AN OPTIMAL SEQUENTIAL ALGORITHM FOR THE UNIFORM APPROXIMATION OF CONVEX FUNCTIONS ON $[0, 1]^2$

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(Received February 16, 1982)

Introduction

Motivations and definitions. In applications (geology, microscopy, engineering design, construction of graphic displays, image transmission (coding) systems, and in mathematical (linear and non-linear) programming, optimal control theory, differential games) the following problem often arises: an unknown, real valued function x is to be uniformly approximated over a domain, G_0, say $G_0 = [0, 1]^p$, based on N (point) evaluations

$$x(t_i) = c_i, \quad i = 1, \ldots, N, \quad t_i \in G_0 \tag{1}$$

– (for other measurement specifications see below, section 4) –, and naturally, based on some a priori knowledge about x.

The latter is usually expressed by $x \in K^0$, where K^0 is a compact subset of the Banach space of continuous functions over G_0, $C(G_0)$.

An N step algorithm of approximation consists in (1): Specifying the choice $t^N = (t_1, \ldots, t_N)$ which might be passive (i.e. simultaneous) or *sequential*, i.e. given by sequence of functions $A^N = (A_1, \ldots, A_N)$ and the rule

$$t_1 := A_N(K^0), \quad c_1 := x(t_1), \tag{2}$$

$$t_i := A_{N-i+1}(t_j, c_j, j = 1, \ldots, i-1, K^0), \quad i = 1, \ldots, N,$$

and (2): in giving a pointwise approximation, a, of x based on the information collected, (i.e. on t^N, c^N and K^0), $c_N = (c_1, \ldots, c_N)$

$$a(t, t_i, c_i, \quad i = 1, \ldots, N, K^0), \quad t \in G_0. \tag{3}$$

Often the main cost of the algorithm is measured by N, i.e. complicated functions (computations), (A^N, a) are allowed in order to achieve a small *global error*

$$e(A^N, K^0) := \sup \{e(A^N, x) | x \in K^0\}, \quad e(A^N, x) := e(t^N(A^N, x) x) \tag{4}$$

$$e(t^N, x) := \sup \{|x(t) - a(t, t_i, x(t_i), \quad i = 1, \ldots, N, K^0)| \; | t \in G\},$$

where $t^N(A^N, x) = \{t_i(A^N, x), i = 1, \ldots, N\}$, are defined by (2).

The global error $e(t^N, K^0)$ of a *passive algorithm* $A^N = t^N$ is defined similarly. Naturally in the construction of the algorithm, especially if we are interested only in the order of magnitude of $e(A^N, K^0)$ with respect to N, N is regarded as a free parameter, and of special interest are those "algorithms in which $A^{N_{k+1}}$ may be regarded as a continuation" of A^{N_k} (that is $t^{N_k}(A^{N_k}, x) \subseteq t^{N_{k+1}}(A^{N_{k+1}}, x)$, for all x) at least for a subsequence $N_k \to \infty$. For a more detailed and general definition of sequential approximation problems in B-spaces, see [2], [4].

If the set K^0 is convex, and has a centre (i.e. is centralsymmetrical, or balanced), then $e_s(N, K^0) := \inf_{A^N} e(A^N, K^0) = \inf_{t^N} e(t^N, K^0) := e_p(N, K^0)$, see [1], [2]. Here we show that for natural classes K^0 of *convex* functions (which are far from being centralsymmetrical, but rather are like simplices, i.e. in linear one to one correspondence to the class of positive *measures*, via solution of the Laplace (or Amper–Monge) equations, see [9]) the use of sequential N-step algorithms allows to achieve *significantly* (with respect to *order in N*) *less global error* than the use of passive ones, see Theorem 1 below.

Here we assume that G_0 is the "interval" $[0, 1]^p$, and

$$(5) \qquad K^0 = \{x \mid x \text{ convex in } t,\ x(t_i^0) = c_i^0,\ i = 0, 1, \ldots, 2^p, \text{ fixed}\} = K^0(x, G_0),$$

where the points t_i^0 are the centre and the vertices of $[0, 1]^p$.

In fact, as a corollary, a good (optimal order) algorithm results, see Remark 1 below, for the more general classes

$$(6) \quad K^0(m, M) = \{x \mid \text{Spectrum } D_2 x \subseteq [m, M],\ x(t_i^0) = c_i^0,\ i = 0, \ldots, 2^p \text{ fixed}\}$$

where $D_2 x(t)$ denotes the Hesse matrix (of second derivatives). (5) is the special case $m = 0$, $M = \infty$, and in (6) $K^0(m, M)$ is in fact understood as the closure in $C(G)$ of the C^2 functions with everywhere defined $D_2 x$; the corrollary coming from the the observation that if z belongs to $K^0(m, M)$, then $x_1 = z - \frac{1}{2} m \|t\|^2$ and $x_2 = \frac{1}{2} M \|t\|^2 - z$, both belong to $K^0(0, \infty)$, (when not stated otherwise we use the usual notations $\|\cdot\|$ and $\langle u, v \rangle$ for euclidean norm and scalar product in R^p).

