SPLINE FUNCTIONS AND THE CAUCHY PROBLEMS, V.

APPLICATION WITH PROGRAMS TO THE METHOD

By

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(Received November 26, 1976)

We have written a program for the ICL 1900 (ODRA 1304) Computer in FORTRAN IV for the method described in [2], [3], [4] and [5] to solve the following examples:

Example 1.

As an application to the approximate solution of the first order differential equation $y' = f(x, y)$, we choose the case when $f \in C^0$ and for this purpose we consider the differential equation

$$y' = 1.5\, x^{1/2} y \quad \text{with} \quad y(0) = 1\,.$$

The exact solution is obviously

$$y(x) = e^{x^{1.5}}\,.$$

Equation (2.1.5), presented in [4], gives the approximate value of $y(x)$ at x_{k+1} and it is as follows:

$$\bar{y}_{k+1} = \bar{y}_k + (\bar{y}_k - x_k \bar{y}'_k)(x_{k+1}^{1.5} - x_k^{1.5}) + 0.6\, \bar{y}'_k (x_{k+1}^{2.5} - x_k^{2.5})$$

and the formula (2.2.9), presented in [4], gives

$$\bar{y}'_{k+1} = f(x_{k+1}, \bar{y}_{k+1}) = 1.5\, x_{k+1} \bar{y}_{k+1}$$

The choosen interval is $0 \leqq x \leqq 1$ and the step h has been choosen to be 0.001.

The flow chart diagram and the program giving the numerical and exact results are presented in flowchart 1, program 1 and results 1.

Example 2.

As an application to the approximate solution of the second order differential equation $y'' = f(x, y, y')$ we consider in this example the case when $f \in C^0$ and so we consider the differential equation

$$y'' = \left(\frac{25}{4}x^3 + \frac{15}{4}x^{0.5}\right)y \quad \text{with } y(0) = 1 \text{ and } y'(0) = 0$$

and in this case the choosen interval is $0 \leqq x \leqq 1$.
Obviously the exact solution is

$$y = e^{x^{2.5}}.$$

Equations (2.1.11), (2.1.12) and (2.1.13), presented in [3], give the approximate values of $y(x)$, $y'(x)$ and $y''(x)$ at $x = x_{k+1}$ and they are as follow:

$$\bar{y}_{k+1} = \bar{y}_k + \bar{y}'_k h + A_k(G_{k+1} - G_k) + B_k(Q_{k+1} - Q_k) + C_k(T_{k+1} - T_k) - h(A_k D_k + B_k E_k + C_k F_k),$$

$$\bar{y}'_{k+1} = \bar{y}'_k + A_k(D_{k+1} - D_k) + B_k(E_{k+1} - E_k) + C_k(F_{k+1} - F_k)$$

and

$$\bar{y}''_{k+1} = \left(\frac{25}{4}x^3_{k+1} + \frac{15}{4}x^{0.5}_{k+1}\right)\bar{y}_{k+1}$$

where,

$$A_k = \bar{y}_k - x_k\bar{y}'_k + 0.5\,x_k^2\bar{y}''_k \qquad \text{with } A_0 = 1,$$

$$B_k = \bar{y}'_k - x_k\bar{y}''_k \qquad \text{with } B_0 = 0,$$

$$C_k = 0.5\,\bar{y}''_k \qquad \text{with } C_0 = 0,$$

$$D_k = \frac{25}{16}x_k^4 + \frac{5}{2}x_k^{1.5} \qquad \text{with } D_0 = 0,$$

$$E_k = \frac{5}{4}x_k^5 + 1.5\,x_k^{2.5} \qquad \text{with } E_0 = 0,$$

$$F_k = \frac{25}{24}x_k^6 + \frac{15}{14}x_k^{3.5} \qquad \text{with } F_0 = 0,$$

$$G_k = \frac{5}{16}x_k^5 + x_k^{2.5} \qquad \text{with } G_0 = 0,$$

$$Q_k = \frac{5}{24}x_k^6 + \frac{3}{7}x_k^{3.5} \qquad \text{with } Q_0 = 0 \text{ and}$$

$$T_k = \frac{25}{128}x_k^7 + \frac{15}{63}x_k^{4.5} \qquad \text{with } T_0 = 0.$$

The flow chart diagram, the program and the results are given in flowchart 2, program 2 and results 2. In this example two numerical results are given in order to show the influence of h on the results.

Example 3.

As an application to the case $f \in C^\infty$, we solve the differential equation

$$y'' = y \qquad \text{with } y(0) = 2 \qquad \text{and } y'(0) = 0.$$

Obviously $f \in C^\infty$ implies $f \in C^r$ where $r \in I^+$.
Our interval will be $0 \leq x \leq 1$ and the method is applied to give the approximate values as follow:

