ON THE REMEZ ALGORITHM

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The aim of this paper is to give a simplified proof of the convergence of the Remez algorithm.

Let $I=[a, b]$ be a bounded and closed interval, U_n an n-dimensional linear space of real functions continuous on I, for which the Haar condition holds, i.e. every non-zero element of U_n has at most $n-1$ roots on I. Let, furthermore Z be a closed subset of I, which has at least $n+1$ points. Under these conditions, every function $f(x)$, which is continuous on Z has a uniquely determined best approximation in U_n, in the sense of $C(Z)$ norm, denoted by $\nu(x)$.

The aim of the Remez algorithm is to construct this element as the limit of an infinite sequence.

We quote the well-known alternation theorem of Čebyšev: $\nu(x)$ is the best approximation in U_n for $f(x)\in C(Z)$, if and only if there exists an ordered set P of $(n+1)$ distinct points of Z, $P=\{x_1, \ldots, x_{n+1}\}$ with the properties

(1) $$|f(x_i)-\nu(x_i)| = \max_{x\in Z} |f(x)-\nu(x)|, \quad i=1, 2, \ldots, n+1,$$

and

$$sg\,(f(x_{i+1})-\nu(x_{i+1})) = -sg\,(f(x_i)-\nu(x_i)), \quad i=1, \ldots, n.$$

Denote by $E_n(f, Z)$ the distance of $f(x)$ from U_n:

$$E_n(f, z) = \min_{u\in U_n} \max_{x\in Z} |f(x)-u(x)|.$$

If Z_1 is a subset of Z, then it is obviously

(2) $$E_n(f, Z_1)\leq E_n(f, Z).$$

This is true specially for the sets of $n+1$ elements, and the alternation theorem of Čebyšev assures the existence of such an ordered system P of $n+1$ elements, for which

$$E_n(f, P) = E_n(f, Z)$$

holds.

If P is an ordered set of $n+1$ distinct elements, then it is easy to find $E_n(f, P)$ and the best approximation on P.

Let $x_1, \ldots, x_{n+1}$ be the (distinct) points of P, then there exist numbers d_j, with the following properties

(3) $$\sum_{j=1}^{n+1} d_j u(x_j) = 0, \quad \text{for every} \quad u \in U_n ;$$

(4) $$sg\, d_j = -sg\, d_{j-1} ;$$

(5) $$d_j \neq 0, \quad j = 1, 2, \ldots, n+1 ;$$

(6) $$\sum_{j=1}^{n+1} |d_j| = 1 ,$$

Let $u_1(x), \ldots, u_n(x)$ be a basis of U_n. Consider the determinant of the matrix, which is obtained from the matrix

$$\begin{pmatrix} u_1(x_1) & \ldots & u_n(x_1) \\ \cdots & \cdots & \cdots \\ u_1(x_{n+1}) & \ldots & u_n(x_{n+1}) \end{pmatrix}$$

by omitting the j-th row. We multiply this determinant by $(-1)^{j-1}$ and denote the received value by $\tilde{d}_j$. Then we have

$$\sum_{j=1}^{n+1} \tilde{d}_j u_k(x_j) = 0 \quad \text{for all} \quad k ,$$

and so for every u in U_n

(7) $$\sum_{j=1}^{n+1} \tilde{d}_j u(x_j) = 0 .$$

The $\tilde{d}_j$'s are not zero and $sg\, \tilde{d}_j = -sg\, \tilde{d}_{j-1}$, the opposite case would contradict to the Haar condition, thus

$$d_j = \tilde{d}_j \Big/ \sum_{j=1}^{n+1} |\tilde{d}_j| \quad \text{have the desired properties.}$$

The construction of the best approximation on P may be the following. The simultaneous equations for the a_i's and E

(8) $$f(x_j) - \sum_{i=1}^{n+1} a_i u_i(x_j) = (-1)^{j-1} E, \quad j = 1, 2, \ldots, n+1$$

are solvable, as a consequence of the alternation theorem of Čebyšev.

After multiplying (8) by d_j and summing over all j, from (7) we have

$$\sum_{j=1}^{n+1} d_j f(x_j) = \left(\sum_{j=1}^{n+1} (-1)^{j-1} d_j\right) E .$$

From (4) and (6) it follows that

$$\sum_{j=1}^{n+1} d_j f(x_j) = E \quad \text{or} \quad -E ,$$

and therefore

$$\left|\sum_{j=1}^{n+1} d_j f(x_j)\right| = E_n(f, P) . \tag{9}$$

Thus E is determined and then one can solve (8) also for the a_i's, and

$$u(x) = \sum_{i=1}^{n} a_i u_i(x)$$

is the best approximation on P.

The Remez-algorithm gives a construction of a sequence of ordered sets P_k of $n+1$ points in Z, in such a way that the best approximation on P_k converges to the best approximation on Z, as $k \to \infty$.

The algorithm starts from an arbitrary ordered set P_0.

