ON THE GENERALIZED SECANT METHOD FOR SOLVING NONLINEAR OPERATOR EQUATIONS IN SEMIORDERED SPACES

By

BÉLA JANKÓ

Dept. of Numerical Analysis and Computer Science, Eötvös University

1088 Budapest, Múzeum krt. 6–8.

(Received 28 December, 1981)

In the present work the iterative way of solving operator equations is studied. For this purpose an extension of the wellknown classical secant method resp. of the method of chords are given. We shall consider a class of nonlinear operators in the conditions of linear semiordered space, namely in Kantorovič's sense, i.e. K-spaces, where the axiom V. is satisfied only for denumerable upper bounded subsets of the space [6; p. 21].

It can be mentioned here that we shall not suppose the existence of certain derivatives or some assumptions for uniform derivability for the above operators [1]. It is known that the problem of effective calculation resp. the assurance of the existence of these derivatives represent generally one of the greatest difficulties connected with numerical applications.

Giving a suitable generalization of the secant method for the above class of operator equations, we can establish more general conditions for the existence and uniqueness of solutions, giving also assumptions for convergence of this method. In this way we can obtain at the same time some estimations concerning the errors of the approximate solutions.

Let us now introduce a notion of the divided difference for the case of nonlinear operators defined in any linear space, particularly valid of course also for K-spaces.

Definition 1. Consider the operator $P(x)$ defined in a certain linear space X and range in Y of the same type. We are going to construct the operator $P_{x'x''}$ defined in the Cartesian product $X \times X$. For this purpose we assume that the following conditions are fulfilled:

A) Let $P_{x'x''}$ be a linear (i.e. addivtive and homogeneous) operator for fixed pair $(x', x'') \in X \times X$, so that

$$P_{x'x''} \in [X \to Y],$$

where $[X \to Y]$ represents the space of all linear operators with domain in X and range in Y;

B) Let the equality

$$(E_1) \qquad P_{x'x''}(x'-x'') = P(x')-P(x'')$$

be satisfied then we say that $P_{x'x''}$ is a divided difference of first order for the operator P concerning the knots x' and x'' [2, 3]. ☒

We mention here that our definition for the divided difference is given for any linear space (not only for normed space); thus this doesn't require any conditions of boundedness in norm.

The divided difference of second order $P_{x'x''x'''}$ may be intorduced in a similar way [2–5].

Definition 2. Let $P_{x'x''x'''}$ be a bilinear operator for fixed knots x', x'', x''' defined in the Cartesian product $X\times X\times X$, where $(x',x'',x''')\in X\times X\times X$, where $P_{x'x'x'''}$ satisfies

$$(E_2) \qquad P_{x'x''x'''}(x''-x''') = P_{x'x''}-P_{x'x'''}. \text{ ☒}$$

The operator $P_{x'x''x'''}(u, v)$ is considered bilinear concerning its arguments u, v (in the case of fixed knots x', x'', x'''), i.e. it is additive

$$P_{x'x''x'''}(u_1+u_2, v) = P_{x'x''x'''}(u_1, v)+P_{x'x''x'''}(u_2, v),$$

$$P_{x'x''x'''}(u, v_1+v_2) = P_{x'x''x'''}(u, v_1)+P_{x'x''x'''}(u, v_2),$$

and homogeneous

$$P_{x'x''x'''}(cu, v) = cP_{x'x''x'''}(u, v),$$

$$P_{x'x''x'''}(u, cv) = cP_{x'x''x'''}(u, v)$$

for any u, u_1, u_2, v, v_1, $v_2\in X$ and for any real number c. ☒

Definition 3. A linear operator L defined in a linear semiordered space X with range in Y of the same type is considered positive, if from $x>\Theta$, it follows $Lx>\tilde{\Theta}$, where Θ is the null-element of X and $\tilde{\Theta}$ is the null-element in Y. In this case we denote $L>0_1$ and 0_1 represents the null-linear operator defined on X. ☒

Definition 4. Let L_1 and L_2 be linear operators defined in X with values in Y. We say $L_1<L_2$, if $L_1x<L_2x$ holds for any $x\in X$. ☒

