# ON THE LEIBNIZIAN QUADRATURE OF THE CIRCLE

By

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The purpose of this paper is to present the original — and probably less known — Leibnizian deduction of the so-called Leibniz-series $\left(\frac{\pi}{4}=\right) 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$ (mostly on the basis [12], [13] and [15], [16], [17], [18]).

The efforts to calculate $\pi$[1] as exactly as possible and the history of mathematics both are almost of the same age[2]. We have chosen the Leibniz' arithmetical quadrature of the circle[3] from the manifold intellectual production accumulated in this field for centuries not only because of its characteristically individual and interesting order of ideas but also thus we can get an insight into the important process of an outstanding period (i.e. the late

[1] Leibniz uses the term "circumferencia (circuli), si diameter circuli sit 1" (= "circumference of a circle, of which the diameter is equal to unity") instead of the symbol $\pi$ ([10], [13], [15], [17]). The $\pi$ (in the sence just quoted) first appeared in print at the beginning of the eighteenth century ([33], p. 243.), but the wide-spread application of it is due to the influence of Euler's works ([42], p. 41.; [48], p. 484.; [49]. p. 347.).

[2] For the details see [35], [42], [44], [50]. Leibniz deduced the infinite series mentioned above in 1673, but it were the Hindu mathematicians, who first obtained it before 1550 ([45], [46], [47]). Similar results were achieved (before Leibniz, though by a methodologically different route from his) by J. Gregory (1638–1675) (in [22], which was first published in print in [34] p. 25.), Newton (1642–1727) ([30]) and Brouncker (1620–1684) ([25]), too. The latter transformed Wallis' infinite product for $\frac{4}{\pi}$ ([25], pp. 181–182) into continued fraction. Euler showed (in [39]) that Brouncker came very close to Leibniz' result because the reciprocal partial fractions of the continued fraction of Brouncker turn out to be the partial sums of the Leibniz-series (see further [36], [37], [38], [40], [41]).

[3] Leibniz remarked: "J'appelle cette Quadrature, A r i t h m e t i q u e, par ce qu'elle exprime e x a c t e m e n t la granduer du Cercle, et de ses portions par un rang infini de nombres rationaux ou commensurables a une grandeur donnée." ([15], p. 338.) (See also [4], [14], [16], [17].)

seventeenth century) in the history of mathematics in the course of which the handling of the mathematical problems developed from the older geometrical methods to the (Leibnizian) analitical solution of them.

Leibniz found the quadrature of the circle "in arithmetical way" in the summer of 1673 ([1], [2])[4]. He sent the first detailed description of his result to Huygens in October 1674 ([12], [13]).

Leibniz' procedure in question can be summarized as follows.

**1.** Leibniz defined the auxiliary curve AEZUF (fig. 1).

Let $O$ be the centrum of circle with radius $AO = a$. Let the tangentes of this circle at $A$ and $B$ meet at $P$. In the fig. 1. $BD \perp AD$, $LQ \| NR \| BD$, $KL \| AO$, $HG \perp AH \perp OU$, $TV \perp AV$, $VZ \perp AV$, $CZ \perp AC$ and $BS \perp AP$. Let us introduce the following notations: $x = AD$, $y = AP = BP = DE$, $z = BD$, $\Psi = AT$, $c = AV$, $b = VZ$, $v = TV$, $M =$ the moment of arc $AT$ about the $AP$-axis, $dx = KL$, $dy =$ the infinitely small element of the interval $AC$[5], $dz = NK$, $d\Psi = NL$.

Moreover, $T_1 =$ the (shaded) area $AVZA$[6],

$T_2 =$ the area of the (shaded) circular segment $ATA$,
$T_3 =$ the area $AZCA$,
$T_4 =$ the area of the circular sector $AOTA$.

If we perform the construction shown in fig. 1. (i.e. the construction of the point $E$) in every point of the semi-circle $ABH$, than the obtained points define a locus, a new curve $AEZUF$ – the so-called "figura segmentorum" or "figura anonyma"[7] – with equation[8]

[4] Leibniz wrote: "Est autem magni momenti haec Circuli reductio ad Rationalitatem, qua nemo quicquam maius ad circuli dimensionem praestitit." ("It is of great importance to reduce the quadrature of the circle to defining it by rational way. This result is the most considerable progress in this field.") ([3], p. 245$^v$) Moreover, Leibniz ascertained: "Quadratura Circuli Arithmetica a nemine ante me data est ..." ("It was I who first found the quadrature of the circle by arithmetical way ...") ([4], p. 7$^v$).
In December of 1673 Leibniz met Huygens and – without giving the details – told him about this invention ([43], pp. 688–689.; [51], p. 63.; [52], p. 62.).
In his correspondence Leibniz first alluded explicitly his quadrature of circle only in summer of 1674 ([10], p. 117.; [11], p. 120.).
The circle-quadrature of Leibniz appeared in print for the first time in 1682 (see [23]).

