EVALUATION OF THE PRODUCT FORM IN CERTAIN QUEUEING NETWORKS

By

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Introduction

After some attempts, very different in efficiency and in aims, mean value analysis turned out to be a computing procedure for closed queueing networks with constant intensity of exponentially distributed service times ([4], [7].)

Several authors discussed the evaluation of the product form stationary distribution obtained by Jackson and generalized by others ([3], [2]). Different recursive algorithms are known for this purpose ([1], [5], [6]). In general the main problem arises when determining the normalizing constant providing

$$\sum_{\mathbf{n}} P(\mathbf{n}) = 1 .$$

Since the sum requires an enormous number of terms a closed form is proposed.

§ 1. Product form

In the following the finite sum and different expectations are discussed for probabilities of type

$$\text{(1)} \quad \tilde{P}(n) = \prod_{i=1}^{K} \varrho_i^{n_i}, \quad \text{where } \mathbf{n}=(n_1, \ldots, n_K) \text{ and } |\mathbf{n}| = \sum_{i=1}^{K} n_i = N .$$

(1) occurs when dealing with queueing networks consisting of K $M/M/1$ type systems as nodes. Customers wander among the K systems i.e. after waiting and being served at a certain node, another one is visited, etc. The routes are supposed to be determined by a matrix P whose element p_{ij} is the probability that after node i node j is entered. In this case in (1)

$$\varrho_i = x_i/\mu_i \quad \text{where} \quad x_i = \sum_{j=1}^{K} x_j \cdot p_{ji}$$

and μ_i is the mean service time at node i, x_i is interpreted as the intensity of visiting node i; n_i is the actual number of customers present at node i.

Unfortunately $\tilde{P}(n)$ fails to be a real distribution: $\sum_{|n|=N} \tilde{P}(n) = 1$ is not sure. Thus a normalization constant $\sum_{|n|=N} \tilde{P}(n) = C$ is needed to obtain the distribution

$$P(n) = C^{-1}\tilde{P}(n). \tag{2}$$

One of our aims is to give an explicit form for this constant with an algorithm simpler than summing up all values $\tilde{P}(n)$.

A model of M nodes with N wandering customers is denoted by (M, N) and let

$$C(M, N) = \sum_{i=1}^{M} \frac{\varrho_i^{M+N-1}}{\prod_{j\neq i}(\varrho_i - \varrho_j)}. \tag{3}$$

§ 2. The normalizing constant

One of the main results is formulated as follows.

Theorem 1. *For an (M, N) model with different intensities $(\varrho_i \neq \varrho_j)$, the normalizing constant satisfies*

$$C = \sum \tilde{P}(n) = C(M, N). \boxtimes \tag{4}$$

Proof. The first equation holds by definition. We prove the second one by induction on $M = 2, 3, \ldots$, with arbitrary N. If $M = 2$ then

$$C(2, N) = \sum_{i+j=N} \varrho_1^i \cdot \varrho_2^j = \frac{\varrho_1^{N+1} - \varrho_2^{N+1}}{\varrho_1 - \varrho_2} = \frac{\varrho_1^{N+1}}{\varrho_1 - \varrho_2} + \frac{\varrho_2^{N+1}}{\varrho_2 - \varrho_1}.$$

Supposing the validity for $M-1$ let us consider

$$C(M, N) = \sum_{i=0}^{N} \varrho_M^i \cdot C(M-1, N-i) = \sum_{i=0}^{N} \varrho_M^i \cdot \sum_{j=1}^{M-1} \frac{\varrho_j^{M+N-i-2}}{\prod_{\substack{K=1\\K\neq j}}^{M-1} (\varrho_j - \varrho_K)}.$$

Changing the summation order we have

$$C = \sum_{j=1}^{M-1} \frac{1}{\prod_{\substack{K=1\\K\neq j}}^{M-1} (\varrho_j - \varrho_K)} \cdot \sum_{i=0}^{N} \varrho_M^i \cdot \varrho_j^{M+N-i-2} =$$

$$= \sum_{j=1}^{M-1} \frac{1}{\prod_{\substack{K=1\\K\neq j}}^{M-1} (\varrho_j - \varrho_K)} \cdot \frac{\varrho_j^{M+N-1} - \varrho_M^{M+N-1}}{\varrho_j - \varrho_M}. \tag{5}$$