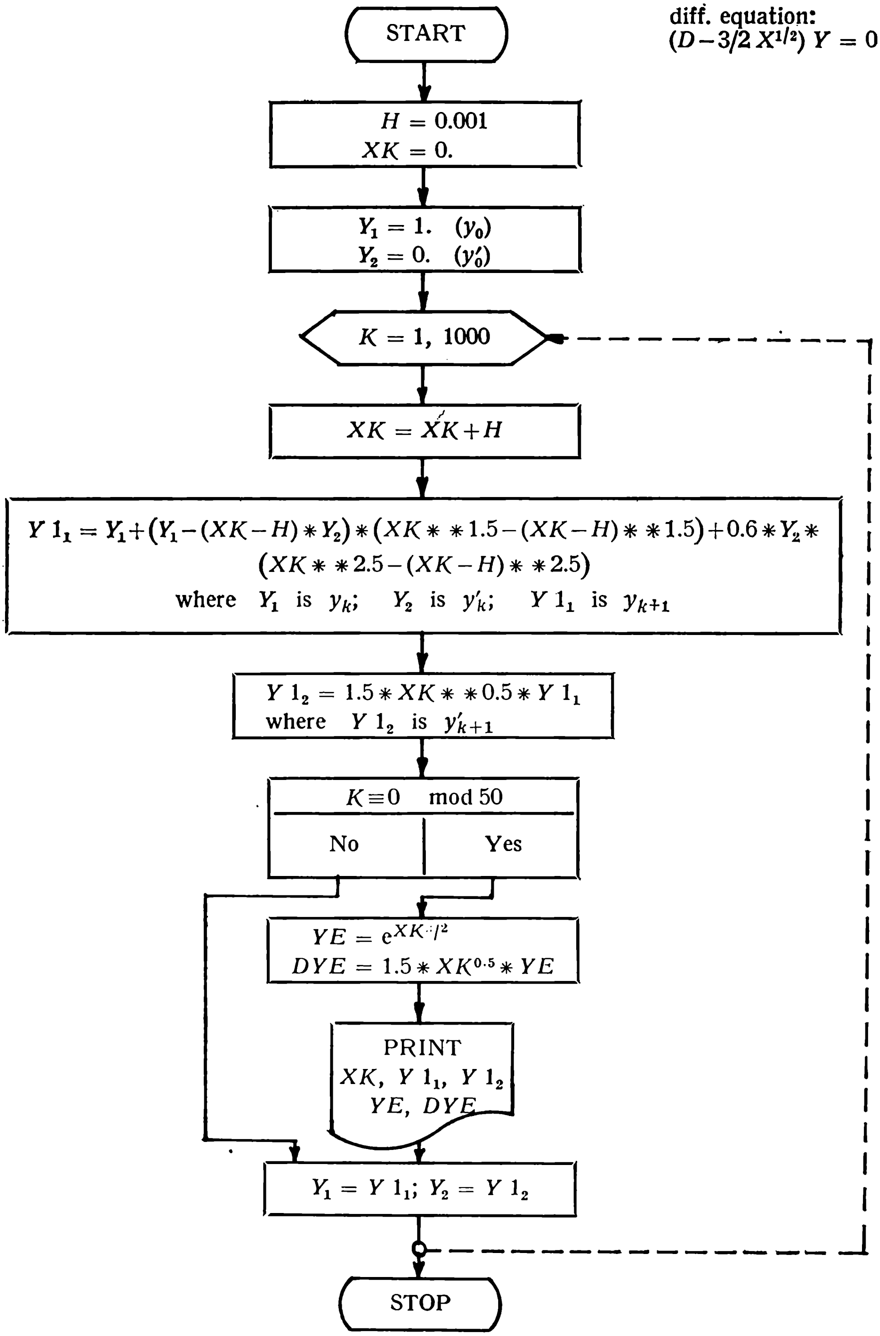
$$\bar{y}_{k+1} = \bar{y}_k + \bar{y}'_k h + \sum_{j=0}^{r+2} \frac{y_k}{(j+2)!} h^{j+2},$$

$$\bar{y}'_{k+1} = \bar{y}'_k + \sum_{j=0}^{r+2} \frac{\bar{y}_k^{(j)}}{(j+1)!} h^{j+1} \quad \text{and}$$

$$\bar{y}''_{k+1} = \bar{y}_{k+1}.$$

The flow chart diagram, program and results are presented in flowchart 3, program 3./a, program 3./b, results 3./a and results 3./b. In the first case 3./a we have choosen r to be $r = 2$. In the second case 3./b r is equal to zero. If r is larger than 2, the results will be more accurate as it was shown in the convergence theorem, (see reference [3]), the error is $O(h^{r+2})$.

Flowchart 1.


diff. equation:
(D−3/2 X1/2) Y = 0
START
H = 0.001
XK = 0.
Y1 = 1. (y0)
Y2 = 0. (y'0)
K = 1, 1000
XK = XK+H
Y 11 = Y1+(Y1−(XK−H)∗Y2)∗(XK∗∗1.5−(XK−H)∗∗1.5)+0.6∗Y2∗
(XK∗∗2.5−(XK−H)∗∗2.5)
where Y1 is yk; Y2 is y'k; Y 11 is yk+1
Y 12 = 1.5∗XK∗∗0.5∗Y 11
where Y 12 is y'k+1
K≡0 mod 50
No
Yes
YE = eXK3/2
DYE = 1.5∗XK0.5∗YE
PRINT
XK, Y 11, Y 12
YE, DYE
Y1 = Y 11; Y2 = Y 12
STOP

