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A NOTE ON THE SCREW LINE FUNCTION ON FUNCTION FIELDS

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Abstract. In this paper, we study the screw line function on function fields and we give special values of its norm. Furthermore, we derive some interesting summation formulas.

1. Introduction

1.1. Background

The famous Riemann zeta function $\zeta(s)$ is a function of a complex variable $s = \sigma + it$, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for $\Re(s) = \sigma > 1$. This function can be extended meromorphically to the entire complex plane $\mathbb{C}$. The zeta function $\zeta(s)$ is holomorphic everywhere except for a simple pole at $s = 1$. The Riemann ξ-function is defined as:

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{s/2}\Gamma(s/2)\zeta(s),$$

and satisfies two functional equations $\xi(s) = \xi(1-s)$ and $\xi(s) = \overline{\xi(\overline{s})}$, where $\Gamma(s)$ is the gamma-function and the bar denotes the complex conjugate. The Riemann Hypothesis (RH, for short), a famous open problem, claims that all non-real zeros of $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$ which is equivalent to all the zeros of $\xi(1/2 - iz)$ being real.

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In [4], Kreîn introduced the class $\mathcal{G}_a$ which is called the screw functions consisting of all continuous functions, $g(t)$ on the interval $(-2a, 2a)$ for $0 < < a \leq \infty$ such that $g(-t) = \overline{g(t)}$ and the kernel function

$$G_g(t, s) = g(t - s) - g(t) - g(-s) + g(0),$$

is nonnegative definite on $(-a, a)$, i.e.,

$$\sum_{i,j=1}^{n} G_g(t_i, t_j)\xi_i\overline{\xi_j} \geq 0 \tag{1.1}$$

for all $n \in \mathbb{N}$, $\xi_i \in \mathbb{C}$ and $|t_i| < a$ $(i = 1, 2, ..., n)$. In literature, a kernel satisfying (1.1) is often referred to as a positive definite kernel or semi-positive definite kernel. However, in this note, we refer to such a kernel as a non-negative definite kernel. If $g(t)$ is a screw function, then there exists a Hilbert space $\mathcal{H}$ and a continuous mapping $t \to x(t)$ from $\mathbb{R}$ into $\mathcal{H}$ such that the inner product

$$\langle x(t + u) - x(u), x(s + u) - x(u)\rangle_{\mathcal{H}}$$

is independent of $u \in \mathbb{R}$ for all $t, s \in \mathbb{R}$. Moreover, the equality

$$\langle x(t) - x(0), x(s) - x(0)\rangle_{\mathcal{H}} = G_g(t, s) \tag{1.2}$$

holds. Consequently,

$$\|x(t) - x(0)\|^2_{\mathcal{H}} = -2g(t)$$

under the condition $g(0) = 0$. A mapping $x : \mathbb{R} \to \mathcal{H}$ endowed with the translation-invariance described above is called a screw line. In [7], Suzuki investigated the screw line corresponding to the screw function associated to the Riemann zeta-function.

In the present paper, we extend Suzuki's work (see [7]) to function fields. Additionally, we study special values of the norm for the screw line. Finally, we derive some interesting summation formulas.

Consider a function field K with a finite field of constants $\mathbb{F}_q$ and let X be its smooth projective algebraic curve of genus g defined over $\mathbb{F}_q$. For more details, refer to [5]. The zeta-function of K is defined as follows

$$Z_K(T) = \sum_{n=0}^{+\infty} C_n T^n = \prod_{D \text{ prime}} \left(1 - T^{\deg(D)}\right)^{-1},$$

where $C_n = \#\{D \in Div(K);\ D \geq 0,\ \deg(D) = n\}$; $Z_K(T)$ is actually a rational function

$$Z_K(T) = \frac{L(T)}{(1 - T)(1 - qT)}, \tag{1.3}$$

where $L(T)$ factors in $\mathbb{C}[T]$ as

$$L(T) = \prod_{j=1}^{2g}(1-\alpha_j T) \in \mathbb{Z}[T]. \tag{1.4}$$

The special value $L(1) = \prod_{i=1}^{2g}(1-\alpha_i)$ is the class number of K, denoted by h_K. The complex numbers $\alpha_1, \ldots, \alpha_{2g}$ are algebraic integers and can be arranged so that $\alpha_j\alpha_{g+j} = q$ holds for $j = 1, \ldots, g$. Since the Riemann hypothesis for function fields (abbreviated to RH) proved by Weil [9] states that α_i, $i = 1, \ldots, 2g$ have absolute value $q^{1/2}$, we may order the indices $j \in \{1, \ldots, g\}$ so that $\alpha_{g+j} = \overline{\alpha_j}$, and we then can write $\alpha_j = q^{1/2}\exp(i\theta_j)$ with $\theta_j \in [0, \pi]$.