Definition 5. A bilinear operator B defined in $X\times X$ with values in Y is positive, if $u>\Theta$, $v>\Theta$ implie $B(u, v)>\tilde{\Theta}$, where $(u, v)\in X\times X$ and $\tilde{\Theta}$ is the null-element of Y. In this case we denote $B>0_2$, 0_2 being the null-bilinear operator in $X\times X$. ☒

Definition 6. Let X and Y be K-spaces and let $\underline{x}_0$, $\bar{x}_0$ be certain given elements in X. The set of elements x satisfying the inequalities

$$\underline{x}_0\leq x\leq \bar{x}_0$$

is called order-segment $[\underline{x}_0, \bar{x}_0]$ in X. ☒

Consider now the operator equation

$$P(x) = \Theta, \tag{1}$$

where $P(x)$ is a nonlinear operator defined on $[\underline{x}_0, \overline{x}_0] \subset X$ with values in Y.

Definition 7. The operator $P(x)$ is called (o)-continuous on $[\underline{x}_0, \overline{x}_0]$ if the following equality [6]

$$(o)-\lim_{n\to\infty} P(x_n) = P\left((o)-\lim_{n\to\infty} x_n\right)$$

is fulfiled for all monotone and convergent sequences on $[\underline{x}_0, \overline{x}_0]$, where $(o)\text{-}\lim\limits_{n\to\infty} x_n$ denotes the limit of sequence $\{x_n\}$, defined in the conditions of K-spaces. ⊠

THEOREM 1. Let X, Y be K-spaces. We consider the nonlinear operator $P(x)$ defined only on the order-segment $[\underline{x}_0, \overline{x}_0] \subset X$ with values in Y. Let us assume that the following conditions are fulfilled

1°. $$P(\underline{x}_0) \leq \Theta \leq P(\overline{x}_0),$$

for the initial approximate solutions $\underline{x}_0$, $\overline{x}_0$, where $\underline{x}_0 \leq \overline{x}_0$ and $\underline{x}_0, \overline{x}_0 \in X$;

2°. $P(x)$ is (o)-continuous and its first divided difference satisfies the inequality

$$P_{x'x''} \leq \Gamma$$

for every $x', x'' \in [\underline{x}_0, \overline{x}_0]$, where Γ is a suitably chosen linear (i.e. additive and homogeneous) operator defined on $[\underline{x}_0, \overline{x}_0]$, having positive inverse $\Gamma^{-1} > 0_1$;

3°. the sequences of the lower approximate solutions $\{\underline{x}_n\}$ respectively the upper approximate solutions $\{\overline{x}_n\}$ are constructed by the following iterative processes

$$\underline{x}_n = \underline{x}_{n-1} - \Gamma^{-1} P(\underline{x}_{n-1}), \tag{2}$$

$$\overline{x}_n = \overline{x}_{n-1} - \Gamma^{-1} P(\overline{x}_{n-1}), \quad (n = 1, 2, \ldots). \tag{2'}$$

Then the operator equation (1) possesses at least one solution x^*, which belongs to $[\underline{x}_0, \overline{x}_0]$, and the monotone increasing sequence $\{\underline{x}_n\}$ resp. the monotone decreasing sequence $\{\overline{x}_n\}$ converge to $\underline{x}^*$, resp. to $\overline{x}^*$; thus

$$\underline{x}^* := (o)-\lim_{n\to\infty} \underline{x}_n \leq x^* \leq (o)-\lim_{n\to\infty} \overline{x}_n =: \overline{x}^* \tag{3}$$

and

$$\underline{x}_{n-1} \leq \underline{x}_n \leq \underline{x}^* \leq \overline{x}^* \leq \overline{x}^* \leq \overline{x}_n \leq \overline{x}_{n-1}, \tag{4}$$

$$P(\underline{x}_n) \leq \Theta \leq P(\overline{x}_n), \quad (n = 0, 1, 2, \ldots), \tag{4'}$$

where $\underline{x}^*$ is the smallest and $\overline{x}^*$ the greatest solution of the equation (1) in the order-segment $[\underline{x}_0, \overline{x}_0]$. ⊠

PROOF. First of all we observe that the inequalities $\underline{x}_0 < \underline{x}_1$ and $\bar{x}_1 < \bar{x}_0$ result directly from (2) resp. (2′) and from the positivity of Γ^{-1}. In the next we follow partially Baluev's way [1]. Really, as a consequence of 2°. we obtain the inequalities used by Baluev, i.e.

$$(5) \qquad P(\bar{x}_0)+\Gamma(x-\bar{x}_0) \leq P(x) \leq P(\underline{x}_0)+\Gamma(x-\underline{x}_0)$$

for all $x \in [\underline{x}_0, \bar{x}_0]$.