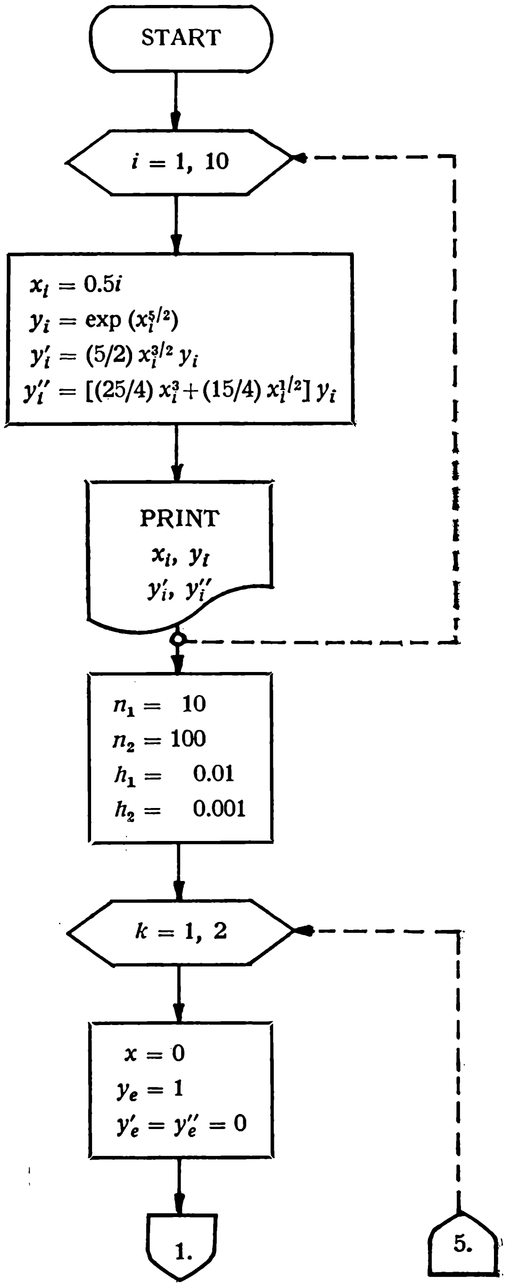
```
      MASTER SPLINE
C**** APPROXIMATE SOLUTION OF THE DIFF. EQUATION
C**** (D-1.5*X**0.5)* Y=0
C**** BY SPLINE FUNCTIONS TECHHIQUE
      DIMENSION Y(2),Y1(2)
      WRITE (2,101)
  101 FORMAT(1H1)
      H=0.001
      XK=0.
      Y(1)=1.
      Y(2)=0.
      DO 1 K=1,1000
      XK=XK+H
      Y1(1)=Y(1)+(Y(1)-(XK-H)*Y(2))*(XK**1.5-(XK-H)**1.5)+0.6*Y(2)*
     1(XK**2.5-(XK-H)**2.5)
      Y1(2)=1.5*XK**0.5*Y1(1)
      IF (K-K/50*50) 2,0,2
      YE=EXP(XK**1.5)
      DYE=1.5*XK**0.5*YE
      WRITE (2,100) XK,Y1(1),Y1(2).YE,DYE
  100 FORMAT(1H0,12HNUM. RESULT,2X,2HX=,F4.2,2X,2HY=,E18.10,2X,
     13HDY=, E18.10/13H EXACT RESULT,12X,E18.10,5X,E18.10)
    2 Y(1)=Y1(1)
    1 Y(2)=Y1(2)
      STOP
      END
```

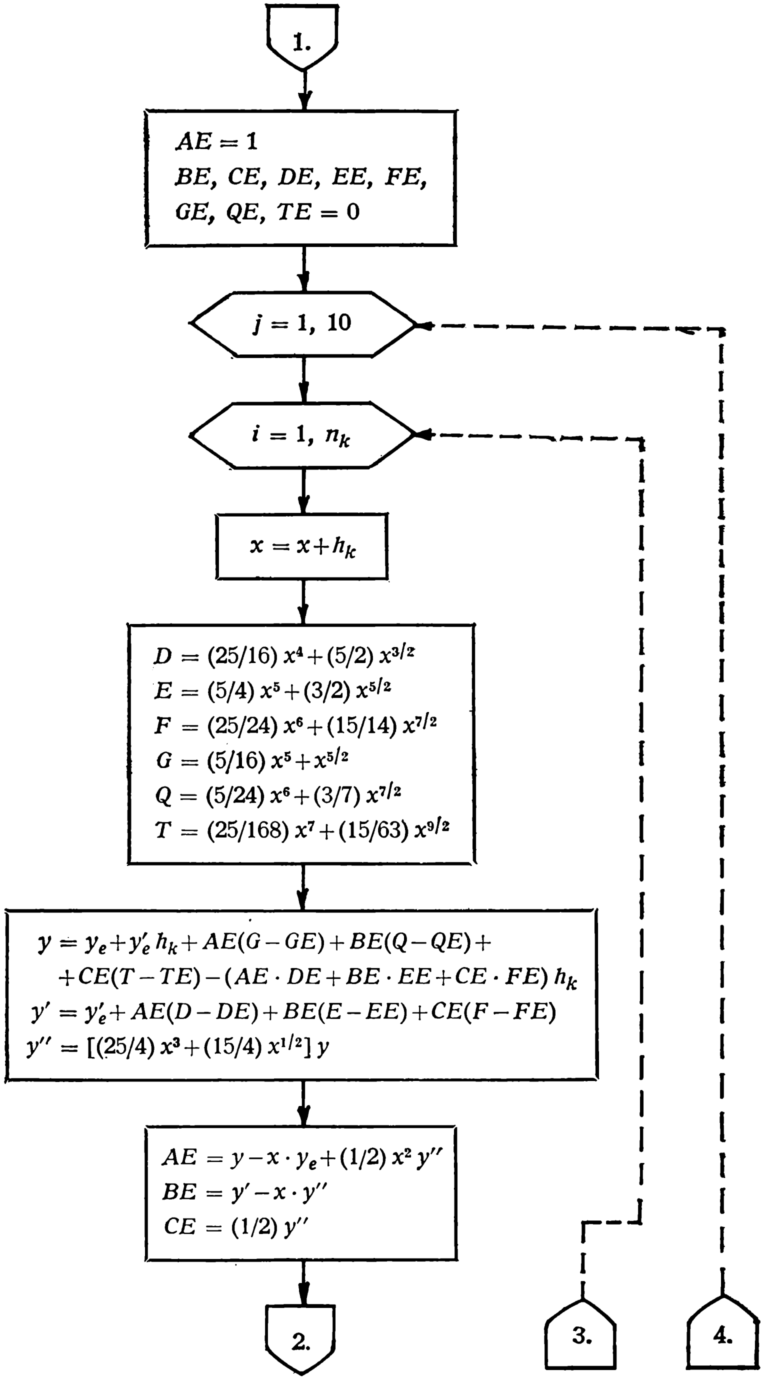
Program 1.