Let us note that a similar approximation problem for convex bodies x of R^{p+1} – (say for the class, K^0, of those lying in the unit ball, B_1) – is reduced to the above one – where each measurement gives the value, $c_i = m(d_i, x)$ of the support function $m(d, x) = \sup\{\langle d, v \rangle \mid v \in x\}$, $d \in R^{p+1}$ in some direction d_i, and the Hausdorff distance $d(x_1, x_2)$ is used. The latter is identical to the uniform norm in $d \in S_1(R^{p+1}) =$ the unit sphere. An equivalent norm arises if d is restricted to the surface of the unit cube, and then we have our problem (1) – (5) because $m(d, x)$ is convex in d, see [2], [5]; $\|\text{grad}_d\, m(d, x)\| \leq 1$, $d \in S_1$, $x \subseteq B_1$.

– In order to have a compact set K^0 in $C(G)$ we should have assumed (because otherwise there are discontinuous functions in K^0), that in (5)

$$(7) \qquad \|\text{grad}\, x(t)\| \leq g_0, \text{ for } t \in G_0,$$

for some constant, g_0, where this, restricted set will be denoted by $K^0(g_0)$. For reasons which will be clear later [:we do not need any information about g_0 in constructing (implementing) the proposed algorithm], we do not include (7) into (5), of course the finiteness of the functional $D(x, G)$ – see below (16), which implies the optimal order $\left(-\frac{2}{p}\right)$ of the error – is guaranteed a priori if (7) is assumed. Of course, by the convexity in t, the functions x in K_0 are continuous and uniformly bounded on any compact subset of the open domain $(0, 1)^p$.

– Instead of computing the optimal function, a, [defined by putting in (4) an infimum, with respect to $a(.)$, before the supremum] we shall – for computational simplicity and without loss (what regards to the order in N of the global error) – define the function a, used in (4), "locally": over the "subintervals" G_i^N, $i = 1, 2, \ldots$, into which G is decomposed (see below) to be the analogons, $a(G_i^N, x)$, of the function $a(G_0, x) := a(K^0, .)$ defined by – for $K^0 = K^0(x, G_0)$ –

$$a(K^0, t) := \sup\{x(t) \mid x \in K^0\}, \quad t \in G_0. \tag{8}$$

Of course $a(K^0, t)$, (thus each of $a(G_i^N, x)$) is a convex, piecewise linear function over G_0, (resp. G_i^N), yet, in general the function a thus constructed will not be convex in t, i.e. coincide with the function

$$a_s(t, t^N, c^N) := \sup\{x(t) \mid x \in K^0,\ x(t_i) = c_i,\ i = 1, \ldots, N\}.$$

Note that for $p = 2$ the function $a(K^0, t)$ is composed from, at most, *4* linear pieces. The function $e(t^N, x)$, from (4) is thus giving the diameter of the set of localization, $l(t^N, x)$, in the space $L_\infty(G_0)$.

Many of the constructions, and methods of proof, which shall be presented below, are valid for arbitrary values of the dimension, p. The solution for the case $p = 1$, which is by no means trivial, has been given in earlier works, [5], [8], and can be easily obtained (reconstructed) using the methods (hints) given below. Here we solve the case $p = 2$ completely, (and also indicate those points, whose generalization, for $p \geq 3$ is not yet obtained, in section 4).

Proposition 1. For the class (5) – even when (7) is assumed – there exist constants $k_i(K^0, g_0)$, $i = 0, 1$, such that

$$e_p(N, K^0) \geq k_0 N^{-\frac{1}{p}} \quad \text{and} \quad e_s(N, K^0) \geq k_1 N^{-\frac{2}{p}}$$

where k_i are positive if K^0 is non-trivial. ⊠

– The class K^0 is non-trivial if there exist two different functions x_1 and x_2 in it. The second inequality is then the consequence of the asymptotics (12), see [10], because one can construct a function, x, which is strongly convex and C^2 (has constant, positive definite Hesse matrix) over a subdomain of G_0. To construct x let $\max(x_2 - x_1)$ be realized at a point, t, which is in-

terior to G_1 – (by the existence of a gradient bound (7) –, $G_1 = \{t \mid x_2(t) > x_1(t)\}$).