To verify the inequality $\underline{x}_1 \leq \bar{x}_1$ we shall show before $\underline{x}_1 \leq \bar{x}_0$. We have indeed

$$\bar{x}_0-\underline{x}_1 = \bar{x}_0-\underline{x}_0+\Gamma^{-1} P(\underline{x}_0) = \Gamma^{-1}[\Gamma(\bar{x}_0-\underline{x}_0)+P(\underline{x}_0)] \geq \Theta,$$

which follows from (5). Since $\underline{x}_0 \leq \underline{x}_1 \leq \bar{x}_0$, i.e. $\underline{x}_1 \in [\underline{x}_0, \bar{x}_0]$. Thus we can apply the relation (5) also for $x=\underline{x}_1$, and from the right side of (5) we get $P(\underline{x}_1) \leq \Theta$; indeed we have

$$P(\underline{x}_1) \leq P(\underline{x}_0)+\Gamma(\underline{x}_1-\underline{x}_0) = \Theta .$$

Now we can verify directly the inequality $\underline{x}_1 \leq \bar{x}_1$; really from the left side of (5) we get

$$\bar{x}_1-\underline{x}_1 = \bar{x}_0-\Gamma^{-1} P(\bar{x}_0)-\underline{x}_1 = -\Gamma^{-1}[P(\bar{x}_0)+\Gamma(\underline{x}_1-\bar{x}_0)] \geq -\Gamma^{-1}P(\underline{x}_1) \geq \Theta .$$

Applying once more the inequalities (5) for $x = \bar{x}_1$, we obtain

$$P(\bar{x}_1) \geq P(\bar{x}_0)+\Gamma(\bar{x}_1-\bar{x}_0) = P(\bar{x}_0)-\Gamma\Gamma^{-1} P(\bar{x}_0) = \Theta .$$

In this way we have establised for $n = 1$ the following inequalities

$$\underline{x}_0 \leq \underline{x}_1 \leq \bar{x}_1 \leq \bar{x}_0 ,$$

respectively

$$P(\underline{x}_1) \leq \quad \leq P(\bar{x}_1) .$$

By complete induction, the above inequalities may be extended for all natural n, i.e.

$$\underline{x}_{n-1} \leq \underline{x}_n \leq \bar{x}_n \leq \bar{x}_{n-1} ,$$

respectively

$$P(\underline{x}_n) \leq \Theta \leq P(\bar{x}_n) .$$

Thus, if the sequence $\{\underline{x}_n\}$ is monotone increasing and upper bounded, then there exists the limit $\underline{x}^*$, i.e.

$$\underline{x}^* := (o)-\lim_{n\to\infty} \underline{x}_n = \sup_n \underline{x}_n .$$

Similarly, we obtain for the decreasing and lower bounded sequence $\{\bar{x}_n\}$ the limit

$$\bar{x}^* := (o)-\lim_{n\to\infty} \bar{x}_n = \inf_n \bar{x}_n .$$

Based on the (o)-continuity of $P(x)$, we get

$$P(\underline{x}^*) = \Theta, \quad P(\overline{x}^*) = \Theta,$$

where $\underline{x}^*$, $\overline{x}^*$ are the smallest respectively the greatest solution of $P(x) = \Theta$ in the order-segment $[\underline{x}_0, \overline{x}_0]$. In this way our theorem is completely proved. $\boxed{\times}$

REMARKS. 1. We mention here that we can assure also the uniqueness of the solution as well; namely in the case, when there exists a linear operator Λ from $[\underline{x}_0, \overline{x}_0] \subset X$ to Y, having positive inverse Λ^{-1}, such that

$$\Lambda \leq P_{x' x''} \tag{6}$$

for all $x', x'' \in [\underline{x}_0, \overline{x}_0]$. Really, starting from the inequality (3), i.e. $\underline{x}^* \leq \overline{x}^*$ and (6), we obtain

$$\Lambda(\overline{x}^* - \underline{x}^*) \leq P_{\overline{x}^* \underline{x}^*}(\overline{x}^* - \underline{x}^*) = P(\overline{x}^*) - P(\underline{x}^*) = \Theta.$$

Based on the positivity of the inverse Λ^{-1} results $\overline{x}^* \leq \underline{x}^*$, and comparing with (3), we obtain $\overline{x}^* = \underline{x}^*$.