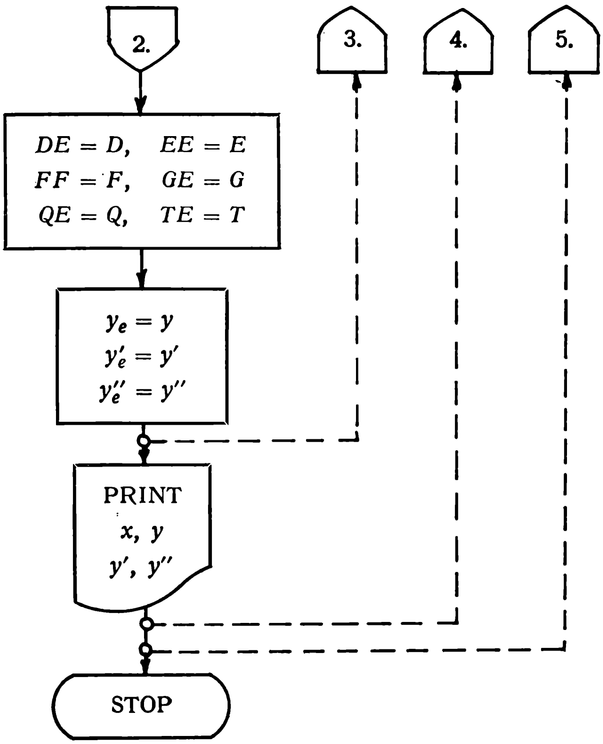
NUM.	RESULT	X=0.05	Y=	0.1011243063E 00	DY=	0.3391812347E 00
EXACT	RESULT			0.1011243073E 01		0.3391812381E 00
NUM.	RESULT	X=0.10	Y=	0.1032128068E 00	DY=	0.4895813300E 00
EXACT	RESULT			0.1032128089E 01		0.4895133988E 00
NUM.	RESULT	X=0.15	Y=	0.1059815376E 00	DY=	0.6156970954E 00
EXACT	RESULT			0.1059815409E 01		0.6156971142E 00
NUM.	RESULT	X=0.20	Y=	0.1093564645E 00	DY=	0.7335854654E 00
EXACT	RESULT			0.1093564691E 01		0.7335854962E 00
NUM.	RESULT	X=0.25	Y=	0.1133148391E 00	DY=	0.8498612936E 00
EXACT	RESULT			0.1133148453E 01		0.8498613399E 00
NUM.	RESILT	X=0.30	Y=	0.1178587514E 00	DY=	0.9683084511E 00
EXACT	RESULT			0.1178587594E 01		0.9683085169E 00
NUM.	RESULT	X=0.35	Y=	0.1230059706E 00	DY=	0.1091569704E 00
EXACT	RESULT			0.1230059808E 01		0.1091569794E 01
NUM.	RESULT	X=0.40	Y=	0.1287860243E 00	DY=	0.1221771503E 00
EXACT	RESULT			0.1287860369E 01		0.1221771623E 01
NUM.	RESULT	X=0.45	Y=	0.1352384136E 00	DY=	0.1360810287E 00
EXACT	RESULT			0.1352384292E 01		0.1360810444E 01
NUM.	RESULT	X=0.50	Y=	0.1424118829E 00	DY=	0.1510506122E 00
EXACT	RESULT			0.1424119020E 01		0.1510506324E 01
NUM.	RESULT	X=0.55	Y=	0.1503642898E 00	DY=	0.1672697128E 00
EXACT	RESULT			0.1503643130E 01		0.1672697386E 01
NUM.	RESULT	X=0.60	Y=	0.1591628690E 00	DY=	0.1849305423E 00
EXACT	RESULT			0.1591628970E 01		0.1849305748E 01
NUM.	RESULT	X=0.65	Y=	0.1688847855E 00	DY=	0.2042389006E 00
EXACT	RESULT			0.168848192E 01		0.2042389412E 01
NUM.	RESULT	X=0.70	Y=	0.1796179293E 00	DY=	0.2254187121E 00
EXACT	RESULT			0.1796179695E 01		0.2254187626E 01
NUM.	RESULT	X=0.75	Y=	0.1914619295E 00	DY=	0.2487163420E 00
EXACT	RESULT			0.1914619775E 01		0.2487164044E 01
NUM.	RESULT	X=0.80	Y=	0.2045293852E 00	DY=	0.2744049651E 00
EXACT	RESULT			0.2045294424E 01		0.2744050417E 01
NUM.	RESULT	X=0.85	Y=	0.2189473202E 00	DY=	0.3027891826E 00
EXACT	RESULT			0.2189473880E 01		0.3027892764E 01
NUM.	RESULT	X=0.90	Y=	0.2348588770E 00	DY=	0.3342100408E 00
EXACT	RESULT			0.2348589573E 01		0.3342101551E 01
NUM.	RESULT	X=0.95	Y=	0.2524252765E 00	DY=	0.3690505883E 00
EXACT	RESULT			0.2524253715E 01		0.3690507272E 01
NUM.	RESULT	X=1.00	Y=	0.2718280700E 00	DY=	0.4077421046E 00
EXACT	RESULT			0.2718281822E 01		0.4077422729E 01

Result 1.

Flowchart 2.


START
i = 1, 10
x_i = 0.5i
y_i = exp (x_i^{5/2})
y'_i = (5/2) x_i^{3/2} y_i
y''_i = [(25/4) x_i^3 + (15/4) x_i^{1/2}] y_i
PRINT
x_i, y_i
y'_i, y''_i
n_1 = 10
n_2 = 100
h_1 = 0.01
h_2 = 0.001
k = 1, 2
x = 0
y_e = 1
y'_e = y''_e = 0
1.
5.