The support hyperplane to the epigraph of x_2 at $(t, x_2(t))$ is over x_1 in a convex subdomain, G_2, of G_1 and a quadratic function, z, is chosen so that

$$z(t) = \frac{1}{2}(x_1(t) + x_2(t)), \quad \operatorname{grad} z(t) = \operatorname{grad} x_2(t)(1-\varepsilon),$$

$$D_2 z(t) \equiv d_2 I, \quad d_2 > 0,$$

ε, d_2 being so small that, on the boundary of G_2, z be smaller than x_1, then $x := \max(z, x_1)$ is convex on G_2 and quadratic over a subdomain, G_3, of G_2. To prove the first inequality – for simplicity for the case $p = 2$ – we first choose in G_3 a square $T = (P_1, P_2, P_3, P_4)$, so that

$$z(P_1) = z(P_2), \quad z(P_3) = z(P_4)$$

which is always possible be turning the original T with some angle about its centre. Important is only that the points $(P_i, z(P_i))$, $i = 1, \ldots, 4$, lie in the same plane. Let w be the maximum of convex functions over G_3 which takes at the vertices $P_1, \ldots, P_n$ and at its centre P_0 the same values as z, i.e. w is linear over the four triangles $[P_0, P_i, P_{i+1}]$, and we can extend w beyond G_3 to the whole G_0, yielding in K^0, a function $\bar{w}$. If N is large then there exists in one of the 4 triangles, formed by the centre and a side of T, a square, U, similar and similarly posed to T, with sidelength not smaller than $r_1 N^{-1/2}$, where r_1 depends only on $K^0(g_0)$, inside of which there are no points from t^N. Let $U = (U_1 U_2 U_3 U_4)$, where $U_1 U_2$ is parallel and nearer to $\overline{P_1 P_2}$ than to $\overline{P_3 P_4}$. Let x_3 to be the maximum of convex functions (defined over G_3) which take the same values at the points P_1, P_2, P_3, P_4, Q, where $P_i - Q = \alpha(U_i - Q)$, as the functions z, z, z, z, w (resp.); and x_4 to be the maximum of convex functions which take the values of x_3 at $P_1, P_2, U_1, U_2, U_3, U_4, P_3, P_4$; continue both of then to be equal to $\bar{w}$ outside G_3. The estimation $\|x_3 - x_4\| \geq r_1 N^{-1/2} r_2$, r_2 depending on d_2 and G_3 – (which corresponds to the one dimensional case of the construction) – is easily proved. For $p > 2$ a similar construction can be given, one better uses simplices (instead of cubes) in th role of the sets G_0, G_3, T.

Thus an algorithm A^N, or more precisely, a sequence of algorithms $\bar{A}^\infty = (\bar{A}^1, \bar{A}^2, \ldots)$ is rightly said to be optimal with respect to order if; for $x \in K^0(g_0)$

$$(10) \qquad e(\bar{A}^N, x) \leq k_2(x) N^{-\frac{2}{p}}, \quad \text{with} \quad k_2(x) \leq k_2(K^0(g_0), \bar{A}^\infty)$$

where the constant, $k_2(K^0(g_0), \bar{A}^\infty)$, is independent of N. In Theorem 1 see (19) we prove, that for $p = 2$, a simple algorithm is optimal (modulo a small factor $\log N$).

The notion of individually best N point approximation, $a(\bar{t}^N, x)$, is defined by the extremal problem for given (known) x

$$(11) \quad e(N, x) := \inf_{t^N} e(t^N, x) = e(\bar{t}^N, x) \geq \|x - a_s(\bar{t}^N, x(t^N), K^0)\|_{C(G_0)}.$$

For the analogous, essentially equivalent problem of approximation of strongly convex, C^2 smooth bodies by polyhedrons in the Hausdorff metrics the asymptotics of $e(N, x)$ for $N \to \infty$, and arbitrary p has been obtained in [10], the result of which implies that

$$e(N, x) = r_p N^{-\frac{2}{p}} \left(\int_G \sqrt{\det D_2 x(t)}\, dt \right)^{\frac{2}{p}} + o\left(N^{-\frac{2}{p}}\right), \tag{12}$$

where the constant r_p depends only on p. Roughly speaking, the best system of N nodes, for N large, must be such that their linear density, at a point t, along the principal directions (eigenvectors) of $D_2 x(t)$ be proportional to the square root of the corresponding eigenvalues.

Our aim is to get exact upper bounds for finite N, and at present we do not know whether for the multidimensional case, $p \geq 2$, is it possible to construct an algorithm satisfying (10) and yielding for each x, with continuous and positive definit second derivative, asymptotically optimal nodes (for $p = 1$ this is true).

The nodes which are yielded by the algorithm 1 below are not such, however they are chosen according to *the principle of equal local errors*, see [1], which is derived from an important property of the optimal set of nodes.