Now, we define the (classical) zeta function ζ_K of K as follows: for $s \in \mathbb{C}$, we substitute T with q^{-s} in $Z_K(T)$ to get the function

$$\zeta_K(s) = Z_K(q^{-s}) = \sum_{n=0}^{+\infty} C_n q^{-ns},$$

which converges for $\Re(s) > 1$. We define the following completed zeta function

$$\xi_K(s) := q^s(1-q^{-s})(1-q^{1-s})q^{(g-1)s}\zeta_K(s) = q^{gs}L(q^{-s}), \tag{1.5}$$

which is an entire function of order one, whose zeros coincide with the zeros of $\zeta_K(s)$. Moreover, $\xi_K(s)$ satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s).$$

By taking the logarithm and subsequently differentiating both sides of (1.5), we obtain

$$(\log \xi_K(s))' = g\log q + \sum_{\widetilde{q}} \frac{\alpha_K(\widetilde{q})\log\widetilde{q}}{\widetilde{q}^s}, \qquad \Re(s) > 1, \tag{1.6}$$

where $\alpha_K(\widetilde{q}) = \sum_{j=1}^{2g} \frac{\alpha_j^n}{n}$, and $\widetilde{q} = q^n$.

Let us recall that all zeros of the zeta function ζ_K lie in the critical strip $0 \le \Re(s) \le 1$, and they are symmetric with respect to the real axis and the line $\Re(s) = 1/2$. Note that the RH in this context is equivalent to saying that the zeros of ζ_K lie on the line $\Re(s) = 1/2$. Let $\mathcal{Z}(K)$ be the set of the zeros ρ of ζ_K. Using (1.3) and (1.4), we obtain

$$\mathcal{Z}(K) = \left\{\frac{1}{2} \pm i\frac{\theta_j}{\log q} + i\frac{2k\pi}{\log q},\ j \in \{1, \ldots, g\},\ k \in \mathbb{Z}\right\}.$$

1.2. Main results

In this subsection, we present the main results of this paper.

Let us define the screw function on function fields $g_K(t)$ to be the even real-valued function on the real line by [1]

$$g_K(t) = tg\log q + \sum_{\tilde{q}\le e^t} \frac{\alpha_K(\tilde{q})\log\tilde{q}}{\tilde{q}^{1/2}}(t-\log\tilde{q})$$

for nonnegative t, where $\alpha_K(\tilde{q})$ is defined in (1.6). Let $\mathfrak{G}_{t,K}(z)$ defined for $z\in\mathbb{C}$ by

$$\mathfrak{G}_{t,K}(z) = \frac{i(1+\mathfrak{D}_K(z))}{2\sqrt{\pi}}\mathfrak{B}_{t,K}(z), \tag{1.7}$$

where for a real positive t

$$\mathfrak{B}_{t,K}(z) = \tag{1.8}$$

$$= \sum_{\tilde{q}\le e^t} \frac{\alpha_K(\tilde{q})\log\tilde{q}}{\tilde{q}^{1/2}}\frac{e^{iz(t-\log\tilde{q})}-1}{iz} - \left(\frac{e^{itz}-1}{iz}\right)\left(\frac{\xi_K^{'}}{\xi_K}(1/2-iz) - g\log q\right)$$

and

$$\mathfrak{D}_K(z) = \overline{\mathfrak{E}_K(z)}/\mathfrak{E}_K(z), \tag{1.9}$$

with

$$\mathfrak{E}_K(z) = \xi_K(1/2-iz) + \xi_K^{'}(1/2-iz). \tag{1.10}$$

We note that for a real negative t, one has $\mathfrak{G}_{t,K}(z) = \mathfrak{G}_{-t,K}(z)$.

In Section 2, we define and study the screw line associated to the screw function on function fields $g_K(t)$ (see Proposition 2.4 and Corollary 2.1) similar to that given in [7] (for the classical Riemann zeta function).

In Section 3, another expression of $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}$ is given in the following theorem.

Theorem 1.1. *Assume that $\xi_K(1/2)\neq 0$ and let $t\ge 0$. We have*

$$\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})} = 2\log q\left[tg + \sum_{n=1}^{[t/\log q]}\sum_{j=1}^{2g}\cos(n\theta_j)(t-n\log q)\right],$$

where $[x]$ denoted by the integer part of x.