[5] The rectangular triangle $KLN$ with infinitely small (that is negligible with respect to finite quantities, but unequal to zero) sides was called by Leibniz "triangulum characteristicum" ("characteristic tringle") ([20], p. 217.; [24], pp. 399–400.; i.e. "triangle caracteristique", [21], p. 259.). Leibniz only introduced the symbols $dx$, $dy$ and $dz$ for the sides of the triangle in question later (see [9]). During 1673–75, under the influence of Pascal's and Wallis' works, Leibniz applied the so-called „infinitesimal unity", therefore he omitted it, when it acted as a multiplier ([4], [9]).

[6] Leibniz did not use indices.

[7] J. Gregory had already introduced this curve ([29], pp. 23–24.), but Leibniz did not noticed it ([51], p. 69.). (Leibniz bought [29], when he was at London for the first time in the spring of 1673, see [51], p. 75.) You can find the "figura segmentorum" (before Leibniz) at P. Fermat (1601–1665), too, but Leibniz could not know this fact in 1674, because the Fermat's paper (i.e. [26]) was in deposit at Carcavy (1600–1684) and it was published only in 1679 for the first time (in [32], pp. 44–57.).

[8] On the basis the equalities $\frac{y}{a} = \frac{x}{z}$ and $z^2 = x(2a-x)$.

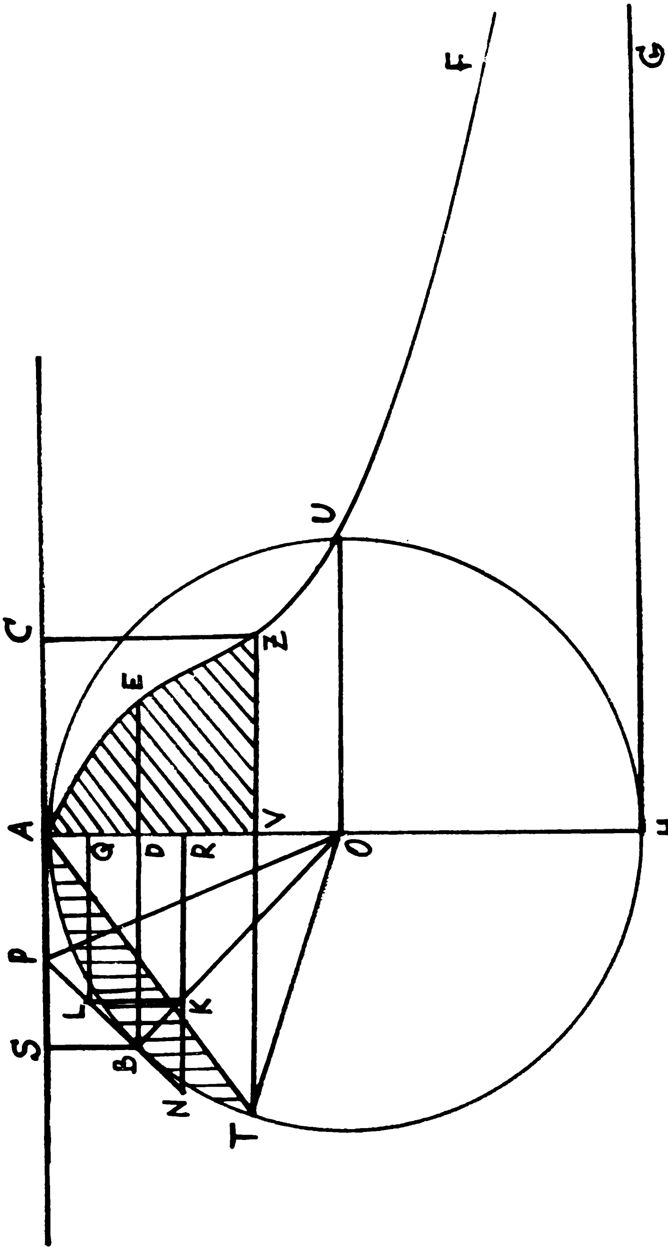


*Fig. 1.*

$$x = \frac{2ay^2}{a^2 + y^2}. \qquad (a)$$

**2.** Leibniz proved that $M = T_1$.

Proof: The moment $M$ is equal to the sum of the elementary moments $(m =) NL \cdot x$[9]. But, since