§ 2. Description of the algorithm

– In order to describe the algorithm 1 (for simplicity, for the case $p = 2$), we need the following preparation. Let x be an arbitrary element of K^0 and let us denote by $K(G, x)$ the set of convex functions, z, defined over on interval $G = [a_1, b_1] \times [a_2, b_2]$, of length $l(G) = b_1 - a_1 = b_2 - a_2$, for which the values $z(S_i^G) = x(S_i^G)$, $i = 1, 2, 3, 4, 5$, are fixed, where S_i^G are the centre, $\frac{1}{2}(a_1 + a_2, b_1 + b_2)$ and vertices of G. Thus $K^0 = K(G_0, x)$. The *diameter*, $d(G, x)$, of $K(G, x)$ in $C(G)$, (more precisely in $L^\infty(G)$) is easily computed as the *maximum* of distances of the value of x at a vertice of G from the value of the *linear* function (at this vertice) which is determined by the values of x at the centre and at the opposite vertice of G. – By a (regular) subdivision of G – for given x – we mean the generation of four "standard" sets $K(G_j, x)$, $j = 1, 2, 3, 4$ from the set $K(G, x)$, where each of G_j is an interval of half size, such that

$$\bigcup_{j=1}^{4} G_j = G \quad \lambda(G_j \cap G_i) = 0 \quad \text{for} \quad i \neq j. \tag{13}$$

(λ is the Lebesque measure). Let us define the indices 1, 2, 3, 4 in such a way, that their growing order corresponds to the clockwise movement round G, G_1 being the down left, G_3 the upper right one. These sets are thus defined if we compute x – in addition to the five already fixed values of x in $K(G, x)$, in 8 new points.

Algorithm 1. Choose a (small) positive number ε. Define by sequential subdivisions a sequence of points, $t_1^n, t_2^n, \ldots, t_{sn+5}^n$, and interval subdivisions $\{G_0^n, \ldots, G_{3n}^n\}$ inductively as follows. Let $G_0^0 = G_0 = [0,1]^2$, $t_1^0, \ldots, t_5^0$ given as in the definition of K^0. Suppose that $G_0^n, \ldots, G_{3n}^n$ are already defined and such that

$$\bigcup_{i=0}^{3n} G_i^n = G_0, \quad \lambda(G_i^n \cap G_j^n) = 0, \quad \text{if} \quad i \neq j. \tag{14}$$

We order the lower indices, i, in the following way: say that G_i^n is before G_j^n, if for an interval, see (13), $G = G_k^m$, $m \leq n$, $G_i^n \subseteq G_\alpha$ and $G_j^n = G_\beta$ holds with $1 \leq \alpha < \beta \leq 4$. Now let $i = i(n)$ be the smallest, in this sense, index such that the uncertainty in the values of x over G_i^n, as computed from the five measurements [points $(s_j, x(s_j))$, $s_j \in G_i^n$, i.e. $d(G_i^n, x)$] is greater than ε. [In fact, as it will be clear below, we could use the following criterion as well (replacing $d(G, x)$ by $\Delta(x, G)$)

$$\Delta(x, G) := x(s_1^G) + x(s_2^G) + x(s_3^G) + x(s_4^G) - 4x(s_0^G) > \varepsilon, \quad \text{for} \quad G = G_{i(n)}^n]. \tag{15}$$

We define now the next new, 8 measurement points, t_{sn+5+j}, $j = 1, \ldots, 8$ as those eight which are needed in the subdivision of $G_{i(n)}^n$. Thus we have defined the new points t_j^{n+1}, $j = 1, \ldots, 8(n+1)+5$ and the new subdivision (14), G_j^{n+1}, $j = 0, \ldots, 3(n+1)$.

The algorithm terminates at some (mega) step $N = N(\varepsilon, x, G_0)$, i.e. after $8N$ function evaluations, when there exists no value, $i(N)$, satisfying (15), then obviously x is approximated by the locally constructed functions, see (8), $a(K(G_j^N), t)$, over each G_j^N, within accuracy ε.

§ 3. Error estimation of the algorithm

An important role will be played by the functional

$$D(x, G) = \int_{\partial G} \langle \operatorname{grad} x(u), n(u) \rangle du, \tag{16}$$

defined for an arbitrary interval G in G_0 and function x in K^0, here

$$\langle \operatorname{grad} x(u), n(u) \rangle,$$

where n is the outer normal to G at the boundary, is interpreted as an arbitrary value of the subgradient of the convex function x of t in direction of the outer normal, thus the smallest value is

$$I_2 + I_1 = \int_{a_2}^{b_2} \left[\frac{\partial x}{\partial s_1}\right] ds_2 + \int_{a_1}^{b_1} \left[\frac{\partial x}{\partial s_1}\right] ds_1, \quad \text{where} \quad t = (s_1, s_2)$$

$$\left[\frac{\partial x}{\partial s_1}\right](s_2) = \frac{\partial x(b_1 - 0, s_2)}{\partial s_1} - \frac{\partial x(a_1 + 0, s_2)}{\partial s_1},$$

$$\left[\frac{\partial x}{\partial s_2}\right](s_1) = \frac{\partial x(s_1, b_2 - 0)}{\partial s_2} - \frac{\partial x(s_1, a_2 + 0)}{\partial s_2}.$$

It is well-known that for C^2 smooth functions

$$D(x, G) = \int_G \Delta x(t)\, dt, \quad \Delta = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} \tag{17}$$

is the Laplace operator, (otherwise Δx should be understood in a generalized sense, see e.g. [9]).