2. The divided difference can be applied usefully for the effective construction of the operator Γ. Indeed, in the next we shall construct the generalized secant method, resp. the method of chords for the iterative solving of operator equations, replacing in (2), (2′) certain suitably chosen divided differences instead of the linear operator Γ. In this way we can state the following theorem.

THEOREM 2. Consider the nonlinear operator $P(x)$, where $x \in [\underline{x}_0, \overline{x}_0] \subset \subset X$ and $P(x) \in Y$. Let us suppose the conditions

1°. for the initial approximate solutions $\underline{x}_0$, $\overline{x}_0$ and $\overline{x}_{-1}$ the following inequalities

$$P(\underline{x}_0) < \Theta < P(\overline{x}_0) < P(\overline{x}_{-1})$$

are fulfiled, where

$$\underline{x}_0 < \overline{x}_0 < \overline{x}_{-1}, \quad \underline{x}_0, \overline{x}_0, \overline{x}_{-1} \in X;$$

2°. there exists a symmetrical divided difference of first resp. of second order, i.e.

$$P_{x' x''} \quad \text{respectively} \quad P_{x' x'' x'''}$$

for all $x', x'', x''' \in [\underline{x}_0, \overline{x}_0]$, where $x' \neq x''$, $x'' \neq x'''$, $x' \neq x'''$, moreover $P(x)$ is (o)-continuous and (o)-convex in the sense

$$P_{x' x'' x'''} > 0_2,$$

0_2 being the null-bilinear operator;

3°. there exists the inverse $P^{-1}_{x' x''}$ for any $x', x'' \in [\underline{x}_0, \overline{x}_0]$ and $P^{-1}_{x' x''} > 0_1$;

4°. the lower and the upper approximate solutions are calculated using the formulas

$$(7) \qquad \underline{x}_n = \underline{x}_{n-1} - P^{-1}_{\bar{x}_{n-1}\underline{x}_{n-1}} P(\underline{x}_{n-1}),$$

$$(7') \qquad \bar{x}_n = \bar{x}_{n-1} - P^{-1}_{\bar{x}_{n-1}\bar{x}_{n-2}} P(\bar{x}_{n-1}), \quad (n = 1, 2, \ldots)$$

then there exists a single solution x^* for the equation $P(x) = \Theta$ in the order-segment $[\underline{x}_0, \bar{x}_0]$, i.e. $x^* := \underline{x}^* = \bar{x}^*$, where

$$\underline{x}^* := (o) - \lim_{n\to\infty} \underline{x}_n; \quad \bar{x}^* := (o) - \lim_{n\to\infty} \bar{x}_n$$

and

$$\underline{x}_{n-1} < \underline{x}_n < x^* < \bar{x}_n < \bar{x}_{n-1}$$

$$P(\underline{x}_n) < \Theta < P(\bar{x}_n). \ \boxtimes$$

PROOF. In order to prove this theorem we require certain inequalities, as

$$(8) \qquad P_{\underline{x}_0\underline{x}_1} < P_{x'x''} < P_{\bar{x}_0\bar{x}_1}$$

for any x' and x'', where $\underline{x}_0 < \underline{x}_1 < x' < x'' < \bar{x}_0 < \bar{x}_{-1}$. For this purpose we can verify easily, step by step, the inequalities

$$P_{\underline{x}_0\underline{x}_1} < P_{\underline{x}_0 x'} < P_{\underline{x}_1 x'} < P_{\underline{x}_1 x''} < P_{x'x''}$$

based of course on the (o)-convexity of P and on the symmetry of the divided differences of first order. In this way we have

