ON FAVOURABLE STOCHASTIC GAMES

by

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1. Introduction

Problems connected with gambling have been constant sources of inspiration to the development of probability theory from the very beginning. The well-known monograph [2] of Dubins and Savage on inequalities for stochastic processes takes its origin from such problems too, and several authors besides them deal and have dealt with similar topics. Most papers are devoted to fair or unfavourable games, while the study of favourable games was disregarded a good while. In this topic we refer to the works Breiman [1] and Móri and Székely [3].

Let us introduce some basic notions. A *stochastic game* is a game of chance in which the gambler's gain is a random multiple of his stake, i.e. the ratio of the gain to the stake is a random variable not less than -1. This random variable describes the game, its positive values mean winning in the strict sense of the word and its negative values mean loss. The gambler may participate in several games at the same time. Thus the sequence of successive *rounds* can be modelled by a sequence of d-dimensional random vectors

$$X_1, X_2, \ldots, \quad X_n' = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(d)}), \quad X_n^{(i)} \geqq -1 .$$

(By vector a column vector is meant, the corresponding transpose is denoted by dash.)

Denote by T_n the fortune of the gambler after n rounds. Let the initial fortune T_0 be equal to 1. In every round the gambler may decide about the proportion of his fortune to be staked and about the ratio of the betting amounts in various games. To do this he can make use of earlier observations. Thus the sequence of random vectors

$$a_1, a_2, \ldots, \quad a_n' = (a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(d)}),$$

is called a *strategy* if

(i) a_n is measurable with respect to the σ-field $\mathcal{F}_{n-1}$ generated by $X_1, X_2, \ldots, X_{n-1}$,

(ii) a_n is an element of the simplex

$$\Delta = \left\{ a \in \mathbf{R}^d : a^{(i)} \geqq 0 , \ \sum_{i=1}^{d} a^{(i)} \leqq 1 \right\}.$$

With the notations just introduced we have

$$T_n = \prod_{j=1}^{n} (1 + a_j' X_j) .$$

The gambler's aim is to increase his fortune to the highest degree. What is meant by it? If no game is favourable in a round i.e. all the components are of non-positive expectation, then the expected gain is maximized by not gambling at all. If there are favourable ones among the games, then we ought to stake all of our money in the most favourable game (in the game of the highest expectation). In this case

$$E(T_n|\mathcal{F}_{n-1}) = T_{n-1}(1 + \max\{0, E(X_n^{(1)}|\mathcal{F}_{n-1}), \ldots, E(X_n^{(d)}|\mathcal{F}_{n-1})\}) .$$

This strategy maximizes the expected gain in every round, but at the same time it can bring to ruin: $T_n \to 0$ a.e.

It was observed even by D. Bernoulli that the optimal strategy should maximize a logarithmic type expectation. In case of independent, identically distributed rounds the constant strategy $a_n = a^*$ provides the highest rate of growth for the total fortune, where a^* is the maximum point of

$$E\left(\log(1 + a' X_1) - \log\left(1 + \sum_{i=1}^{d} (1 + X_1^{(i)})\right)\right)$$

in the simplex Δ. [The negative term does not play any role but enables to avoid restrictions on the finiteness of $E(\log(1 + X_1^{(i)}))$.] The optimality of this strategy was given a precise mathematical meaning in [1] and [3]. The latter work contains also a characterization for "nearly optimal" strategies.

The aim of the present note is to extend the results of [3] to the case when neither independence nor identical distribution are supposed for the successive rounds. This generalization does have a practical value, since the i.i.d. theory is not applicable to the phenomena of everyday life that can be described in terms of favourable games (Stock Exchange, etc.).

2. Results

Since the rounds are not independent, the previous outcomes can provide informations on the result of the forthcoming round. Hence the optimal strategy is in a real connection with the past, it can't be expected to be constant.