[1]Our $g_K(t)$ differ from that considered by Suzuki [6, Equation 1.1] by a sign $-$.

As an application of Theorem 1.1 we study special values of $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}$ at some points (see Corollary 3.1). Furthermore, we derive some interesting formulas (see Theorem 3.1 and Proposition 3.3). As a consequence of Proposition 3.3, we get the following interesting summation formulas.

Corollary 1.1. *Assume that $\xi_K(1/2) \neq 0$ and $4\min_j(\theta_j/\log q)^2 > 1$. We have*

$$\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{\cos(\theta_j)}{(\theta_j+2k\pi)^2} = \frac{-g}{2}.$$

Finally, in Corollary 3.2 we determine an upper bound of $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}$ for any $t \geq 0$.

2. Preliminary

The purpose of this section is to give some results for the screw function and screw line associated to the zeta function on function fields similar to that given in [6, Pages 1449, 1452] and [7] for the classical Riemann zeta function.

Let us recall that the screw function on function fields $g_K(t)$ to be the even real-valued function on the real line by

$$g_K(t) = tg\log q + \sum_{\tilde{q}\leq e^t} \frac{\alpha_K(\tilde{q})\log\tilde{q}}{\tilde{q}^{1/2}}(t-\log\tilde{q}) \tag{2.1}$$

for nonnegative t, where $\alpha_K(\tilde{q})$ is defined in (1.6).

In the following proposition, we study some proprieties of $g_K(t)$.

Proposition 2.1.

(i) *If $\Im(z) > 1/2$, we have*

$$\int_0^{+\infty} g_K(t)e^{izt}dt = -\frac{1}{z^2}\frac{\xi_K'}{\xi_K}\left(\frac{1}{2}-iz\right).$$

(ii) *Let $t \geq 0$, we have*

$$g_K(t) = \sum_{\gamma\in\Gamma}\frac{1-\cos(\gamma t)}{\gamma^2} = \sum_{\gamma\in\Gamma}\frac{1-e^{i\gamma t}}{\gamma^2}, \tag{2.2}$$

where Γ is the set of all zeros of $\xi_K(1/2-iz)$ counting with multiplicity.

Proof. The proof closely follows the same approach as done in [6, Theorem 1.1]. We will adapt that proof to our situation and keep some notations of it.

(i) We use the change of variables $s = 1/2 - iz$ with $z \in \mathbb{C}$. By (1.6) we obtain

$$(\log \xi_K(s))' = g \log q + \sum_{\tilde{q}} \frac{\alpha_K(\tilde{q}) \log \tilde{q}}{\tilde{q}^s}, \qquad \Re(s) > 1,$$

which is equivalent to $\Im(z) > 1/2$ in terms of z.

By [6, Equations 2.2 and 2.3], if $\Im(z) > 1/2$ we get

$$\int_0^\infty \left(\sum_{\tilde{q} \le e^t} \frac{\alpha_K(\tilde{q}) \log \tilde{q}}{\tilde{q}^{1/2}} (t - \log \tilde{q}) \right) e^{izt} dt =$$
$$= \int_0^\infty \sum_{\tilde{q}} \frac{\alpha_K(\tilde{q}) \log \tilde{q}}{\tilde{q}^{1/2}} (t - \log \tilde{q}) \mathbf{1}_{[\log \tilde{q}, \infty)}(t) e^{izt} dt =$$
$$= -\frac{1}{z^2} \left(\sum_{\tilde{q}} \frac{\alpha_K(\tilde{q}) \log \tilde{q}}{\tilde{q}^s} \right).$$

Using (1.6), we obtain

$$\int_0^\infty \left(\sum_{\tilde{q} \le e^t} \frac{\alpha_K(\tilde{q}) \log \tilde{q}}{\tilde{q}^{1/2}} (t - \log \tilde{q}) \right) e^{izt} dt = -\frac{1}{z^2} \left(\frac{\xi_K'}{\xi_K}(1/2 - iz) - g \log q \right).$$

Therefore, we have (i).