$$P_{\underline{x}_0\underline{x}_1} < P_{x'x''}.$$

Similarly we can also establish

$$P_{x'x''} < P_{\bar{x}_0\bar{x}_{-1}}$$

using the inequalities

$$P_{x'x''} < P_{x'\bar{x}_{-1}} < P_{\bar{x}_0\bar{x}_{-1}}.$$

Moreover we need for the next the following inequality, which results also directly from the (o)-convexity of P:

$$(8') \qquad P^{-1}_{\bar{x}_0\bar{x}_{-1}} < P^{-1}_{\bar{x}_0 x_0}$$

as well. Really, considering that if we have two linear positive operators $L_1 > 0_1$ and $L_2 > 0_1$, then the result is $L_1L_2 > 0_1$. Thus we obtain the inequality (8′) in the following way: we observe that

$$P^{-1}_{\bar{x}_0\bar{x}_{-1}}, \quad P_{\bar{x}_0\bar{x}_{-1}} - P_{\bar{x}_0\underline{x}_0} \quad \text{and} \quad P^{-1}_{\bar{x}_0\underline{x}_0}$$

are positive operators and as a consequence we get

$$0_1 < P^{-1}_{\bar{x}_0\bar{x}_{-1}} [P_{\bar{x}_0\bar{x}_{-1}} - P_{\bar{x}_0\underline{x}_0}] P^{-1}_{\bar{x}_0\underline{x}_0} = P^{-1}_{\bar{x}_0\underline{x}_0} - P^{-1}_{\bar{x}_0\bar{x}_{-1}}.$$

Now based on the positivity of the inverses $P_{\bar{x}_0 \bar{x}_{-1}}^{-1}$ respectively $P_{\bar{x}_0 \underline{x}_0}^{-1}$ we obtain directly from (7′), (7) the inequalities $\bar{x}_1 < \bar{x}_0$ resp. $\underline{x}_0 < \underline{x}_1$. We can also verify $\underline{x}_1 < \bar{x}_1$. Really, using the inequality (8′) and the property (E_1) we obtain

$$\bar{x}_1 - \underline{x}_1 = \bar{x}_0 - \underline{x}_0 + P_{\bar{x}_0 \underline{x}_0}^{-1} P(\underline{x}_0) - P_{\bar{x}_0 \bar{x}_{-1}}^{-1} P(\bar{x}_0) >$$

$$> \bar{x}_0 - \underline{x}_0 + P_{\bar{x}_0 \underline{x}_0}^{-1} P(\underline{x}_0) - P_{\bar{x}_0 \underline{x}_0}^{-1} P(\bar{x}_0) =$$

$$= P_{\bar{x}_0 \underline{x}_0}^{-1} [P_{\bar{x}_0 \underline{x}_0}(\bar{x}_0 - \underline{x}_0) + P(\underline{x}_0) - P(\bar{x}_0)] = \Theta .$$

From the above inequalities we get $\underline{x}_0 < \underline{x}_1 < \bar{x}_1 < \bar{x}_0$. i.e. $\underline{x}_1, \bar{x}_1 \in [\underline{x}_0, \bar{x}_0]$.

Now we are going to establish the inequalities $P(\bar{x}_1) > \Theta$ and $P(\underline{x}_1) < \Theta$ For this purpose we start from the relations

$$P_{\bar{x}_0 \bar{x}_1} - P_{\bar{x}_0 \bar{x}_{-1}} = P_{\bar{x}_0 \bar{x}_1 \bar{x}_{-1}}(\bar{x}_1 - \bar{x}_{-1})$$

respectively

$$P(\bar{x}_0) - P(\bar{x}_1) - P_{\bar{x}_0 \bar{x}_{-1}}(\bar{x}_0 - \bar{x}_1) = P_{\bar{x}_0 \bar{x}_1 \bar{x}_{-1}}(\bar{x}_1 - \bar{x}_{-1})(\bar{x}_0 - \bar{x}_1) < \Theta .$$

Using (7′) for $n = 0$ and the convexity of P and the inequalities $\bar{x}_1 < \bar{x}_0 < \bar{x}_-$ we get $P(\bar{x}_1) > \Theta$.