1.
$AE = 1$
$BE, CE, DE, EE, FE, GE, QE, TE = 0$
$j = 1, 10$
$i = 1, n_k$
$x = x + h_k$
$D = (25/16)\,x^4 + (5/2)\,x^{3/2}$
$E = (5/4)\,x^5 + (3/2)\,x^{5/2}$
$F = (25/24)\,x^6 + (15/14)\,x^{7/2}$
$G = (5/16)\,x^5 + x^{5/2}$
$Q = (5/24)\,x^6 + (3/7)\,x^{7/2}$
$T = (25/168)\,x^7 + (15/63)\,x^{9/2}$
$y = y_e + y'_e h_k + AE(G - GE) + BE(Q - QE) + CE(T - TE) - (AE \cdot DE + BE \cdot EE + CE \cdot FE)\,h_k$
$y' = y'_e + AE(D - DE) + BE(E - EE) + CE(F - FE)$
$y'' = [(25/4)\,x^3 + (15/4)\,x^{1/2}]\,y$
$AE = y - x \cdot y_e + (1/2)\,x^2 y''$
$BE = y' - x \cdot y''$
$CE = (1/2)\,y''$
2.
3.
4.


2.
3.
4.
5.
DE = D, EE = E
FF = F, GE = G
QE = Q, TE = T
y_e = y
y'_e = y'
y''_e = y''
PRINT
x, y
y', y''
STOP

```
      MASTER HEAD
      DIMENSION XZ(10),Z(10),Z1(10),Z2(10),N(2),H(2),ISTR(7,3)
      DATA ISTR(1,1)/28H       EXACT VALUES           /
      DATA ISTR(1,2)/28HNUMERICAL VALUES, H=0.01  /
      DATA ISTR(1,3)/28HNUMERICAL VALUES, H=0.001/
      H(1)=0.01
      H(2)=0.001
      N(1)=10
      N(2)=100
      CALL PRINT1
      CALL EXACT(XZ,Z,Z1,Z2)
      CALL PRINT2(XZ,Z,Z1,Z2,1,ISTR)
      DO 1 I=1,2
      CALL APPROX(XZ,Z1,Z2,H(I),N(I))
    1 CALL PRINT2(XZ,Z,Z1,Z2,I+1,ISTR)
      STOP
      END
      SUBROUTINE PRINT1
      WRITE(2,100)
  100 FORMAT(1H1//20X,62HAPPROXIMATE SOLUTION OF D. E.
     1(D**2-4.25*X**3-3.75*X**0.5)*Y=0/41X,19HBY SPLINE
     2FUNCTIONS/)
      RETURN
      END
      SUBROUTINE PRINT2(XZ,Z,Z1,Z2,I,ISTRING)
      DIMENSION XZ(10),Z(10),Z1(10),Z2(10),ISTRING(7,3)
      WRITE(2,100)(ISTRING(J,I),J=1,7)
  100 FORMAT(///,30X,7A4/20X,61(1H-)/
     120X,7HI   X   I,7X,4HY(X),6X,1HI,6X,5HY'(X),6X,1HI,
     26X,6HY"(X),5X,1HI/20X,61(1H-))
      DO 1 J=1,10
    1 WRITE(2,101) XZ(J),Z(J),Z1(J),Z2(J)
  101 FORMAT(20X,1HI,1X,F3.1,1X,1HI,3(1X,F15.12,1X,1HI))
      WRITE(2,102)
  102 FORMAT(20X,61(1H-))
      RETURN
      END
      SUBROUTINE APPROX(XZ,Z,Z1,Z2,H,N)
      DIMENSION XZ(10),Z(10),Z1(10),Z2(10)
      YE=1
      Y1E,Y2E,X=0
      AE=1
      BE,CE,DE,EE,FE,GE,QE,TE=0
      DO 2 J=1,10
      DO 1 I=1,N
      X=X+H
      D=(25./16.)*X**4+2.5*X**1.5
      E=1.25*X**5+1.5*X**2.5
      F=(25./24.)*X**6*(15./14.)*X**3.5
      G=(5./16.)*X**5+X**2.5
      Q=(5./24.)*X**6+(3./7.)*X**3.5
      T=(25./168.)*X**7+(15./63.)*X**4.5
      Y=YE+Y1E*H+AE*(G-GE)+BE*(Q-QE)+CE*(T-TE)
     1-(AE*DE+BE*EE+CE*FE)*H
      Y1=Y1E+AE*(D-DE)+BE*(E-EE)+CE*(F-FE)
      Y2=(6.25*X**3+3.75*SQRT(X))*Y
      AE=Y-X*Y1+0.5*X*X*Y2
      BE=Y1-X*Y2
```

```
      CE=0.5*Y2
      DE=D
      EE=E
      FE=F
      GE=G
      QE=Q
      TE=T
      YE=Y
      Y1E=Y1
      Y1E=Y1
    1 Y2E=Y2
      XZ(J)=X
      Z(J)=Y
      Z1(J)=Y1
      Z2(J)=Y2
    2 CONTINUE
      RETURN
      END
      SUBROUTINE EXACT(XZ,Z,Z1,Z2)
      DIMENSION XZ(10),Z(10),Z1(10),Z2(10)
      DO 1 I=1,10
      XZ(I)=I*0.1
      X=XZ(I)
      Z(I)=EXP(X**2.5)
      Z1(I)=2,5*Z(I)*X**1.5
    1 Z2(I)=Z(I)*(6.25*X**3+3.75*SQRT(X))
      RETURN
      END
```

Program 2.