(ii) We replace in the proof of [6, Theorem 1.1 (2)] $\xi(z)$ by $\xi_K(z)$ which is an entire function of order one. By Hadamard's factorization theorem, we get

$$\xi_K\left(\frac{1}{2} - iz\right) = e^{a+bz} \prod_{\gamma} \left[\left(1 - \frac{z}{\gamma}\right) e^{\frac{z}{\gamma}} \right].$$

Taking the logarithmic derivative of both sides, we obtain

$$\frac{\xi_K'}{\xi_K}\left(\frac{1}{2} - iz\right) = ib + i \sum_{\gamma} \left(\frac{1}{z - \gamma} + \frac{1}{\gamma} \right), \tag{2.3}$$

where the sum on the right-hand side converges absolutely and uniformly on every compact subset of $\mathbb{C}$ outside the zeros γ. Equation (2.3) with

$z = 0$ yields $ib = \frac{\xi_K'}{\xi_K}(1/2)$. On the other hand, taking the logarithmic derivative of $\xi_K(s) = \xi_K(1-s)$ and replacing s by $1/2$, we get $ib = = \frac{\xi_K'}{\xi_K}(1/2) = 0$. So, for each term on the right-hand side of (2.3), we obtain

$$-\frac{i}{z^2}\left(\frac{1}{z-\gamma}+\frac{1}{\gamma}\right) = \int_0^{+\infty} \frac{1-e^{-it\gamma}}{\gamma^2} e^{izt}dt, \qquad \Im(z) > \Im(\gamma),$$

where $|\Im(\gamma)| \leq 1/2$ (see [8, Theorem 2.12]). Therefore, (2.2) is derived by interchanging summation and integration, and applying the symmetry $\gamma \to -\gamma$. ■

Let us recall that the RH holds on function fields. Then, we prove the positivity of $g_K(t)$ in the following theorem.

Proposition 2.2. *The function $g_K(t)$ is pointwise nonnegative, that is $g_K(t) \geq \geq 0$ for every $t \in \mathbb{R}$. Further, $g_K(t) > 0$ for every $t \neq 0$.*

Proof. The proof forward that of [6, Theorem 1.7]. ■

Let us define the function $\Delta_K(z)$ for $z \in \mathbb{C}$ by

$$\Delta_K(z) = (\mathfrak{E}_K(z) + \overline{\mathfrak{E}_K(\overline{z})})/2 = \xi_K(1/2 - iz),$$

where $\mathfrak{E}_K(z)$ is defined in (1.10). Hence, Γ coincides with the set of all zeros of both $\Delta_K(z)$ and $1 + \mathfrak{D}_K(z)$ where $\mathfrak{D}_K(z)$ is defined in (1.9). Let t a positive integer, we define

$$\mathcal{P}_{t,K}(z) = \sum_{\gamma\in\Gamma} ord(\gamma)\frac{e^{i\gamma t}-1}{\gamma}\frac{1}{z-\gamma}. \tag{2.4}$$

Then, $\mathcal{P}_{t,K}(z)$ is a meromorphic function on $\mathbb{C}$ such that all poles are simple and Γ is the set of all poles (see [7, Equation 2.2] with $\mathcal{P}_{t,K}(z)$ instead $P_t(z)$. Moreover, for negative t we have $\mathcal{P}_{t,K}(z) = \mathcal{P}_{-t,K}(z)$.

Let $\mathbb{H}^2$ be the Hardy space [2] on the upper half plane. As usual, we identify $\mathbb{H}^2$ with a closed subspace of $L^2(\mathbb{R})$ via boundary values. Consequently, the inner product of $\mathbb{H}^2$ coincides with the standard inner product of $L^2(\mathbb{R})$. Under the RH, the function $\mathfrak{E}_K(z)$ defined in equation (1.10) is an entire function

[2] Denote by $H(\mathbb{D})$ the space of analytic function $F : \mathbb{D} \longrightarrow \mathbb{C}$. We define The Hardy spaces by

$$H^p(\mathbb{D}) = \{F \in H(\mathbb{D}) : \|F\|_{H^p(\mathbb{D})} < \infty\},$$

where $\|F\|_{H^p(\mathbb{D})} = \sup_{0<r<1} \|F_r\|$.

satisfying $|\mathfrak{E}_K(\overline{z})| < |\mathfrak{E}_K(z)|$ if $\Im(z) > 0$. Hence, it generates the de Branges space $\mathbb{H}(\mathfrak{E}_K)$, which is Hilbert space of entire functions isomorphic to the model subspace $\mathcal{K}(\mathfrak{D}_K(z)) := \mathbb{H}^2 \ominus \mathfrak{D}_K \mathbb{H}^2$, by the mapping $F(z) \to F(z)/\mathfrak{E}_K(z)$ from $\mathbb{H}(\mathfrak{E}_K)$ into $\mathbb{H}^2$, where $\mathfrak{D}_K(z)$ is defined in (1.9). The model subspace $\mathcal{K}(\mathfrak{D}_K)$ is a subspace of $L^2(\mathbb{R})$. In particular, the inner product of $\mathcal{K}(\mathfrak{D}_K)$ matches that of $L^2(\mathbb{R})$ (see [7, Section 3.1]).