$$\frac{NL}{KL} = \frac{a}{BD},$$

therefore

$$NL \cdot x = \frac{ax}{BD} \cdot KL,$$

further considering

$$y = \frac{ax}{BD},$$

we have $M =$ summa omnium $NL \cdot x =$ summa omnium $\frac{ax}{BD} \cdot KL =$ summa omnium $y \cdot KL = T_1$.[10]

**3.** Proposition: $M = 2T_2$.

Proof: From the equality $\frac{NL}{NK} = \frac{a}{a-x}$ we have $(m =) NL \cdot x = a(NL - NK)$, so that

$$M = \text{summa omnium } m = \text{summa omnium } NL \cdot x =$$
$$= \text{summa omnium } a \cdot NL - \text{summa omnium } a \cdot NK = a\Psi - av = a(\Psi - v).$$[11]

This completes the proof of the proposition in question because the area $T_2$ is equal to $\frac{1}{2} a(\Psi - v)$.

**4.** From the latter two propositions it follows that $T_1 = 2T_2$.[12]

**5.** Leibniz calculates the arae $T_3$ that $T_1$ and (finally) $T_2$ are to be known.

[9] In Leibnizian term: $(M =)$ summa omnium $NL \cdot x$.

[10] In modern form: $M = \int_0^\Psi x \, d\Psi = \int_0^c y \, dx = T_1$.

[11] In modern style: $M = \int_0^\Psi x \, d\Psi = \int_0^\Psi a \, d\Psi - \int_0^v a \, dz = a(\Psi - v)$.

[12] This proposition is one of the most useful special cases of Leibniz' notable "transmutation theorem", see later.

$T_3 =$ summa omnium $x\,dy$. But $x = \dfrac{2ay^2}{a^2+y^2}$ (see (a)) and

$$\frac{y^2}{a^2+y^2} = \frac{y^2}{a^2+y^2}\cdot\frac{a^2-y^2}{a^2-y^2} = \frac{(ay)^2}{a^4-y^4} - \frac{y^4}{a^4-y^4} =$$

$$= \left(\frac{y}{a}\right)^2\cdot\frac{1}{1-\left(\frac{y}{a}\right)^4} - \left(\frac{y}{a}\right)^4\cdot\frac{1}{1-\left(\frac{y}{a}\right)^4} =$$

$$= \left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)^4 + \left(\frac{y}{a}\right)^6 - \left(\frac{y}{a}\right)^8 + \ldots,$$

so that

$$T_3 = \text{summa omnium } \frac{2ay^2}{a^2+y^2}dy = 2\left(\frac{b^3}{3a} - \frac{b^5}{5a^3} - \frac{b^7}{7a^5} - \ldots\right).$$[13]

Thus, we have $T_2 = \dfrac{1}{2}T_1 = \dfrac{1}{2}(bc - T_3)$.

**6.** Lastly, $T_4 = T_2 + \dfrac{av}{2} = \dfrac{bc+av}{2} - \dfrac{T_3}{2}$.

But, in our case: $bc + av = 2ab$,[14] so

$$T_4 = ab - \frac{b^3}{3a} + \frac{b^5}{5a^3} - \frac{b^7}{7a^5} + \ldots . \qquad (b)$$

**7.** In the special case $a = b = 1$, we get from (b) the area $\dfrac{\pi}{4}$ of the quarter circle with unity radius:

$$\left(\frac{\pi}{4} =\right) 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots .$$

* * *

It is remarkable that after 1674 Leibniz proved the equality $T_1 = 2T_2$ using no more the moments (as we have shown), but he considered it ([15], [16], [17], [18], [19] and [24]) as an immediate consequence of his so-called "transmutation theorem".

Leibniz obtained this theorem as follows:

Let an arbitrary (smooth convex) curve $AT$ be given (fig. 2.). Now we are going to find the area of the curve-segment $ATA$.

[13] Leibniz performed the long-division and the integration term by term on the basis of Mercator's and Wallis' works ([27], [25]).

[14] It follows from the formulae of the remark (8), if $x = c$, $y = b$, $z = v$.