In fact, for a function x in $K^0(x, G)$, which might be discontinuous at the boundary of G, the value (16) can be defined (and be finite!) by a limit procedure

$$D(x, G) = \lim_{n \to \infty} D(x_n, G); \quad \text{where} \quad x_n \in C(G), \quad x_n \to x, \quad \text{for} \quad n \to \infty,$$

uniformly on compact subsets of the interior of G. However, without assuming an upper bound, g_0, for the gradients over G_0, $e(A^N, K^0)$ cannot be even made to converge to zero for $N \to \infty$.

Theorem 1. *The number of function evaluations, i.e. steps, $N = N(\varepsilon, x, G)$ necessary to finish the algorithm 1 over G, for an arbitrary convex function, x, with gradients less than g_0, is estimated from above by*

$$N(\varepsilon, x, G) \leq N_0(\varepsilon, x, G) := \left[8 \frac{D(x, G)}{\varepsilon} \log \frac{2^{5/8} l(G) g_0}{\varepsilon} \right]_+ \tag{18}$$

where $[\ ]_+$ denotes the non-negative entire part, log is of base 2. ☒

Corollary 1. Choosing ε to be, for given N, the solution of (18): $N_0(\varepsilon, x, G_0) = N$, we obtain an N-step algorithm, $\bar{A}^N$, whose global error is almost of order $-1 = -\frac{2}{p}$,

$$\left(\text{because of } D(x, G_0) \leq 4 l(G_0) g_0, -D \lg D \leq \frac{1}{2} \right)$$

$$e(\bar{A}^N, K^0) \leq \frac{4}{N} + \frac{32 g_0 l(G_0)}{N} \log 2^{5/8} N l(G_0) g_0 . \tag{19}$$ ☒

Choosing $\varepsilon = \varepsilon_k = 4^{-k} \varepsilon_0$ and performing (with memory) algorithm 1 with $\varepsilon = \varepsilon_k$, $k = 1, 2, \ldots$, we obtain an infinitely continuable algorithm, for which essentially the same inequality (19) holds, at each step N, for the global error. For the proof of the theorem we need the following two lemmas.

Lemma 1. *For arbitrary G and x the following inequalities hold*

$$d(G, x) \leq \Delta(x, G) \leq D(x, G) . \tag{20}$$ ☒

Proof. The first (from the left) inequality was noted already in the description of algorithm 1 [by noting that the maximum of two non-negative

numbers is less than (or equal to) their sum]. The second part can be interpreted as the statement that the simplest discrete approximation of the integral (17) – (which is based on five evaluation of x, to get a discrete approximation for Laplace operator at the centre of G) is always less than the exact value of the integral thus approximated. The idea of the following proof was communicated to the author by T. Fiala.

Let v_i be the unit vector pointing from the centre of G, s_0^G, into the vertice s_i^G, then

$$x(s_i^G) = x(s_0^G) + \sqrt{2} \int_0^c \langle \operatorname{grad} x(s_0^G + t\sqrt{2}\, v_i), v_i \rangle \, dt, \quad c = \frac{l(G)}{2} \tag{21}$$

now because of $\langle \operatorname{grad} x(s), v_i \rangle = \frac{1}{\sqrt{2}} \left(\pm \frac{\partial x(s)}{\partial s_1} \pm \frac{\partial x(s)}{\partial s_2} \right)$ the signs taking all possible combination depending on i, using the *monotonity* of $\frac{\partial x(s)}{\partial s_1}$ $\left(\text{resp. } \frac{\partial x(s)}{\partial s_2} \right)$ in s_1 (resp. s_2) and summing (18) with respect to i, we obtain (20). It is not difficult to construct many functions x, for which

$$2\Delta(x, G) = D(x, G). \quad \boxtimes$$

Lemma 2. *For arbitrary non-negative numbers z_1, z_2, z_3, z_4, L such that*

$$z_1 + z_2 + z_3 + z_4 > 1, \quad \text{and} \quad L \geq 1,$$

$$\sum_i [8 z_i \log L]_+ \leq \left[8 \left(\sum_i z_i \right) \log 2L \right]_+ - 8. \quad \boxtimes \tag{22}$$

Proof. By the simple inequalities $\log L \geq 0$, and $\log 2 = 1$

$$\sum [u_i] \leq [\sum u_i], \quad \text{for} \quad u_i \geq 0. \quad \boxtimes$$