Let us define

$$F_\gamma(z) = \sqrt{\frac{ord(\gamma)}{\pi}} \frac{i(1+\mathfrak{D}_K(z))}{2(z-\gamma)}, \qquad \gamma \in \Gamma, \tag{2.5}$$

where $\mathfrak{D}_K(z)$ is defined in (1.9) and Γ denoted by the set of all ordinates of distinct zeros $\rho = 1/2 - i\gamma$ of $\xi_K(s)$ for $\gamma \in \Gamma(\subset \mathbb{R})$.

In the following proposition, we state some results on $\mathcal{P}_{t,K}(z)$, $\mathfrak{B}_{t,K}(z)$, $\mathfrak{G}_{t,K}(z)$ and $\{F_\gamma(z)\}_{\gamma\in\Gamma}$.

Proposition 2.3. [7]

(i) *Let $\mathcal{P}_{t,K}(z)$ and $\mathfrak{B}_{t,K}(z)$ be functions defined in (2.4) and (1.8), respectively. Then, both coincide.*

(ii) *For any fixed $t \in \mathbb{R}$, the function $\mathfrak{G}_{t,K}(z)$ belongs to $L^2(\mathbb{R})$ as a function of z.*

(iii) *The family of functions $\{F_\gamma(z)\}_{\gamma\in\Gamma}$ defined in (2.5) forms an orthonormal basis of the Hilbert space $\mathcal{K}(\mathfrak{D}_K)$.*

Proof. The proof follows the lines of that [7, Proposition 2.1] we just replace $P_t(z)$ by $\mathcal{P}_{t,K}(z)$. The only difference is the use of an explicit formula demonstrated by Bllaca and Mazhouda in [3, Theorem 1]. ■

Since RH holds on function fields, we obtain the following results.

Proposition 2.4. *The mapping $t \longrightarrow \mathfrak{G}_{t,K}(z)$ from $\mathbb{R}$ to $L^2(\mathbb{R})$ is a screw line corresponding to $g_K(t)$.*

Proof. The proof closely follows the same approach as [7, Theorem 1.1]. We will adapt that proof to our situation and keep some notations of it.

By the proof of [7, Theorem 1.1] we replace $\mathfrak{G}_t(z)$ by $\mathfrak{G}_{t,K}(z)$ and using Proposition 2.3 (i), we obtain

$$\begin{aligned} \mathfrak{G}_{t,K}(z) &= \sum_{\gamma\in\Gamma} \sqrt{ord(\gamma)} \frac{e^{i\gamma t}-1}{\gamma} . \sqrt{\frac{ord(\gamma)}{\pi}} \frac{i(1+\mathfrak{D}_K(z))}{2(z-\gamma)} = \\ &= \sum_{\gamma\in\Gamma} \sqrt{ord(\gamma)} \frac{e^{i\gamma t}-1}{\gamma} F_\gamma(z) \end{aligned} \tag{2.6}$$

unconditionally. Then, by Proposition 2.3 (ii) the coefficients in the right-hand side converge in L^2-sence

$$\sum_{\gamma\in\Gamma}\left|\sqrt{ord(\gamma)}\frac{e^{i\gamma t}-1}{\gamma}\right|^2 \leq \sum_{\gamma\in\Gamma}\frac{ord(\gamma)}{|\gamma|^2}<\infty.$$

Therefore, under RH and from Proposition 2.3 (iii) with $\mathfrak{G}_{t,K}(z)$, equation (2.6) yields that $\mathfrak{G}_{t,K}(z)$ belongs to the subspace $\mathcal{K}(\mathfrak{D}_K)$ of $L^2(\mathbb{R})$ and

$$\langle \mathfrak{G}_{t+u,K}-\mathfrak{G}_{u,K},\mathfrak{G}_{s+u,K}-\mathfrak{G}_{u,K}\rangle_{L^2(\mathbb{R}} = \sum_{\gamma\in\Gamma} ord(\gamma)\frac{e^{i\gamma t}-1}{\gamma}\frac{e^{-i\gamma s}-1}{\gamma}. \tag{2.7}$$

