| \left(1 - \frac{K(B_0 \eta_0 + MB_0 \eta_0)}{\|F_{x_0 u_0}\|} \right) \geq \frac{1 - h_0}{B_0},$$

whence it results

$$(12) \quad |L_1| = \frac{1}{\|F_{x_1 u_1}\|} \leq \frac{B_0}{1 - h_0} =: B_1.$$

From (11) we obtain

$$|F(x_1)| \leq K \|x_1 - u_0\| \|x_1 - x_0\| \leq 2KB_0^2 \eta_0^2 (M+1),$$

so we have

$$(13) \quad h_0 \eta_0 =: \eta_1.$$

Considering the formulas (12), (13)

$$(14) \quad h_1 := 2KB_1^2 \eta_1 (M+1) \leq \frac{h_0^2}{(1-h_0)^2}$$

is given. The inequality $h_1 < \alpha$ is satisfied iff $\alpha < \frac{3-\sqrt{5}}{2}$. By induction we obtain

$$\begin{aligned}
|L_n| &\leq B_n \leq \frac{B_{n-1}}{1-\alpha} \leq \frac{1}{(1-\alpha)^n} B_0, \\
(15) \quad |F(x_n)| &\leq \eta_n \leq h_{n-1} \eta_{n-1} \leq \left(\frac{\alpha}{(1-\alpha)^2} \right)^{2^{n-1}} (1-\alpha)^{2^n} \eta_0, \\
h_n &\leq \frac{h_{n-1}^2}{(1-h_{n-1})^2} \leq \left(\frac{1}{1-\alpha} \right)^{2(2^{n-1})} h_0^{2^n}.
\end{aligned}$$

On the basis of the formulas (15) we have

$$\begin{aligned}
\|x_{n+p} - x_n\| &\leq B_{n+p-1} \eta_{n+p-1} + \dots + B_n \eta_n \leq \\
&\leq B_0 \eta_0 \sum_{k=1}^p \left(\frac{\alpha}{(1-\alpha)^2} \right)^{2^{n+k-1}-1} (1-\alpha)^{n+k-1},
\end{aligned}$$

whence using the inequality

$$(2^{n+k-1} - 1) - (2^n - 1) \geq 2(2^{n-1} - 1), \quad n \geq 1,$$

we obtain

$$\begin{aligned}
\|x_{n+p} - x_n\| &\leq (1-\alpha)^n \left(\frac{\alpha}{(1-\alpha)^2} \right)^{2^{n-1}} \cdot \delta_p, \\
(16) \quad \delta_p &:= B_0 \eta_0 \sum_{k=1}^p (1-\alpha)^{k-1} \left(\frac{\alpha}{(1-\alpha)^2} \right)^{2(2^{k-1}-1)}.
\end{aligned}$$

Considering the completeness of X it results that the sequence $\{x_n\}$ has a limit $x^* \in X$. Since $\lim_{n \rightarrow \infty} \eta_n = 0$ by the continuity of the functional F we obtain

$F(x^*) = 0$. From (16) if $p \rightarrow \infty$ the rapidity of convergence is given.

Using the inequality $2^{k-1} \geq k$, $k \geq 1$ we can estimate $\|x_n - x_0\|$ and $\|u_n - x_0\|$ in the same way as (16). So we have

$$\|x_n - x_0\| \leq B_0 \eta_0 \sum_{k=1}^n (1-\alpha)^{k-1} \left(\frac{\alpha}{(1-\alpha)^2} \right)^{2^{k-1}-1} \leq \frac{1-\alpha}{1-2\alpha} B_0 \eta_0,$$

(17)

$$\|u_n - x_0\| \leq \frac{MB_0 \eta_0 (1-\alpha)}{1-2\alpha} + \eta_0.$$

Choosing r as in the 3°. condition of the Theorem it results $x^* \in \in S(x_0, r)$. This completes the proof.

Computational example

$$F(x) = (x^2 + y^2 - 2)^2 + (x^2 - y^2)^2 = 0,$$

initial values: $x_{-1} (.7; .7)$
 $x_0 (.8; .8)$ } for the secant method,
 $x_0 (.7; .7)$ for the Steffensen-method.

	Secant method	Steffensen method
1	.89931 .89931	.959508 .959508
2	.938356 .938356	.976706 .976706
3	.963359 .963359	.987196 .987196
4	.977575 .977575	.993213 .993213
5	.98629 .98629	.99649 .99649
6	.991566 .991566	.998227 .998227
7	.994807 .994807	.99913 .99913

The exact solution is (1; 1), the secant method will accurate to three decimal places after 11 step.

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REFERENCES

- [1] Altman, M.: Concerning Approximate Solutions of Non-Linear Functional Equations. Bull. Acad. Polon. Sci. 5. (1957) pp. 461 – 465.
- [2] Balázs, M. – Goldner, G.: On the Approximate Solution of Non-Linear Functional Equations. Anal. Sti. Univ. "Al. I. Cuza" Sect. Mat. 15 (1969) pp. 369 – 373.
- [3] Jankó, B.: On the Solving of the Non-Linear Operator Equations in Banach Spaces (in Roumanian) Bucharest, 1969.