APPROXIMATE SOLUTION OF D. E.
(D * * 2 – 4.25 * X * * 3 – 3.75 * X * * 0.5) * Y = 0
BY SPLINE FUNCTIONS

EXACT VALUES

X	Y(X)	Y′ (X)	Y″ (X)
0.1	1.003167282935	0.079307337208	1.195879853621
0.2	1.018049502162	0.227642789125	1.758223393582
0.3	1.050530243140	0.431549333640	2.335023646708
0.4	1.106490046658	0.699805751159	3.066867585497
0.5	1.193364759448	1.054795233197	4.096701777248
0.6	1.321615432606	1.535578368202	5.623126754535
0.7	1.506762656208	2.206134146894	7.957552758975
0.8	1.772575196053	3.170878906850	11.617638577939
0.9	2.156385566864	4.602885687435	17.496507881904
1.0	2.718281828477	6.795704571179	27.182818284721

NUMERICAL VALUES, H = 0.01

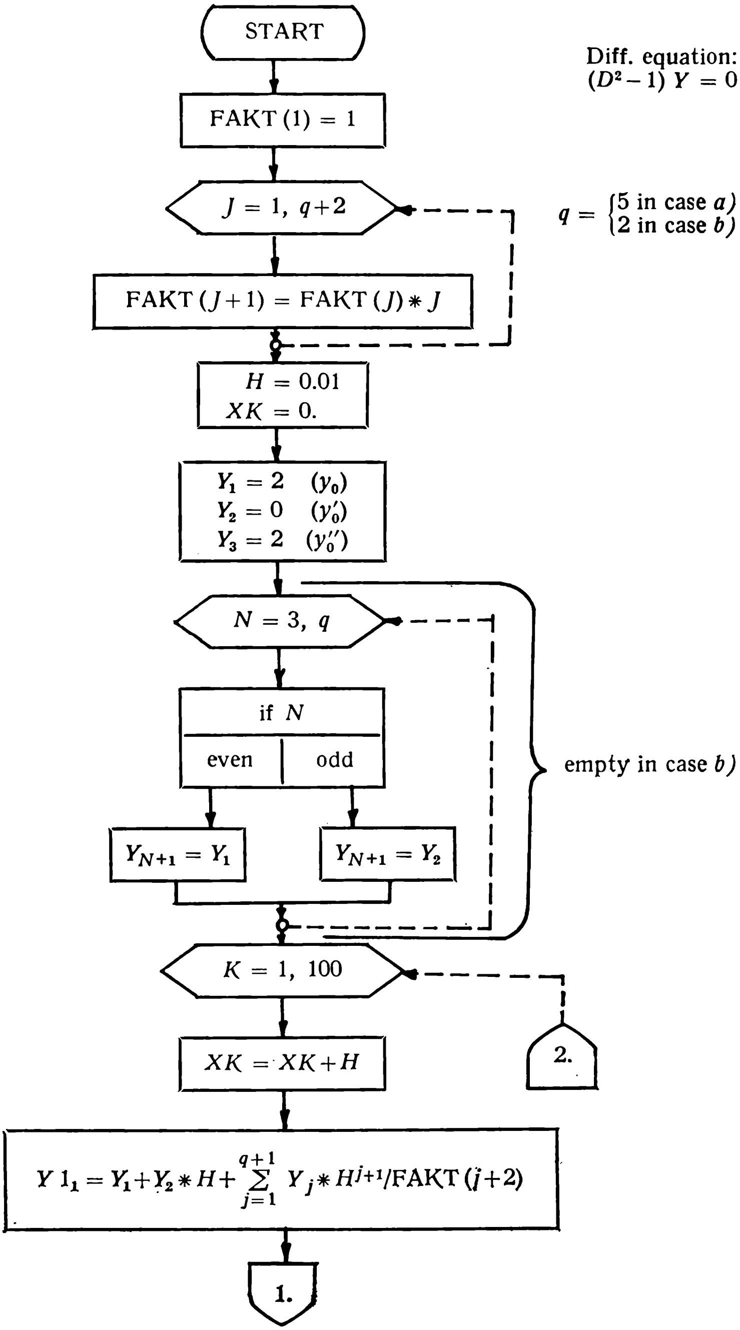
X	Y(X)	Y′ (X)	Y″ (X)
0.1	1.003167280854	0.079307298794	1.195879851132
0.2	1.018049494625	0.227642715237	1.758223380573
0.3	1.050530226027	0.431549209628	2.335023608640
0.4	1.106490013582	0.699805544769	3.066867493674
0.5	1.193364519494	1.054794879717	4.096701571134
0.6	1.321615325708	1.535577739501	5.623126299293
0.7	1.506762464239	2.206132986233	7.957551744998
0.8	1.772574844115	3.170876689953	11.617636271517
0.9	2.156384903849	4.602881316327	17.496502503749
1.0	2.718280542117	6.795695703533	27.182805424317