From (1.2) and (2.2) we get $G_{g_K}(t,s) = \sum_{\gamma\in\Gamma} ord(\gamma)\frac{(e^{i\gamma t}-1)(e^{-i\gamma s}-1)}{\gamma^2}$ and $\mathfrak{G}_{t,K}:\mathbb{R}\longrightarrow L^2(\mathbb{R})$, is a screw line of $g_K(t)$. Hence, $\mathfrak{G}_{0,K}$ is identically zero by (1.7) and (1.8). Moreover, by (2.7) with $u=0$, we get

$$\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})} = \sum_{\gamma\in\Gamma} ord(\gamma)\left|\frac{e^{i\gamma t}-1}{\gamma}\right|^2 = 2\sum_{\gamma\in\Gamma} ord(\gamma)\frac{1-\cos(\gamma t)}{\gamma^2}.$$

On the other hand, by equation (2.2) we have

$$g_K(t)=\sum_{\gamma\in\Gamma} ord(\gamma)\frac{1-\cos(\gamma t)}{\gamma^2}.$$

This complete the proof of Proposition 2.4. ■

As a consequence of Proposition 2.4, we get the following corollary.

Corollary 2.1. *For all $t\geq 0$, we have*

$$\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})} = 2g_K(t). \tag{2.8}$$

Proof. We only sketch the proof since it follows the lines of that [7, Corollary 1.1].

Proposition 2.4 states that (2.8) is a necessary condition for the RH. Therefore, to prove the RH, it is sufficient to show that (2.8) is also a sufficient condition.

Conversely, if we assume that equality (2.8) holds for all $t=t_0$, then $g_K(t)$ is nonnegative on $[t_0,\infty)$. This implies the truth of the RH by Proposition 2.2.

■

3. Further results on $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}$

In this section, we study special values of $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}$ at some points. Furthermore, we derive some interesting summation formulas. Finally, we determine an upper bound of this norm.

Let us recall that the zeros ρ of the function ζ_K are denoted by

$$\rho = \frac{1}{2} + i\tau^{\pm}_{k,j} \quad where \quad \tau^{\pm}_{k,j} = (\pm\theta_j + 2k\pi)/\log q, \quad j = 1, \ldots, 2g \quad and \quad k \in \mathbb{Z},$$

and Γ the set of all zeros of $\xi_K(1/2 - iz)$ with counting multiplicity and the multiplicity of $\gamma \in \Gamma$ by $ord(\gamma)$. Throughout this section, we replace γ by $\tau^{\pm}_{k,j}$ where $j = 1, ..., 2g$ and $k \in \mathbb{Z}$.

Remark 3.1. We have $\xi_K(1/2) = 0$ if and only if for some $j = 1, 2, ..., g$, $\theta_j = 0$; in this case, instead of ξ_K, we may take the function $F_K(s) = \xi_K(s)/(s-1/2)^m$, where m is the multiplicity of the eventual zero of ξ_K at $s = 1/2$. Functions F_K and ξ_K have the same zeros with $\Im(\rho) > 0$. For this reason, we assume in this section $\xi_K(1/2) \neq 0$.

The norm of screw line is defined for $t \geq 0$ by

$$\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})} = 2\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{1-\cos(\tau^{\pm}_{k,j}t)}{(\tau^{\pm}_{k,j})^2}. \tag{3.1}$$

Note that we have $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})} = \|\mathfrak{G}_{-t,K}\|^2_{L^2(\mathbb{R})}$ and a simple computation yields

$$\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})} = 4\sum_{j=1}^{g}\left(\sum_{k=1}^{+\infty} \frac{1-\cos((\tau^{\pm}_{k,j})t)}{(\tau^{\pm}_{k,j})^2} + \frac{1-\cos((\tau^{+}_{0,j})t)}{(\tau^{+}_{0,j})^2}\right). \tag{3.2}$$

Proof of Theorem 1.1. By (2.1) and $\widetilde{q} = q^n$, we have

$$\begin{aligned} g_K(t) &= tg\log q + \sum_{\tilde{q}\leq e^t} \frac{\alpha_K(\tilde{q})\log\tilde{q}}{\tilde{q}^{1/2}}(t - \log\tilde{q}) = \\ &= tg\log q + \sum_{q^n\leq e^t} \frac{\alpha_K(q^n)\log q^n}{q^{n/2}}(t-\log q^n) = \\ &= tg\log q + \sum_{n=1}^{[t/\log q]} \frac{\alpha_K(q^n)\log q^n}{q^{n/2}}(t - \log q^n). \end{aligned}$$

Let us recall that $\alpha_j = q^{1/2}\exp(i\theta_j)$ with $\theta_j \in [0,\pi]$ and $\alpha_K(q^n) = \sum_{j=1}^{2g} \frac{\alpha_j^n}{n}$. Since the function $g_K(t)$ is real, we obtain

$$g_K(t) = tg\log q + \log q \sum_{n=1}^{[t/\log q]} \sum_{j=1}^{2g} \cos(n\theta_j)(t - n\log q).$$

From Corollary 2.1, we obtain

$$\begin{aligned}\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})} &= 2g_K(t) = \\ &= 2\log q \left[tg + \sum_{n=1}^{[t/\log q]} \sum_{j=1}^{2g} \cos(n\theta_j)(t - n\log q) \right].\end{aligned}$$ ∎

In the following corollary, we give some special values of $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}$.

