ON THE UNITARY THEORY OF ITERATION METHODS FOR SOLVING NONLINEAR OPERATOR EQUATIONS CONSIDERED IN SEMIORDERED SPACES

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In an earlier paper of ours [2] we tried to give a unitary theory for a certain class of iteration methods, used for solving nonlinear operator equations defined in Banach spaces. In the present work we resume this problem in the conditions of a B_k-space, i.e. linear semiordered complete space, normed in a general sense or in the L.V. Kantorovič's sense [3].

For this purpose we consider the equation

$$P(x) = \Theta, \tag{1}$$

where $P(x)$ is a non-linear operator defined in a domain D of the B_k-space X with values also in X. It is well-known that in an isolated manner certain iteration methods have already been treated in semiordered space [1, 4, 5]. We had shown in our above mentioned work that in the conditions of Banach spaces the order of convergence has a direct influence on the structure of the iteration methods. The order of convergence permits to generate and classify as well, certain well-known iteration methods and also enables the common treatment of these methods at the same time giving common conditions for convergence. The situation is analogous in the conditions of B_k-spaces.

For this purpose we introduce a notion of order of convergence.

Definition. Let x^* be a solution of the operator equation (1). We shall say that the lower iteration method (3), (see below) has the order of convergence k, *if*

(i) the generalized norm $\| x^* - x_n \|$ tends in the Kantorovič's sense to the null-element, when $n \to \infty$;

(ii) the derivatives of the iteration-operator $\Psi(x)$ satisfy the equalities

$$\Psi'(x^*) = 0_1,\ \Psi''(x^*) = 0_2,\ \dots,\ \Psi^{(k-1)}(x^*) = 0_{k-1},\ \Psi^{(k)}(x^*) \neq 0_k,$$

where 0_i, $(i = 1, 2, \dots, k)$ are i-linear null-operators.

We consider now the following iteration operator, analogous to that given in [2]:

$$\Psi(x) \equiv x - [P'(x) + \lambda_1(x)\, P(x)]^{-1}\, P(x) + \lambda_2(x)\, P^2(x), \tag{2}$$

where $\Psi(x)$ is a nonlinear operator with domain $D \subset X$ and range in X; moreover, $\lambda_1(x)$ and $\lambda_2(x)$ represent bilinear operators for fixed x, defined on the domain $D \times D \subset X \times X$, having values also in X.

Using the iteration operator $\Psi(x)$, we can construct the following general iteration method

$$x_{n+1} = \Psi(x_n) \tag{3}$$

and we shall apply it for solving equation (1).

Certain methods, such as the well-known Newton-Kantorovič's method, the Čebyšev's method and the method of tangent hyperbolas can be generated in a similar way as in the conditions of Banach space [2].

Thus, if we choose

$$\lambda_1(x) \equiv \lambda_2(x) \equiv 0_2,$$

where 0_2 is the null bilinear operator and besides this we impose the condition

$$\Psi'(x^*) = 0_1,$$

0_1 being the null linear operator, then in this case the general iteration method (3) is reduced precisely to the Newton–Kantorovič's method, which is related to the generalized Čaplygin's method [1].

It is worth mentioning here that if $\lambda_1(x) \not\equiv 0_2$ and $\lambda_2(x) \not\equiv 0_2$, then we obtain a class of transfinite number of iteration methods of second order.

In the case, in which we put $\Psi''(x^*) = 0_2$ and choosing $\lambda_2(x) \equiv 0_2$, we obtain the method of tangent hyperbolas

$$x_{n+1} = x_n - \left[P'(x_n) - \frac{1}{2} P''(x_n) \Gamma_n P(x_n)\right]^{-1} P(x_n);$$

$$\Gamma_n = [P'(x_n)]^{-1}.$$

If we choose $\lambda_1(x) \equiv 0_2$ and impose also $\Psi''(x^*) = 0_2$, then we find the method of Čebyšev [5],

$$x_{n+1} = x_n - \Gamma_n P(x_n) - \frac{1}{2} \Gamma_n P''(x_n)[\Gamma_n(x_n)]^2; \quad \Gamma_n = [P'(x_n)]^{-1}.$$

Of course, besides these two methods there exists a class of a transfinite number of iteration methods of third order. In this case one of the operators $\lambda_1(x)$ and $\lambda_2(x)$ is arbitary.