The first $M-1$ terms of (4) are on the right-hand side of (5) and the M-th term is easy yielded since (cf. [8])

$$\frac{\varrho_M^{M-2}}{\prod_{j\neq M}(\varrho_M-\varrho_j)} = -\sum_{i=1}^{M-1} \frac{\varrho_i^{M-2}}{\prod_{j\neq i}(\varrho_i-\varrho_j)} . \boxtimes$$

Remarks. 1. The value of the constant C, however, is not independent of the choice of the ϱ_i. Namely, the limit

$$C^* = \lim_{N\to\infty} C(M, N)$$

depends on $\varrho = \max \varrho_i$ i.e.

$$C^* = \begin{cases} 0 & \text{if} \quad \varrho < 1 \\ \text{finite} & \text{if} \quad \varrho = 1 \\ \infty & \text{if} \quad \varrho > 1 . \end{cases}$$

Table 1. demonstrates that if $\varrho = 1$, the convergence of C^* is far from being slow. The well-known result

$$C(M, \infty) = \frac{1}{\prod_{j\neq 1}(1-\varrho_j)}$$

is easy obtained from (4).

2. The weakness of the present algorithm seems to be in the differences $\varrho_i - \varrho_j$ when they are close to zero. Fortunately

$$\frac{\varrho_i^{N+M-1}}{\prod_{K\neq i}(\varrho_i-\varrho_K)} + \frac{\varrho_j^{M+N-1}}{\prod_{K\neq j}(\varrho_j-\varrho_K)} = \frac{1}{\varrho_i-\varrho_j} \cdot \frac{\varrho_i^{N+M-1}\cdot P_j - \varrho_j^{N+M-1}\cdot P_i}{P_j \cdot P_i}$$

where

$$P_l = \prod_{\substack{K\neq i\\ K\neq j}} (\varrho_l-\varrho_K) \quad (l = i, j)$$

and in this case the numerator is "close to zero", too. Some numerical examples shed light on this fact.

Five models are evaluated with $\varrho_1 = 1$, $\varrho_2 = .8$, $\varrho_3 = .5$ and with $\varrho_4 = .2+10^{-L}$, $\varrho_5 = .2-10^{-L}$.

Table 1

	$L=1$	$L=2$	$L=3$	$L=4$	$L=\infty$
(5,1)	2.7	2.7	2.7	2.7	2.7
(5,2)	4.6301	4.63	4.63	4.63	4.63
(5,5)	9.5566	9.5557	9.5557	9.5557	9.5557
(5,10)	13.595	13.593	13.593	13.593	13.593
(5,20)	15.408	15.406	15.406	15.406	15.406
(5,100)	15.627	15.625	15.625	15.625	15.625
(5,1000)	15.627	15.625	15.625	15.625	15.625

As visible from the last column of Table 1 there is no theoretical hindrance to solve the restriction $\varrho_i \neq \varrho_j$ if $i \neq j$. Dealing with finite sums, we have

$$\sum_{i=0}^{N} \varrho_1^i \varrho_2^{N-i} = (N+1)\varrho^N \quad \text{if} \quad \varrho_1 = \varrho_2 = \varrho .$$

Thus supposing $\varrho_M = \varrho_{M-1} = \varrho$ and $\varrho_i \neq \varrho_j$ if $i_{i,j} < M$ then after some connections we have

$$C(M,N) = \sum_{j=1}^{N-2} \frac{\varrho_j^{M+N-1}}{\prod_{\substack{K=1\\K\neq j}}^{M} (\varrho_j - \varrho_K)} +$$

$$+\varrho^{N+1} \cdot \sum_{j=1}^{M-2} \frac{\varrho_j^{M-3}}{\prod_{\substack{K=1\\K\neq j}}^{M} (\varrho_j - \varrho_K)} [(N+1)(\varrho - \varrho_j) - \varrho_j] .$$

It is obvious that if $\varrho < 1$ the second term tends very quickly to zero as N increases.