Analogously with the i.i.d. case we give the following definition: let $a = a_n^*(X_1, X_2, \ldots, X_{n-1})$ maximize the function

$$f_n(a, X_1, \ldots, X_{n-1}) = E\left(\log(1 + a' X_n) - \log\left(1 + \sum_{i=1}^{d} (1 + X_n^{(i)})\right) \middle| \mathcal{F}_{n-1}\right)$$

in the simplex Δ. (Use the regular variant of conditional distribution for the expectation.) The uniqueness of a_n^* follows from Lemma 2 of [3] provided the conditional linear independence of the coordinates of the random vector X_n.

Denote by T_n^* the fortune obtained by the strategy $\{a_n^*\}$. Now let $\{a_n\}$ be an arbitrary strategy, we will examine the asymptotic behaviour of the ratio T_n/T_n^*. The optimality of the strategy $\{a_n^*\}$ is indicated by the following

THEOREM 1.

$\lim_{n\to\infty} T_n/T_n^* = T$ exists and it is finite with probability one, further $E(T)\leqq 1$. ⊠

A strategy $\{a_n\}$ is called *quasioptimal* on the event A if $T(\omega)>0$ for almost every $\omega\in A$. We aim at finding conditions that provide the quasi-optimality of $\{a_n\}$. A "good" strategy must not risk going to ruin, hence we can suppose

$$P(a_n' X_n > -1) = 1 .$$

This requirement is quite natural and it is met if

$$\sum_{i=1}^{d} a_n^{(i)} I(P(X_n^{(i)} = -1 \mid \mathcal{F}_{n-1}) > 0) < 1 .$$

(Here and in the sequel $I(.)$ denotes the indicator of the event in the brackets.)

For $z\in\Delta$, $n\in\mathbf{N}$ let us define a stochastic function $W_n(z) = W_n(z, \omega)$ by

$$W_n(z) = \min\left\{1, E\left(\frac{(a_n^* - z)' X_n}{1+z' X_n} \,\middle|\, \mathcal{F}_{n-1}\right)\right\}.$$

As it turns out from the proof of Theorem 1, $W_n(z)\geqq 0$ with probability 1.

THEOREM 2.

The strategy $\{a_n\}$ is quasioptimal on the event $\{\sum_n W_n(a_n)<\infty\}$. ⊠

We remark that in case of independent rounds a_n is constant with probability 1 and $W_n(z)$ is also a deterministic function. In [3] it is proved that in the i.i.d. situation the condition $\sum_n W_n(a_n)<\infty$ is necessary and sufficient for the strategy $\{a_n\}$ to be quasioptimal. The proof given there is not applicable to the general case and concerning the necessarity – although it seems to be valid – we have partial results only.

3. Proofs

PROOF OF THEOREM 1. The proof below is a straight adaptation of the method given in [1] and refined in [3]. We repeat it merely for the sake of completeness.

Denote

$$U_n = \log\left(1 + \sum_{i=1}^{d} (1 + X_n^{(i)})\right).$$

For $0 \leqq t \leqq 1$ we have

$$E\left(\log(1 + a_n^{*\prime} X_n) - U_n \mid \mathcal{F}_{n-1}\right) \geqq E\left(\log\left(1 + (ta_n + (1-t)\, a_n^*)' X_n\right) - U_n \mid \mathcal{F}_{n-1}\right)$$

by the definition of a_n^*. For the expression on the right-hand side the following lower bound can be obtained:

$$\log(1-t) + E\left(\log(1 + a_n^{*\prime} X_n) - U_n \mid \mathcal{F}_{n-1}\right).$$

From these inequalities it follows that

$$0 \geqq E\left(\frac{1}{t} \log\left(1 - t + t \frac{1 + a_n' X_n}{1 + a_n^{*\prime} X_n}\right) \middle| \mathcal{F}_{n-1}\right) \geqq \frac{1}{t} \log(1-t).$$