Similarly we can obtain $P(\underline{x}_1) < \Theta$. Indeed, considering

$$P_{\underline{x}_1 \underline{x}_0} - P_{\bar{x}_0 \underline{x}_0} = P_{\underline{x}_1 \bar{x}_0 \underline{x}_0}(\underline{x}_1 - \bar{x}_0)$$

respectively

$$P(\underline{x}_1) - P(\underline{x}_0) - P_{\bar{x}_0 \underline{x}_0}(\underline{x}_1 - \underline{x}_0) = P_{\underline{x}_1 \bar{x}_0 \underline{x}_0}(\underline{x}_1 - \bar{x}_0)(\underline{x}_1 - \underline{x}_0) < \Theta .$$

In this way we have $P(\underline{x}_1) < \Theta < P(\bar{x}_1)$.

By complete induction we obtain

$$\underline{x}_{n-1} < \underline{x}_n < \bar{x}_n < \bar{x}_{n-1}$$

respectively

$$P(\underline{x}_n) < \Theta < P(\bar{x}_n) .$$

Thus the sequences $\{\underline{x}_n\}$ resp. $\{\bar{x}_n\}$ being monotone increasing and upper bounded resp. monotone decreasing and lower bounded, there exist the limits

$$\underline{x}^* := (o) - \lim_{n \to \infty} \underline{x}_n , \quad \bar{x}^* := (o) - \lim_{n \to \infty} \bar{x}_n$$

and based on the continuity, we have

$$P(\underline{x}^*) = P(\bar{x}^*) = \Theta ,$$

$\underline{x}^*$ resp. $\bar{x}^*$ being the smallest resp. the greatest solution of $P(x) = \Theta$ in the order-segment $[\underline{x}_0, \bar{x}_0]$.

For the uniqueness of the solution we can choose a linear operator Λ with positive inverse from Theorem 1., such that $\Lambda \leq P_{uv}$ for any $u, v \in [\underline{x}_0, \bar{x}_0]$. Indeed, keeping in mind the inequalitites (8) we can choose $\Lambda := P_{\underline{x}_0 \underline{x}_1}$

Using the same reasoning as at the end of the proof for the first theorem, we get the uniqueness of the solution. So our theorem is completely proved. ☒

REMARS. 1. Comparing the method given in (2) resp. (2′) from Theorem 1. we can mention that the operator Γ is replaced in (7) resp. (7′) by $P_{\bar{x}_n \underline{x}_n}$ resp. $P_{\bar{x}_n \bar{x}_{n-1}}$ and of course Γ does not depend on n. However these divided differences can be fixed at a certain n. In this way the operator Γ from the first theorem can be constructed for example $\Gamma := P_{\bar{x}_0 \bar{x}_{n-1}}$.

2. We observe that instead of $P_{\bar{x}_{n-1} \underline{x}_{n-1}}$ in (7) can use $P_{\underline{x}_{n-1} \bar{x}_n}$. Of course in this case it is necessary to calculate before $\bar{x}_n$ by (7′) and then use

$$\underline{x}'_n = \underline{x}_{n-1} - P^{-1}_{\underline{x}_{n-1} \bar{x}_n} P(\underline{x}_{n-1}),$$

which gives better approximations. This fact can be verified directly.

3. In a similar manner we can study the case when we have

$$P_{x' x'' x'''} < 0_2 \quad \text{and} \quad P^{-1}_{x' x''} > 0_1 .$$

4. In this way certain results in [5] were generalized, without supposing conditions concerning the derivability respectively uniformly derivability for the operator P.

Let us indicate at last some illustrative examples for the established theorems.

We consider first the following class of functional equations

$$F(t; x(t)) = g(t), \tag{9}$$

F being a given measurable bounded real function of arguments t resp. of x; the unknown function $x(t)$ resp. the given function $g(t)$ belong to a given order-segment of X, where X represents the K-space of measurable bounded real functions defined on a finite closed interval $[a, b]$. In this case of course the axiom V. of the K-space is satisfied also for denumerable upper bounded subsets.