Proof of the Theorem 1. We show, by induction with respect to the value of N_0, that (18) holds. Let us assume first that $N_0(\varepsilon, x, G) = 0$, then because of the fact that

$$l(G) g_0 \frac{1}{\varepsilon} \geq \sqrt{2}^{-1} \tag{23}$$

can be assumed (otherwise the formula (16) shows, by Lemma 1, and a simple inequality that, $d(x, G) < \varepsilon$, thus $N(\varepsilon, x, G) = 0$), we obtain that $D(x, G)$ must be less than ε, thus again by Lemma 1, $d(x, G) < \varepsilon$, i.e. $N(\varepsilon, x, G) = 0$. The function $N(\varepsilon, x, G)$ satisfies – for a subdivision (13) –

$$8 + \sum_{j=1}^{4} N(\varepsilon, x, G_j) \geqq N(\varepsilon, x, G).$$

Thus we have to prove that if $N(\varepsilon, x, G) > 1$, i.e. $d(x, G)$, (or $\Delta(x, G)) > \varepsilon$, then

$$8 + \sum_{j=1}^{4} N_0(\varepsilon, x, G_j) \leq N_0(\varepsilon, x, G). \tag{24}$$

This is by the additivity of the functional $D(x, G)$ with respect to subdivisions, a consequence of Lemma 2: put

$$z_i = D(x, G_i)\,\varepsilon^{-1},\ L = L(G_i) = \frac{g_0}{\varepsilon} \cdot 2^{5/8}\, l(G_i) = \frac{1}{2} L(G). \quad \square$$

Remark 1. For the class $K^0(m, M)$, see (6), with finite m, M the transformations $z \to x_1(z, m)$, $z \to x_2(z, M)$ lead to the following algorithm. Perform simultaneously, for x_1 and x_2 the algorithm 1, as described, replacing the criterion of subdivision (15) by

$$\min(\Delta(x_1, G), \Delta(x_2, G)) > \varepsilon, \quad \text{for} \quad G = G^n_{l(n)}$$

and define the approximation $a(G, z)$, for each $G = G^N_i$, as

$$\frac{1}{2}\left(\frac{1}{2} m \|t\|^2 + a(G, x_1) + \frac{1}{2} M |t|^2 - a(G, x_2)\right).$$

Remark 2. A more simple, infinitely continuable variant of the algorithm 1 is the following. Instead of fixing a number ε, let us perform the sequence of subdivisions according to the following rule: let $i(n)$ be that value of i for which $d(x, G^n_i)$, (or $\Delta(x, G^n_i)$) is maximal. The number of subdivisions (steps), $N_1(\varepsilon, x, G)$, needed to reach a global accuracy ε, can be estimated with the same inductive method, and turns out to be majorated by the same quantity $N_0(\varepsilon, x, G)$.

We conjecture that, as the previous proof indicates, the logarithmic factor cannot be eliminated from the error estimations (18), (19). It is easy to see, that for each, fixed x, in K^0, with continuous and positive definite Hesse matrix, asymptotically, for $N \to \infty$, (see (17))

$$\varrho(t^N, x) = \frac{3}{4N} D(x, G_0) + o(N^{-1}), \quad t^N = t^N(\bar{A}^N, x). \tag{25}$$

Indeed, because of (15), (17), for large N, and continuous Δx

$$\int \Delta x(t)\, dt \cong \left(3\frac{N}{8} + 1\right) 2\varepsilon. \tag{26}$$

It seems true that no (similarly simple) optimal order algorithm (in the global sense, (10)) exists for which the asymptotic error would be expressible with the same main part as in (12); note that

$$\Delta x(t) \geq 2(\det D_2 x(t))^{1/2}, \quad \text{with equality iff} \quad \frac{\partial^2 x}{\partial s_1 \partial s_2} = 0, \tag{27}$$

and

$$\frac{\partial^2 x}{\partial s_1^2} = \frac{\partial^2 x}{\partial s_2^2}.$$

The reasons behind this statement and further motivation concerning our algorithm 1 (its stability properties) will be given in the next section.

§ 4. Generalizations and further comments on the algorithm

The most interesting question is, of course, whether or not the straightforward generalization of algorithm 1 to the $p \geq 3$ dimensional case yields an optimal (with respect to order of the global error) algorithm and if so is that order equal to $\left(-\frac{2}{p}\right)$, the lower bound, given by (12). While the answer to the first question seems to be positive, concerning the second one the following remarks (negative results, resp. conjectures) are stated, which indicate, that the algorithm 1 yields the optimal order $\left(-\frac{2}{p}\right)$ only for such classes of functions x, for which $\|\Delta x\|_{L_{p/2}}(G) < \infty$ uniformly.