The expression behind the sign of expectation is a decreasing function of t, hence the conditional version of the monotone convergence theorem implies

$$0 \geqq E\left(\frac{1 + a_n' X_n}{1 + a_n^{*\prime} X_n} \middle| \mathcal{F}_{n-1}\right) - 1$$

as $t \downarrow 0$. Using this we arrive at

$$E\left(\frac{T_n}{T_n^*} \middle| \mathcal{F}_{n-1}\right) = \frac{T_{n-1}}{T_{n-1}^*} E\left(\frac{1 + a_n' X_n}{1 + a_n^{*\prime} X_n} \middle| \mathcal{F}_{n-1}\right) \leqq \frac{T_{n-1}}{T_{n-1}^*} \quad \text{a.s.}.$$

Thus, being a non-negative supermartingale, T_n/T_n^* converges with probability one. Applying the Fatou lemma to the limit T we get

$$E(T) \leqq \lim_{n\to\infty} ET_n/T_n^*) \leqq T_0/T_0^* = 1. \quad \boxtimes$$

PROOF OF THEOREM 2. First we remark that

$$E\left(\frac{(a_n^* - a_n)' X_n}{1 + a_n' X_n} \middle| \mathcal{F}_{n-1}\right) = E\left(\frac{1 + a_n^{*\prime} X_n}{1 + a_n' X_n} - 1 \middle| \mathcal{F}_{n-1}\right) \geqq$$

$$\geqq \left[1 \Big/ E\left(\frac{1 + a_n' X_n}{1 + a_n^{*\prime} X_n} \middle| \mathcal{F}_{n-1}\right)\right] - 1 \geqq 0$$

by the conditional Jensen inequality and by (1), thus $W_n(a_n) \geqq 0$.

As a starting point for our proof we use the identity

$$\frac{T_n^*}{T_n} = \prod_{j=1}^{n} \left(\frac{1 + a_j^{*\prime} X_j}{1 + a_j' X_j}\right) = \prod_{j=1}^{n} \left(1 + \frac{(a_j^* - a_j)' X_j}{1 + a_j' X_j}\right).$$

Taking the logarithm of both sides we obtain

$$\log \frac{T_n^*}{T_n} \leqq \sum_{j=1}^{n} \frac{(a_j^* - a_j)' X_j}{1 + a_j' X_j}. \tag{2}$$

$T>0$ means $\lim_{n\to\infty} \log(T_n^*/T_n)<\infty$, hence it suffices to show that if $\sum_n W_n(a_n)<<\infty$ then the sum on the right-hand side of (2) does not tend to $+\infty$.

Let us introduce the strategy

$$\tilde{a}_n = a_n^* + (a_n - a_n^*)\, I\,(W_n(a_n)<1).$$

Then

$$E\left(\frac{(a_n^*-\tilde{a}_n)'X_n}{1+\tilde{a}_n' X_n}\,\middle|\,\mathcal{F}_{n-1}\right) = I\,(W_n(a_n)<1)\,W_n(a_n) = W_n(\tilde{a}_n). \tag{3}$$

On the event $\sum W_n(a_n)<\infty$ clearly $\tilde{a}_n = a_n$ holds for sufficiently large values of n. Therefore it suffices to show that the sum

$$\sum_n \frac{(a_n^*-\tilde{a}_n)'X_n}{1+a_n' X_n}$$

converges on the event where $\sum W_n(a_n)$ does so. Consider the martingale

$$Z_n = \sum_{j=1}^{n} \frac{(\tilde{a}_j-a_j^*)'X_j}{1+\tilde{a}_j' X_j} + W_j(\tilde{a}_j).$$

Its differences are bounded from above:

$$Z_n - Z_{n-1} = 1 - \frac{1+a_n^{*\prime}X_n}{1+a_n' X_n} + W_n(\tilde{a}_n) \leqq 2.$$

In virtue of [4, Theorem IV. 6.3] Z_n converges a.e. on the event $\{\limsup Z_n<<\infty\}$. Finally, combining (2) with Theorem 1 we obtain

$$\limsup_{n\to\infty} \sum_{j=1}^{n} \frac{(a_j-a_j^*)'X_j}{1+a_j' X_j} < +\infty,$$

hence $\sum_n W_n(a_n)<\infty$ implies $\limsup Z_n<+\infty$, from which the assertion to be proved immediately follows.

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