Let us assume that the following conditions are satisfied:

1°. $$F(t; \underline{x}_0(t)) < g(t) < F(t; \bar{x}_0(t))$$

and

$$x_0(t) < \bar{x}_0(t), \quad \text{for} \quad \forall t \in [a, b],$$

where $\underline{x}_0(t), \bar{x}_0(t) \in X$ represent some initial approximate solutions;

2°. let $F(t, x(t))$ be defined on the order-interval $[\underline{x}_0(t), \bar{x}_0(t)] \subset X$ and range in X. We consider the first resp. second order of divided differences for F in usual manner:

$$F_{\xi\eta} := \frac{F(t; \xi(t)) - F(t; \eta(t))}{\xi(t) - \eta(t)}$$

respectively

$$F_{\xi\eta\zeta} := \frac{F_{\xi\eta} - F_{\xi\zeta}}{\eta - \zeta}$$

for all $\xi < \eta < \zeta$, where $\xi, \eta, \zeta \in [\underline{x}_0(t), \bar{x}_0(t)]$. Let F be (o)-continuous with respect to x and moreover (o)-convex in the sense

$$F_{\xi\eta\zeta} > 0\,,$$

for all $\xi < \eta < \zeta$, and $\xi, \eta, \zeta \in [\underline{x}_0(t), \ddot{x}_0(t)]$;

3°. $$0 < F_{\bar{x}_0 \bar{x}_{1-1}} < +\infty\,,$$

where $\bar{x}_{-1}(t)$ denotes a chosen initial approximate solution and $\bar{x}_{-1} > \bar{x}_0$;

4°. the lower resp. the upper approximate solutions are defined by formulas

$$\underline{x}_n(t) = \underline{x}_{n-1}(t) - \frac{1}{F_{\bar{x}_{n-1}\underline{x}_{n-1}}}\,[F(t; \underline{x}_{n-1}(t)) - g(t)]$$

respectively

$$\bar{x}_n(t) = \bar{x}_{n-1}(t) - \frac{1}{F_{\bar{x}_{n-1}\bar{x}_{n-2}}}\,[F(t; \bar{x}_{n-1}(t)) - g(t)]\,, \quad (n = 1, 2, \ldots).$$

In this large conditions we can assure the existende and the uniqueness of the solution $x^*(t)$ for the equation (9) on the order-segment $[\underline{x}_0(t), \bar{x}(t)]$, having the inequalities

$$\underline{x}_{n-1}(t) < \underline{x}_n(t) < x^*(t) < \bar{x}_n(t) < \bar{x}_{n-1}(t)$$

and

$$F(t; \underline{x}_n(t)) < g(t) < F(t; \bar{x}_n(t))\,.$$

Indeed, we can apply directly Theorem 2. for this case. Of course we can use Theorem 1. as well – choosing the operator Γ in a convenient manner.

Similar conditions can be imposed for the following class of nonlinear integral equations:

$$F\left(t; x(t); \int_a^b K(t, s; x(s))\,ds\right) = 0\,,$$

where F resp. K are measurable functions.

REFERENCES

[1] *Балуев, А. Н.:* О методе С. А. Чаплыгина, Вестник Ленинград. Унив. № 13, 27–42 (1956).

[2] *Jankó, B.:* On Solving the Operator Equations (in Romanian), Dissertation; Cluj – Kolozsvár, 1967.

[3] *Jankó, B.:* Solving Nonlinear Operator Equations in Banach Spaces (in Romanian); Romanian Academy Publishing House, Bucharest, 1969.
[4] *Jankó, B. – Groze, S.:* On the Generalized Divided Differences (in Romanian); Analele Stiint. Univ. Iasi, XVII, 2, 375–379 (1971).
[5] *Jankó, B.:* Applications for Solving Functional Equations Defined in Semiordered Spaces (in Romanian); Studii si Cercetari de Matematica (Cluj), X, 1, 51–57 (1959).
[6] *Канторович, Л. В.–Вулих, Б. З.–Пинскер, А. Г.:* Функциональный анализ в полуупорядоченных пространствах, Москва–Ленинград, 1950.
[7] *Schmidt, J. W.:* Die Regula Falsi für Operatoren in Banach Räumen, ZAMM, Sonderheft, 61–63 (1961).
[8] *Сергеев, А. С.:* О методе хорд, Сибирский Мат. Журнал 2, 2, 282–289 (1961).
[9] *Улъм, С.:* Об интерполяционных методах решения нелинейных уравнений в пространстве Банаха, Известия Акад. Эстонской ССР, 12, 1, 24–30 (1963).