We mention that the iteration operator can be chosen in another form

$$\tilde{\Psi}(x) \equiv x - [P'(x) + \mu(x) P(x)]^{-1} \left(P(x) + \Lambda_2(x) P^2(x)\right). \tag{2'}$$

If we now impose $\tilde{\Psi}''(x^*) = 0_2$, i.e.

$$\tilde{\Psi}''(x^*) \Delta x_1 \Delta x_2 = \Theta,$$

for any $\Delta x_1, \Delta x_2 \in D$, then we get a relation for the bilinerar operators $\mu(x)$ and $\Lambda_2(x)$:

$$P''(x^*)\Delta x_1 \Delta x_2 + 2\mu(x^*)[P'(x^*)\Delta x_1]\Delta x_2 = \\ = 2\Lambda_2(x^*)[P'(x^*)\Delta x_1][P'(x^*)\Delta x_2].$$

Thus we obtain a new class of iteration methods of third order having the form

$$x_{n+1} = \tilde{\Psi}(x_n). \tag{3'}$$

Putting in relation (3′)

$$\mu(x^*)\Delta x_1 \Delta x_2 = -P''(x^*)[\Gamma(x^*)\Delta x_1]\Delta x_2$$

for any $\Delta x_1, \Delta x_2 \in D$, then it follows

$$\Lambda_2(x^*)\Delta x_1 \Delta x_2 = -\frac{1}{2}P''(x^*)[\Gamma(x^*)\Delta(x_1)][\Gamma(x^*)\Delta x_2],$$

where $\Gamma(x^*) \equiv [P'(x^*)]^{-1}$.

If we admit

$$\Lambda_2(x)\Delta x_1 \Delta x_2 = -\frac{1}{2}P''(x)[\Gamma(x)\Delta x_1][\Gamma(x)\Delta x_2]$$

for any $x \in D$, then we recover L. K. Võhandu's method [7]. The class of iteration method (3′) contains also Ü. Kaasik's method

$$x_{n+1} = x_n - (I + \alpha R_n)^{-1}[\Gamma_n P(x_n) + (\alpha+1)R_n \Gamma_n P(x_n)],$$

where I is the identity operator and

$$R_n = \frac{1}{2}\Gamma_n P''(x_n)\Gamma_n P(x_n), \quad \Gamma_n \equiv [P'(x_n)]^{-1},$$

α being a real parameter. For $\alpha = -1, 0, -2$ we get the generalized method of tangent hyperbolas, the Čebyšev's method and the method of L.K. Võhandu.

We notice that the methods of Kaasik and Võhandu have not been treated in the conditions of semiordered spaces.

At last we mention that the general form of the iteration operator can be given as follows:

$$\Psi(x) \equiv x - \{P'(x) + \mu_1(x)P(x) + \ldots + \mu_{i+1}(x)[P(x)]^{i+1}\}^{-1}P(x) + \\ + \Lambda_2(x)[Px]^2 + \ldots + \Lambda_{j+1}(x)[P(x)]^{j+1},$$

where the multilinear operators

$$\mu_r(x), \Lambda_s(x), \quad (r = 1, 2, \ldots, i;\ s = 1, 2, \ldots, j;\ i+j = k-1)$$

can be determined by means of the conditions

$$\Psi^{(p)}(x^*)\,\Delta x_1 \ldots \Delta x_p = \Theta, \quad (p = 2, 3, \ldots, k-1)$$

and the operators $\mu_{i+1}(x)$, $\Lambda_{j+1}(x)$ are arbitrary; these can be chosen in a convenient manner for the necessities of the numerical calculations.

In this way we obtain a class of iteration methods of order k, analogously as in [2].

The general expression of the iteration operator $\tilde{\Psi}(x)$ can be considered in the following form

$$\tilde{\Psi}(x) \equiv x - \{P'(x) + \mu_1(x)P(x) + \ldots + \mu_{i+1}(x)\,[P(x)]^{i+1}\}^{-1} \cdot$$
$$\cdot \{P(x) + \Lambda_2(x)\,[P(x)^2 + \ldots + \Lambda_{j+1}(x)\,[P(x)]^{j+1}\}.$$

Now we are going to establish some common conditions of the convergence of order k for the general iteration method (3). For this purpose we shall use the generalized Taylor's formula, considered in X for the iteration operator

$$(4) \qquad \Psi(x_n) = \Psi(x^*) + \Psi'(x^*)(x_n - x^*) + \frac{1}{2}\Psi''(x^*)(x_n - x^*)^2 + \ldots$$
$$\ldots + \frac{1}{(k-1)!}\int_{x^*}^{x_n} \Psi^{(k)}(\tilde{x})\,(x_n - \tilde{x})^{k-1}\,d\tilde{x}$$

where $\tilde{x} = x^* + t(x_n - x^*)$, $(0 \leq t \leq 1)$.