Corollary 3.1. *Assume that* $\xi_K(1/2) \neq 0$. *We have*

(i)
$$\|\mathfrak{G}_{\log q,K}\|^2_{L^2(\mathbb{R})} = 2g\log^2 q.$$

(ii)
$$\|\mathfrak{G}_{2\log q,K}\|^2_{L^2(\mathbb{R})} = 2\log^2 q \left[2g + \sum_{j=1}^{2g} \cos(\theta_j) \right].$$

(iii)
$$\|\mathfrak{G}_{3\log q,K}\|^2_{L^2(\mathbb{R})} = 2\log^2 q \left[g + 2\sum_{j=1}^{2g} \left(\cos(\theta_j) + \cos^2(\theta_j)\right) \right].$$

Proof. The proof yields from Theorem 1.1 for $t = \log q$, $t = 2\log q$ and $t = 3\log q$. ∎

In the following theorem, we derive interesting summation formula.

Theorem 3.1. *Assume that* $\xi_K(1/2) \neq 0$. *We have*

$$\sum_{k=1}^{+\infty} \left(\sum_{j=1}^{g} \frac{1 - \cos(\pm\theta_j)}{(\pm\theta_j + 2k\pi)^2} \right) = \frac{g}{2} + \sum_{j=1}^{g} \frac{\cos(\theta_j) - 1}{(\theta_j)^2}.$$

Proof. By Corollary 3.1 (i) we have

$$\|\mathfrak{G}_{\log q,K}\|^2_{L^2(\mathbb{R})} = 2g\log^2 q.$$

On the other hand, (3.2) with $t=\log q$ yields

$$\|\mathfrak{G}_{\log q,K}\|^2_{L^2(\mathbb{R})} = 4\log^2 q \sum_{j=1}^{g}\left(\sum_{k=1}^{+\infty}\frac{1-\cos(\pm\theta_j)}{(\pm\theta_j+2k\pi)^2}+\frac{1-\cos(\theta_j)}{(\theta_j)^2}\right).$$

Therefore

$$\sum_{j=1}^{g}\left(\sum_{k=1}^{+\infty}\frac{1-\cos(\pm\theta_j)}{(\pm\theta_j+2k\pi)^2}+\frac{1-\cos(\theta_j)}{(\theta_j)^2}\right)=\frac{g}{2}.$$

■

Let us recall that the superzeta functions on function fields of the second kind (see [1, section 5]) is defined by

$$\mathcal{Z}_K(s,t)=\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{1}{((\tau^{\pm}_{k,j})^2+t^2)^s},\quad \Re(s)>1/2, \tag{3.3}$$

where $t\in\mathbb{C}$ such that $t^2+(\tau^{\pm}_{k,j})^2\notin\mathbb{R}_-$ for all k.

In the following proposition, we express $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}$ in terms of special values of $\mathcal{Z}_K(s,t)$.

Proposition 3.2. *Assume that $\xi_K(1/2)\neq 0$ and let $t\geq 0$. We have*

$$\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}=2\left[\mathcal{Z}_K(1,0)-\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{\cos((\tau^{\pm}_{k,j})t)}{(\tau^{\pm}_{k,j})^2}\right]. \tag{3.4}$$

Proof. By (3.1), we obtain

$$\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}=2\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{1-\cos((\tau^{\pm}_{k,j})t)}{(\tau^{\pm}_{k,j})^2}=$$
$$=2\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{1}{(\tau^{\pm}_{k,j})^2}-2\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{\cos((\tau^{\pm}_{k,j})t)}{(\tau^{\pm}_{k,j})^2}.$$

From (3.3) for $s=1$ and $t=0$, we have

$$\mathcal{Z}_K(1,0)=\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{1}{(\tau^{\pm}_{k,j})^2}.$$