If the set P_k is given, one can find $E_n(f, P_k)$ and $v_k(x)$, the best approximation of $f(x)$ on P_k, as it was described.

If $\max_{x \in Z} |f(x) - v_k(x)| = E_n(f, P_k)$,

then $v_k(x)$ is the best approximation also on Z, according to the Čebyšev theorem. In the opposite case, one chooses a "maximum" point y_k in Z for the function $|f(x) - v_k(x)|$.

Let the (new) P_{k+1} be constructed by replacing one element of P_k by y_k, so that after ordering the equalities

$$sg\,(f(x_j^{(k+1)}) - v_k(x_j^{(k+1)})) = -sg\,(f(x_{j-1}^{(k+1)}) - v_k(x_{j-1}^{(k+1)})) \tag{10}$$

hold.

It is easy to prove the inequality:

$$E_n(f, P_{k+1}) > E_n(f, P_k) . \tag{11}$$

In fact from (9), (6), (3), (4) and (11) it follows that

$$E_n(f, P_{k+1}) - E_n(f, P_k) = \left|\sum_{j=1}^{n+1} d_j^{(k+1)} f(x_j^{(k+1)})\right| - \sum_{j=1}^{n+1} |d_j^{(k+1)}|\, E_n(f, P_k) =$$

$$= \left|\sum_{j=1}^{n+1} d_j^{(k+1)} (f(x_j^{(k+1)}) - v_k(x_j^{(k+1)}))\right| - \sum_{j=1}^{n+1} |d_j^{(k+1)}|\, E_n(f, P_k) =$$

$$= \sum_{j=1}^{n+1} |d_j^{(k+1)}| \left(|f(x_j^{(k+1)}) - v_k(x_j^{(k+1)})| - E_n(f, P_k)\right) .$$

Consequently

$$E_n(f, P_{k+1}) - E_n(f, P_k) = \tag{12}$$
$$= |d_{j^*}^{(k+1)}| \left(\max_{x \in Z} |f(x) - v_k(x)| - E_n(f, P_k)\right)$$

is valid, since the numbers in the parentheses are zero except the j^*-th one, finally from (5) we get (11).

If an inequality

$$|d_j^{(k)}| > \varepsilon \tag{13}$$

is valid for all k with a positive ε independent of k, then we have from (12)

$$\max_{x \in Z} |f(x) - v_k(x)| \le E_n(f, P_k) + \frac{1}{\varepsilon}\left(E_n(f, P_{k+1}) - E_n(f, P_k)\right).$$

Therefore from (2) we have

$$\max_{x \in Z_1} |f(x) - v_k(x)| \le E_n(f, Z) + \frac{1}{\varepsilon}\left(E_n(f, P_{k+1}) - E_n(f, P_k)\right).$$

But the numbers $E_n(f, P_{k+1}) - E_n(f, P_k)$ converge to zero, as they are the differences of a bounded and increasing sequence.

Thus the inequality

$$\limsup_{k} \max_{x \in Z} |f(x) - v_k(x)| \le E_n(f, z) \tag{14}$$

holds.

If $v(x)$ is an accumulation point of the $v_k(x)$-s, then the relation

$$\max_{x \in z} |f(x) - v(x)| \le E_n(f, z)$$

is valid, but the inequality is not possible, hence we have

$$\max_{x \in Z} |f(x) - v(x)| = E_n(f, Z),$$

i.e. $v(x)$ is a best approximation on Z.

The sequence of $v_k(x)$ must have at least one accumulation point, since they are elements of a finite dimensional space and form a bounded set. The uniqueness of the best approximation assures the convergence $v_k(s)$ to the element of best approximation on Z.

It remained to prove the validity of (13) with ε independent of k.

If such an ε does not exist, one can select a subsequence k' of indices k, for which the following relations hold:

$$d_{j^*}^{(k')} \to 0 \quad \text{for a fixed} \quad j^*, \tag{15}$$

$$x_j^{(k')} \to \xi_j \quad (\in Z) \quad \text{for all } j, \quad j = 1, 2, \ldots, n+1. \tag{16}$$

Let $\tilde{u}(x)$ be the element of U_n, which interpolates $f(x)$ in the points ξ_j, except ξ_{j*}

(17) $$\tilde{u}(\xi_j) = f(\xi_j), \quad j = 1, 2, \ldots, n+1, \quad j \neq j^*.$$

Such an element exists because of the Haar condition. Taking into account (9), (3), (15)–(17) and the continuity of f and $\tilde{u}$, we get

$$E_n(f, P_{k'}) = \left| \sum_{j=1}^{n+1} d_j^{(k')} f(x_j) \right| =$$

$$= \left| \sum_{j=1}^{n+1} d_j^{(k')} \left(f(x_j) - \tilde{u}(x_j) \right) \right| \to 0 \quad \text{for} \quad k' \to \infty ,$$

But this contradicts to (11).

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