We assume the following conditions:

1°. Let $P(x)$ be defined on the segment D, given by the elements x satisfying the inequalities $\underline{x}_0 \leq x \leq \bar{x}_0$, where $\underline{x}_0$, $\bar{x}_0$ are given initial approximative solutions. There exist certain additive and homogeneous operators Λ and Γ with positive inverses Λ^{-1}, Γ^{-1}, such that

a) $$\Lambda \Delta x \leq P(x + \Delta x) - P(x) \leq \Gamma \Delta x$$

for any positive Δx and for any $x, x + \Delta x \in [\underline{x}_0, \bar{x}_0]$;

b) $\Gamma^{-1}P(x)$ is (o)-monotone and (o)-continuous;

c) $$P(\underline{x}_0) < \Theta < P(\bar{x}_0)$$

for the initial approximate solutions $\underline{x}_0$, $\bar{x}_0$, where $\underline{x}_0 < \bar{x}_0$;

2°. The iteration operator $\Psi(x)$ defined by the equality (2) is uniformly differentiable of order k;

3°. The relations

$$\Psi'(x^*) = 0_1, \ \Psi''(x^*) = 0_2, \ldots, \Psi^{(k-1)}(x^*) = 0_{k-1} \ \text{and} \ \Psi^{(k)}_{(x)} > 0_k$$

are satisfied for $k = 2\nu + 1$, ν being a natural number.

Theorem. Let us assume that the conditions 1°.–3°. are fulfilled. Then the operator equation $P(x) = \Theta$ possesses a single solution x^* and the general process $x_{n+1} = \Psi(x_n)$ is convergent of order k. The monotone decreasing upper approximations $\{\bar{x}_n\}$ resp. the monotone increasing lower approximations $\{\underline{x}_n\}$, defined by the algorithms

$$\bar{x}_{n+1} = \Psi(\bar{x}_n) \quad \text{resp.} \quad \underline{x}_{n+1} = \Psi(\underline{x}_n)$$

converge to the solution x^*,

$$x^* = (B_k) - \lim_{n\to\infty} \underline{x}_n = (B_k) - \lim_{n\to\infty} \bar{x}_n ,$$

where $\underline{x}_n$ resp. $\bar{x}_n$ satisfy the following inequalities

(5) $$\underline{x}_0 < \underline{x}_1 < \ldots < \underline{x}_{n-1} < \underline{x}_n < x^* < \bar{x}_n < \bar{x}_{n-1} < \ldots \bar{x}_1 < \bar{x}_0 .$$

Proof. First of all we observe that the condition 1°. ensure the existence and the unicity for the solution of the equation (1) as well [6]. Further, based on the condition 2° the generalized Taylor's formula (4) can be constructed, which is important in the establishment of the inequalities (5) [1].

In fact the relations (4) and $\Psi^{(k)}(x) > 0_k$ and $\bar{x}_0 - x^* > \Theta$ imply $\bar{x}_1 - x^* > \Theta$. By induction we get $\bar{x}_n - x^* > \Theta$ for all natural n.
Using the equality

$$\bar{x}_{n+1} = x^* + \frac{1}{(k-1)!} \int_{x^*}^{\bar{x}_n} \Psi^{(k)}(\tilde{x})(\bar{x}_n - \tilde{x})^{k-1}\, d\tilde{x}$$

which results also from (4), we get

$$\bar{x}_{n+1} - \bar{x}_n = -\frac{1}{(k-1)!} \int_{x^*}^{\bar{x}_{n-1}} \Psi^{(k)}(\tilde{x})(\bar{x}_{n-1} - \tilde{x})^{k-1}\, d\tilde{x} +$$

$$+ \frac{1}{(k-1)!} \int_{x^*}^{\bar{x}_n} \Psi^{(k)}(\tilde{x})(\bar{x}_n - \tilde{x})^{k-1}\, d\tilde{x} = \frac{1}{(k-1)!} \int_{x^*}^{\bar{x}_n} [\Psi^{(k)}(\tilde{x})(\bar{x}_n - \tilde{x})^{k-1} -$$

$$- \Psi^{(k)}(\tilde{x})(\bar{x}_{n-1} - \tilde{x})^{k-1}]\, d\tilde{x} - \frac{1}{(k-1)!} \int_{\bar{x}_n}^{\bar{x}_{n-1}} \Psi^{(k)}(\tilde{x})(\bar{x}_{n-1} - \tilde{x})^{k-1}\, d\tilde{x} \leq \Theta .$$

Similarly we obtain $x^* - \underline{x}_n \geq \Theta$ and $\underline{x}_{n+1} - \underline{x}_n \geq \Theta$.

We mention that we can establish a similar theorem also for $k = 2\nu$ by imposing the condition $\Psi^{(k)}(x) < 0_k$ instead of $\Psi^{(k)}(x) < O_k$. It can also be mentioned here that the theorem proved above is true for the case of $\tilde{\Psi}(x)$.

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