A generalization of the inequality (20), i.e. Lemma 1, would be

(28) $d(x, G) \leq \Delta_p(x, G) \leq c_p l^{2-p}(G)\, D(x, G)$, for some constant, c_p

where $D(x, G)$ is given by the obvious generalization of (17) and (16),

$$\Delta_p(x, G) := \sum_1^{2^p} x(P_i^G) - 2^p x(P_0^G).$$

The first part is again easy, what regards to the second part, the exact value of c_1 is $\frac{1}{2}$, (here $d(x, G) = \Delta(x, G)$ and — as noted after (21) — the exact value of c_2 seems to be also $\frac{1}{2}$. Now, in order to prove that the algorithm 1 has an error of optimal order, $\left(-\frac{2}{p}\right)$,

$$N_0(\varepsilon, x, G) := k_p \left(\frac{l^{2-p}(G) \cdot D(x, G)}{\varepsilon}\right)^{\frac{2}{p}} (1 + o(\varepsilon^{-\delta}))$$

$\delta > 0$ the following inequality should hold (generalization of Lemma 2, i.e. of the subadditive property of N_0), (modulo of course some logarithmic factor)

$$D_p(x, G) := \left(l^{2-p}(G)\, D(x, G)\right)^{\frac{p}{2}}$$

(29) $$\sum_{j=1}^{2^p} D_p(x, G_j) \leq D_p(x, G); \quad \text{if} \quad G = \bigcup_{j=1}^{2^p} G_j$$

for regular subdivisions, and all x. Now (29) requires nothing else as the norm inequality, $(a_j = D_p^{2/p}(x, G_j))$, $\|a\|_{L_{p/2}} \leq \|a\|_{L_1}$, which is not true when $p \geq 3$. By Holder'-s inequality $D_p(x, G_j) \leq \|\Delta x\|^{p/2}_{L_{p/2}(G)}$, and the latter functional, $\Delta^{p/2}(x, G)$, is obviously additive in G. The conjecture that the main part of N_0 is $k_p \times \varepsilon^{-p/2} \Delta^{p/2}(x, G)$, specially that $\Delta x \in L_{p/2}(G)$ is needed, is seen from Hadamard'-s inequality $\Delta^{p/2} x(t) \geq l_p \sqrt{\det D_2 x(t)}$ and from the fact that the Green function for the Laplace operator - having the singularity $\left(\frac{1}{|t-s|}\right)^{p-2}$, $p > 2$ - is only in $L_{\frac{p-\delta}{p-2}}$ the dual of $L_{p/2} + \lambda$; $\delta, \lambda > 0$.

Let us see whether there exists an other upper bound – see (27) – for the functional giving the main part of the individually best approximations, (12),

$$A_p(x, G) := \left(\int_G \sqrt{\det D_2(x, t)}\right)^{\frac{p}{2}} \tag{30}$$

which is expressible in terms of (the values and) the gradients of x, at the boundary of G. If these values are fixed along the boundaries of the intervals belonging to subdivision, then the behaviour of the function, i.e. its values become independent for the different subintervals, thus subdivision means decomposition, in a well defined sense. If one applies the Cauchy-Schwarz inequality to the integral (30), then one obtains that

$$A_p(x, G) \leq H_p(x, G) := \left(\text{vol grad } x(G) \cdot \text{vol}(G)\right)^{\frac{1}{p}} \tag{31}$$

where vol is for the Lebesque measure and grad $x(G)$ is the image of the set G in the "monoton" map $t \to \text{grad } x(t)$. Notice that $H_1(x, (a, b)) = (x'(b) - x'(a))(b-a) = D_1(x, G)$. Because of the monotonity of grad x:

$$\text{grad } x(G) = \bigcup_{i=1}^{2^p} \text{grad } x(G_i), \quad \text{vol}\left(\text{grad } x(G_i) \cap \text{grad } x(G_j)\right) = 0, \quad j \neq i,$$

$H_p(x, G)$ satisfies the required subadditive property

$$\sum_{j=1}^{2^p} H_p^{p/2}(x, G_j) \leq H_p^{p/2}(x, G), \tag{32}$$

because of $\|a\|_{L_q} \leq \|a\|_{L_1}$, for $q = \frac{1}{2}$, $a \in R^{2^p}$.

Thus in order to prove

$$N(\varepsilon, x, G) \leq k_p \left(\frac{H_p(x, G)}{\varepsilon}\right)^{\frac{p}{2}}, \tag{33}$$

for some constant k_p for an algorithm using subdivisions for a measurement pattern in which at each step over an interval, G_n we get and use an infor-

mation, I, concerning the values of x over G, $I(x, G_n) = c_n$, about x, and accepting (or subdividing) it, depending on the value of some "barrier" functional $\Delta_p^I(x, G_n) \leq \varepsilon$, (or not), it would be enough to prove an inequality (the analogon of Lemma 1)

$$d_p^I(x, G_n) := d\{z \mid I(z, G_n) = I(x, G_n),\ z \in K^0\} \leq \tag{34}$$

$$\leq q_p \Delta_p^I(x, G_n) \leq r_p H_p(x, G_n)$$

where d stands for the diameter of a set in $L_\infty(G_n)$, q_p, r_p are constants.