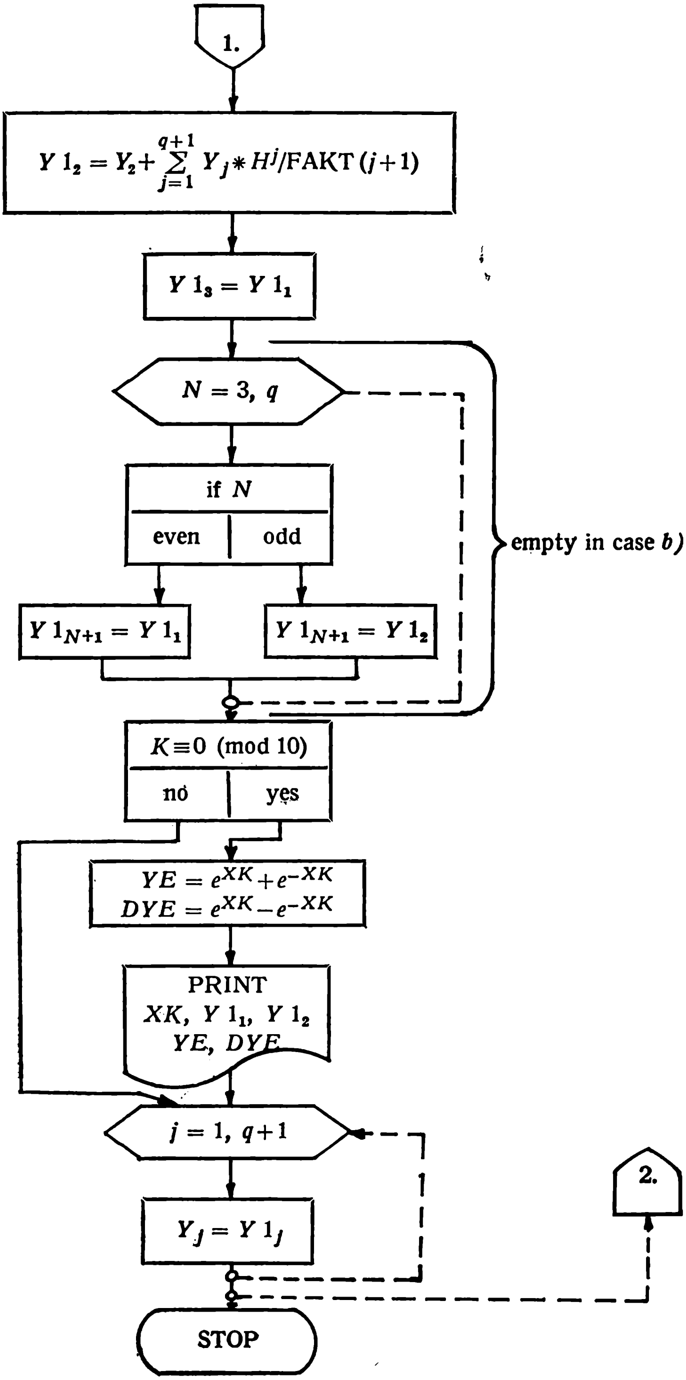
NUMERICAL VALUES, H = 0.001

X	Y(X)	Y′ (X)	Y″ (X)
0.1	1.003167282833	0.079307337187	1.195879853563
0.2	1.018049502133	0.227642789112	1.758223393698
0.3	1.050530243039	0.431549333622	2.335023646766
0.4	1.106490046571	0.699805751116	3.066867585729
0.5	1.193364579259	1.054795233154	4.096701777481
0.6	1.321615431661	1.535578366106	5.623126747317
0.7	1.506762654098	2.206134140578	7.957552737555
0.8	1.772575191861	3.170878892938	11.617638526949
0.9	2.156385559093	4.602885657281	17.496507771542
1.0	2.718281813780	6.795704515590	27.182818046767

Result 2.

Flowchart 3.


START
Diff. equation:
$(D^2-1)\,Y=0$
FAKT (1) = 1
$J = 1,\ q+2$
$q = \begin{cases} 5 \text{ in case } a) \\ 2 \text{ in case } b) \end{cases}$
FAKT $(J+1)$ = FAKT $(J) * J$
$H = 0.01$
$XK = 0.$
$Y_1 = 2 \quad (y_0)$
$Y_2 = 0 \quad (y_0')$
$Y_3 = 2 \quad (y_0'')$
$N = 3,\ q$
if N
even
odd
$Y_{N+1} = Y_1$
$Y_{N+1} = Y_2$
empty in case $b)$
$K = 1,\ 100$
2.
$XK = XK+H$
$Y\,1_1 = Y_1 + Y_2 * H + \sum_{j=1}^{q+1} Y_j * H^{j+1}/\text{FAKT}\,(j+2)$
1.


1.
Y 1₂ = Y₂ + Σ_{j=1}^{q+1} Y_j * H^j/FAKT (j+1)
Y 1₃ = Y 1₁
N = 3, q
if N
even
odd
Y 1_{N+1} = Y 1₁
Y 1_{N+1} = Y 1₂
empty in case b)
K ≡ 0 (mod 10)
no
yes
YE = e^{XK} + e^{-XK}
DYE = e^{XK} − e^{-XK}
PRINT
XK, Y 1₁, Y 1₂
YE, DYE
j = 1, q+1
Y_j = Y 1_j
2.
STOP

```
      MASTER SPLINE
C**** APPROXIMATE SOLUTION OF THE DIFF. EQATION (D**2-1)*Y=0
C**** BY SPLINE FUNCTIONS TECHNIQUE.
      REAL FAKT(8),Y(6).Y1(6)
      WRITE (2,101)
  101 FORMAT(1H1)
      FAKT(1)=1.
      DO 8 J=1,7
    8 FAKT(J+1)=FAKT(J)*J
      H=0.01
      XK=0.
      Y(1)=2
      Y(2)=0
      Y(3)=2
      DO 200 N=3,5
      IF (N-N/2*2.EQ.0) Y(N+1)=Y(1)
      IF (N-N/2*2.NE.0) Y(N+1)=Y(2)
  200 CONTINUE
      DO 6 K=1,100
      XK=XK+H
      S=0.
      DO 201 J=1,6
  201 S=S+Y(J)*H**(J+1)/FAKT(J+2)
      Y1(1)=Y(1)+Y(2)*H+S
      S=0.
      DO 202 J=1,6
  202 S=S+Y(J)*H**J/FAKT(J+1)
      Y1(2)=Y(2)+S
      Y1(3)=Y(1)
      DO 203 N=3,5
      IF (N-N/2*2.EQ.0)Y1(N+1)=Y1(1)
      IF (N-N/2*2.NE.0)Y1(N+1)=Y1(2)
  203 CONTINUE
      IF (K-K/10*10) 5,0,5
      YE=EXP(XK)+EXP(-XK)
      DYE=EXP(XK)-EXP(-XK)
      WRITE (2,100) XK,Y1(1),Y1(2),YE,DYE
  100 FORMAT(1H0,12HNUM. RESULT,2X,2HX=,F4.2,2X,2HY=,D18.10,2X,
     13HDY=,D18.10/13H EXACT RESULT,12X,D18.10,5X,D18.10)
    5 DO 6 J=1,6
    6 Y(J)=Y1(J)
      STOP
      END
```