Table 1 seems to support that a model with equal intensities can be well approximated by those with different ones.

§ 3. The expectations

In the reality the number of customers at a node is a random variable ν_i. $P(\mathbf{n})$ was the probability that variable ν_i takes the value n_i for all i. Let E_i denote the expectation of variable ν_i.

Theorem 2. *For a model in Th. 1 the mean number of customers at node i is*

$$E_i = \frac{\varrho_i}{C(M,N)} \cdot \sum_{j\neq i} \frac{\varrho_j^{M-2}}{\prod_{K\neq j} (\varrho_j - \varrho_K)} \left[\frac{\varrho_j(\varrho_j^N - \varrho_i^N)}{\varrho_j - \varrho_i} - N\varrho_i^N \right] . \quad \boxtimes$$

Proof. For simplicity's sake let $i = M$.

$$E_M = C(M,N)^{-1} \cdot \sum_{i=0}^{N} i \cdot \varrho_M^i \cdot C(M-1, N-i) =$$

$$= C(M,N)^{-1} \cdot \sum_{i=0}^{N} i \cdot \varrho_M^i \cdot \sum_{j=1}^{M-1} \frac{\varrho_j^{M+N-i-2}}{\prod_{K\neq j}^{M-1} (\varrho_j - \varrho_K)} =$$

$$= C^{-1} \cdot \sum_{j=1}^{M-1} \frac{\varrho_j^{M-2}}{\prod_{\substack{K=1\\K\neq j}}^{M-1} (\varrho_j - \varrho_K)} \cdot \sum_{i=0}^{N} i \cdot \varrho_M^i \varrho_j^{N-i} .$$

The inner sums are easily evaluated as follows:

$$\left[\frac{\varrho_j(\varrho_j^N-\varrho_M^N)}{\varrho_j-\varrho_M}-N\cdot\varrho_M^N\right].$$

Thus the theorem is proved. ☒

Remark. If $N\to\infty$ and $\varrho_1=\max\varrho_i=1$ the well-known result $E_i=$ $=\frac{\varrho_i}{1-\varrho_i}$ can be obtained since all terms in the sum vanish except the first one:

$$\frac{\varrho_1^{M+N-1}}{\varrho_1-\varrho_i}=\frac{1}{\varrho_1-\varrho_i} \quad \text{and} \quad C(M,N)=\frac{1}{\prod\limits_{i=2}^{M}(1-\varrho_i)}.$$

The convergence of the finite means when N increases is demonstrated in Tables 2–3.

Table 2

N	E_1	E_2	E_3	E_4	E_5
1	.370	.296	.185	.0925	.0555
2	.7987	.6044	.3454	.1592	.0922
5	2.401	1.513	.6770	.2642	.1449
10	5.9145	2.698	.9039	.315	.1684
50	44.49	3.999	.99998	.3333	.1764
100	94.49	4	1	.3333	.1764
1000	994.49	4	1	.3333	.1765

$M=5;\ \varrho_1=1,\ \varrho_2=.8,\ \varrho_3=.5,\ \varrho_4=.25,\ \varrho_5=.15$

Table 3.

N	E_1	E_2	E_3	E_4	E_5
10	3.02	3.009	3.04	.6724	.2550
100	33.75	32.89	32,07	.660	.324
1000	420.33	322.10	256.18	.997	.332
5000	3561.76	944.42	492.48	1	1/3
10^4	8501	998.18	498.95	1	1/3
10^5	98500	999.0	499.0	1	1/3

$M=5;\ \varrho_1=1,\ \varrho_2=.999,\ \varrho_3=.998,\ \varrho_4=.5,\ \varrho_5=.25$

§ 4. Conclusions

The significance of the present paper seems to be in the basic importance of the factors $\varrho_i - \varrho_j$. The small number of parameters (m intensities and M transition probabilities in P) and the simple algorithms offer reason for beginning each investigation, also in the case of more general models, with an approximative consideration neglecting more detailed dependencies. Not only the clumsiness of algorithms increases but the uncertainty of the measured parameters as well, when dealing with more complicated models.

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