– Indeed for $p = 1$ in this way, using at each steps three points

$$\Delta_1^I := \Delta_1(x, a, b) = 2d(x, (a, b)) = x(a) + x(b) - 2x\left(\frac{a+b}{2}\right), \tag{35}$$

one obtains (33) with $k_1 = \frac{1}{2}$; in the inductive proof, (for arbitrary p), $N_0(\cdot)$, is chosen in the form $N_0(\varepsilon, x, G) = \left[u\left(\frac{H_p(x, G)}{\varepsilon}\right)^{p/2} - v\right]_+$ with appropriate positive constants u, v, the constant v is needed (used) in order to compensate for the number of measurements needed in a subdivision and not included into the left side of (32), see (24).

– In the case $p = 2$, the inequality (34) is not true, i.e. cannot be satisfied for the information pattern used in Algorithm 1.

It seems true that the only information pattern, I for which (34) holds is the complete one, I^C, i.e. when – at step n – we measure all the values $x(t)$ and subgradient $\operatorname{grad} x(t)$ along the boundary of some interval G_n then of course, (as e.g. in the case $p = 1$), we could choose Δ_p^I to be d_p^I or H_p as well.

Any discretized version of the latter algorithm, i.e. one in which the subgradients are approximated by finite differences seems – except the case $p = 1$ – to be far from being optimal order with respect to the number of function evaluations, this will be partly explained below when we speak about the instabilities.

In the case $p = 2$, when the information, I^m, used at step n consists of the values of x along the boundary of G_n and along its two middle lines, Δ_2^I taken as d_2^I, (34) is not satisfied, for any r_2, yet the role of the functional H_2 can be taken by

$$F_2(x, G) := (I_1(x, G) \cdot I_2(x, G))^{1/2} \tag{36}$$

where I_1, I_2 are the integrals defined earlier in (16), (17). Concerning this functional, F_2, let us prove that in the case of the complete information pattern, I^C, the inequality (34) with H_2 replaced by F_2 holds, with $d_2^{I^C} = \Delta_2^{I^C}$

and $r_2 = 2$. Indeed for arbitrary x, z, with $I^C(x) = I^C(z)$, $(x, z \in C^2(G))$

$$|x(t) - z(t)| \leq 2 \sup_z \int\int_G \left| \frac{\partial^2 z(t)}{\partial s_1 \partial s_2} \right| dt \leq$$

$$\leq 2 \sup_z \int\int \sqrt{\frac{\partial^2 z}{\partial s_1^2} \frac{\partial^2 z}{\partial s_2^2}}\, dt \leq 2F_2(x, G),$$

by the convexity of z, and then the Cauchy-Schwarz inequality yields the last part by the definition of F_2.

If not the values of the gradients (along the boundary) but instead the values of x along the two middle lines are computed (used), then for the corresponding algorithm as in the proof of Theorem 1 (using the subadditivity of F_2) one can prove that

$$N_2^m(x, \varepsilon, G) \leq \left[w_1 \frac{F_2(x, G)}{\varepsilon} \log \frac{w_2 l(G) g_0}{\varepsilon} \right]_+ . \tag{37}$$

Even this algorithm cannot be discretized (to give an optimal order one), because of the following *instability*. If the values of x are known – along the middle lines and the boundary of x – only within some accuracy ε_0 [e.g. this results if for their approximation the one dimensional algorithm (35) is used], then the perturbation of the value of d_2^m is not bounded by a quantity proportional to ε_0.

– Our Algorithm 1 is *stable* with respect to errors in the measurement of $x(t_i) = c_i$, more precisely: suppose that the values c_i, $i = 1, \ldots, N$ are measured within accuracy ε_0, then using, as in Algorithm 1, five such values, over the current interval, G, the value of the uncertainty in $x(t)$ over G, $d(x, G, \varepsilon_0)$, can be computed easily. Let us subdivide G whether (for a pregiven ε)

$$d(x, G, \varepsilon_0) \leq \varepsilon + 4\varepsilon_0 .$$

For this algorithm, the number of steps $N(\varepsilon, x, G)$ is estimated by the same expression as when $\varepsilon_0 = 0$, thus in the case of measurement errors of magnitude M^{-1}, to obtain over G_0 a global accuracy $5M^{-1}$ (choose $\varepsilon = \varepsilon_0 = M^{-1}$) no more than $k(g_0, K^0) M \log M$ steps are needed, by (19).

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