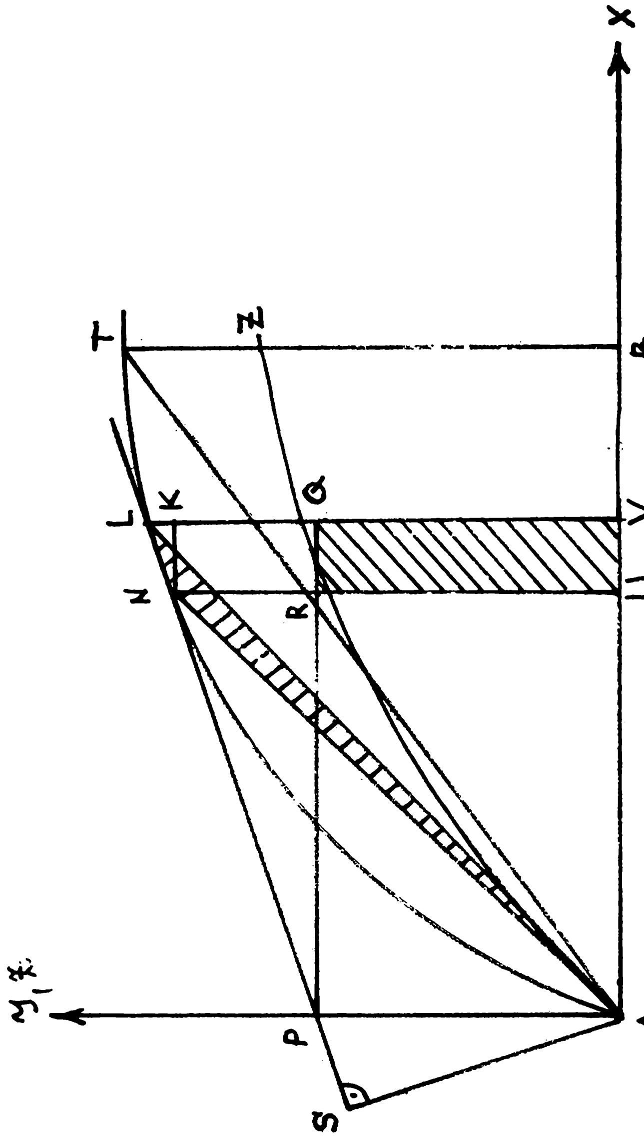


*Fig. 2.*

For this purpose let us divide the given arc $AT$ into infinitely small parts (whose number is infinite). Let us join the points of this partition to the origin $A$, at the same time through them draw a set of parallels to the $z$-axis to their intersection with the $x$-axis.

Let now $N$, $L$ be two neighbouring points on the arc $AT$. Let the straight line $NL$ (that is the one of the tangents[15] of the arc $AT$) meet the $z$-axis at $P$. Let the parallel to the $x$-axis through $P$ meet $NU$ in $R$, $LV$ in $Q$.

From our construction, it follows that the area of the (shaded) rectangle $RQUV$ is equal to the double-area of the (shaded) triangle $ANL$.[16]

If we perform the construction just mentioned in every tangent of the arc $AT$, then the corresponding points $R$ (or – because of the partition of the arc $AT$ into infinitely small parts – points $Q$) define a new curve $AZ$. One half of the area under the arc $AZ$ of this curve is equal to the area of the segment $ATA$. This statement is the famous transmutation theorem (of Leibniz).[17]

The above equality $2T_2 = T_1$ is one of the special cases of this theorem.

The transmutation theorem was the first truly independent achievement of Leibniz.[18] Leibniz was stimulated by the efficiency of this theorem to solve further quadrature-problems. These investigations of his considerably contributed to invent convenient symbols for infinitesimals ([7], [9]).

At the same time, the experience gained by him in calculations with moments led him to discover the fundamental, general rules of the (Leibnizian) infinitesimal calculus ([5], [6], [7], [8]).

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[15] This tangent-concept of Leibniz see e.g. [4], pp. $1^r$, $2^v$.

[16] Because of the similarity of the triangles $NLK$, $APS$.

[17] Leibniz discovered this theorem is 1673 ([1], [2]) inspired by the works of Pascal and Desargues (1593–1662) (see [15], p. 342.; [17], p. 358.).

[18] But, in a slightly different way, it had occured in the works of Gregory and Barrow (1630–1677) ([28], [31]), too, though Leibniz studied the details (in question) of their works only later. (See [51], pp. 69, 76.)

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