Program 3./a

NUM.	RESULT	X=0.10	Y= 0.2010008336E 01	DY=	0.2002335000E 00
EXACT	RESULT		0.2010008336E 01		0.2003335000E 00
NUM.	RESULT	X=0.20	Y= 0.2040133511E 01	DY=	0.4026720051E 00
EXACT	RESULT		0.2040133511E 01		0.4026720051E 00
NUM.	RESULT	X=0.30	Y= 0.2090677028E 01	DY=	0.6090405869E 00
EXACT	RESULT		0.2090677028E 01		0.6090405869E 00
NUM.	RESULT	X=0.40	Y= 0.2162144744E 01	DY=	0.8215046516E 00
EXACT	RESULT		0.2162144744E 01		0.8215046516E 00
NUM.	RESULT	X=0.50	Y= 0.2255251930E 01	DY=	0.1042190611E 01
EXACT	RESULT		0.2255251930E 01		0.1042190611E 01
NUM.	RESULT	X=0.60	Y= 0.2370930437E 01	DY=	0.1273307164E 01
EXACT	RESULT		0.2370930436E 01		0.1273307164E 01
NUM.	RESULT	X=0.70	Y= 0.2510338011E 01	DY=	0.1517167404E 01
EXACT	RESULT		0.2510338011E 01		0.1517167404E 01
NUM.	RESULT	Y=0.80	Y= 0.2674869893E 01	DY=	0.1776211964E 01
EXACT	R3SULT		0.2674869893E 01		0.1776211964E 01
NUM.	RESULT	X=0.90	Y= 0.2866172771E 01	DY=	0.2053033451E 01
EXACT	RESULT		0.2866172771E 01		0.2053033452E 01
NUM.	RESULT	X=1.00	Y= 0.3086161270E 01	DY=	0.2350402287E 01
EXACT	RESULT		0.3086161270E 01		0.2350402387E 01

Result 3./a

```
      MASTER SPLINE
C**** APPROXIMATE SOLUTION OF THE DIIFF. EQUATION (D**2-1)*Y=0
C**** BYSPLIINE FUNCTIONS TECHNIQUE.
      REAL FAKT(5),Y(3),Y1(3)
      WRITE (2,101)
  101 FORMAT(1H1)
      FAKT(1)=1.
      DO 8 J=1,4
    8 FAKT(J+1)=FAKT(J)*J
      H=0.01
      XK=0.
      Y(1)=2
      Y(2)=0
      Y(3)=2
      DO 6 K=1,100
      XK=XK+H
      S=0.
      DO 201 J=1,3
  201 S=S+Y(J)*H**(J+1)/FAKT(J+2)
      Y1(1)=Y(1)+Y(2)*H+S
      S=0.
      DO 202 J=1,3
  202 S=S+Y(J)*N**J/FAKT(J+1)
      Y1(2)=Y(2)+S
      Y1(3)=Y1(1)
      IF(K-K/10*10) 5,0,5
      YE=EXP(XK)+EXP(-XK)
      DYE=EXP(XK)-EXP(-XK)
      WRITE (2,100) XK,Y1(1),Y1(2),YE,DYE
  100 FORMAT (1H0,12HNUM. RESULT;2X,2HX=,F4.2,2X,2HY=,D18.10,2X,
     13HDY=,D18.10/13H EXACT RESULT,12X,D18.10,5X,D18.10)
    5 DO 6 J=1,3
    6 Y(J)=Y1(J)
      STOP
      END
```

Program 3./b

NUM.	RESULT	X=0.10	Y=	0.2010008336E 01	DY=	0.2003334996E 00
EXACT	RESULT			0.2010008336E 01		0.2003335000E 00
NUM.	RESULT	X=0.20	Y=	0.2040133511E 01	DY=	0.4026720035E 00
EXACT	RESULT			0.2040133511E 01		0.4026720051E 00
NUM.	RESULT	X=0.30	Y=	0.2090677028E 01	DY=	0.6090405832E 00
EXACT	RESULT			0.2090677028E 01		0.6090405869E 00
NUM.	RESULT	X=0.40	Y=	0.2162144743E 01	DY=	0.8215046449E 00
EXACT	RESULT			0.2162144744E 01		0.8215046516E 00
NUM.	RESULT	X=0.50	Y=	0.2255251929E 01	DY=	0.1042190600E 00
EXACT	RESULT			0.2255251930E 01		0.1042190611E 01
NUM.	RESULT	X=0.60	Y=	0.2370930434E 01	DY=	0.1273307149E 00
EXACT	RESULT			0.2370930436E 01		0.1273307164E 01
NUM.	RESULT	X=0.70	Y=	0.2510338006E 01	DY=	0.1517167382E 00
EXACT	RESULT			0.2510338011E 01		0.1517167404E 01
NUM.	RESULT	X=0.80	Y=	0.2674869885E 01	DY=	0.1776211935E 00
EXACT	RESULT			0.2674869893E 01		0.1776211964E 01
NUM.	RESULT	X=0.90	Y=	0.2866172760E 01	DY=	0.2053033413E 00
EXACT	RESULT			0.2866172771E 01		0.2053033452E 01
NUM.	RESULT	X=1.00	Y=	0.3086161255E 01	DY=	0.2350402339E 00
EXACT	RESULT			0.3086161270E 01		0.2350402387E 01

Result 3./b

Acknowledgement: The authors wish to thank Professor János Balázs for many helpful and stimulating discussions concerning this work. The authors are also very gratefull to the Numerical and Computing Mathematics Department in The Eötvös Loránd University of Science in Budapest for achieving the programs in its computing centre.

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