Hence, the proof of Proposition 3.2 is complete. ■

Remark 3.2. From [2, Section 5], assume that

$$\xi_K(1/2)\neq 0\quad\text{and}\quad 4\min_j(\theta_j/\log q)^2>1,$$

one has

$$\lambda_{K,1}^{V}(1) = \sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{1}{(\tau_{k,j}^{\pm})^2} = $$
$$= \mathcal{Z}_K(1,0),$$

where $\lambda_{K,1}^{V}(1)$ denoted the first Voros–Li coefficients on function fields[3]. Therefore, formula (3.4) can be written as follows

$$\|\mathfrak{G}_{t,K}\|_{L^2(\mathbb{R})}^2 = 2\left[\lambda_{K,1}^{V}(1) - \sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{\cos((\tau_{k,j}^{\pm})t)}{(\tau_{k,j}^{\pm})^2}\right]. \tag{3.5}$$

Now, an interesting summation formula given by the following theorem.

Proposition 3.3. *Assume that* $\xi_K(1/2) \neq 0$ *and* $4\min_j(\theta_j/\log q)^2 > 1$. *We have*

$$\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{\cos(\theta_j)}{(\theta_j + 2k\pi)^2} = \frac{1}{2\log^2 q}\lambda_{K,1}^{V}(1) - \frac{g}{2}$$

or, equivalently

$$\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{\cos(\theta_j)}{(\theta_j + 2k\pi)^2} = \frac{1}{2q}\left[\frac{L''}{L}(q^{-1/2}) - \left(\frac{L'}{L}(q^{-1/2})\right)^2 + q^{1/2}\frac{L'}{L}(q^{-1/2})\right] - \frac{g}{2}.$$

Proof. By Corollary 3.1 (i), we obtain

$$\|\mathfrak{G}_{\log q,K}\|_{L^2(\mathbb{R})}^2 = 2g\log^2 q.$$

[3] We recall that the Voros–Li coefficients $\lambda_{K,w}^{V}(n)$ are the coefficients in the Taylor series expansion (see [2, Equations (7) and (11)]

$$\log \xi_K\left(\frac{1}{2} + \frac{\sqrt{ws}}{1-s}\right) - \log \xi_K(1/2) = \sum_{n=1}^{+\infty} \frac{\lambda_{K,w}^{V}(n)}{n} s^n$$

or

$$\lambda_{K,w}^{V}(n) = \sum_{\rho\in\mathcal{Z}(K),\Im(\rho)>0} \left[2 - \left(\frac{1+\sqrt{4w(\rho-1/2)^2+w^2}}{2(\rho-1/2)}\right)^{2n} - \right.$$
$$\left. - \left(\frac{1-\sqrt{4w(\rho-1/2)^2+w^2}}{2(\rho-1/2)}\right)^{2n}\right].$$

On the other hand, from (3.5) with $t = \log q$ we get

$$\|\mathfrak{G}_{\log q,K}\|^2_{L^2(\mathbb{R})} = 2\lambda^V_{K,1}(1) - 2\log^2 q \sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{\cos(\pm\theta_j)}{(\pm\theta_j + 2k\pi)^2}.$$

Therefore

$$2\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{\cos(\theta_j)}{(\theta_j + 2k\pi)^2} = \frac{1}{\log^2 q}\lambda^V_{K,1}(1) - g.$$

In addition, from [2, Section 5], assume that $\xi_K(1/2) \neq 0$ and $4\min_j(\theta_j/\log q)^2 > 1$, one has

$$\lambda^V_{K,1}(1) = \frac{\log^2 q}{q}\left[\frac{L''}{L}(q^{-1/2}) - \left(\frac{L'}{L}(q^{-1/2})\right)^2 + q^{1/2}\frac{L'}{L}(q^{-1/2})\right]. \tag{3.6}$$

Hence, the proof of Theorem 3.3 is complete. ■

Proof of Corollary 1.1. Using the lower and the upper bounds of $\lambda^V_{K,1}(1)$ stated in [2, Proposition 10], assume that

$$\xi_K(1/2) \neq 0 \quad \text{and} \quad 4\min_j(\theta_j/\log q)^2 > 1,$$

we obtain

$$\begin{aligned} \max\left\{0, \frac{1}{2\log^2 q}\left(\frac{2g}{\gamma_0}\left[\frac{2\log q}{\pi} - \frac{1}{\gamma_0}\right]\right)\right\} - \frac{g}{2} &\leq \sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{\cos(\theta_j)}{(\theta_j + 2k\pi)^2} \leq \\ &\leq \frac{1}{2\log^2 q}\left(\frac{2g}{\gamma_0}\left[\frac{2\log q}{\pi} + \frac{1}{\gamma_0}\right]\right) - \frac{g}{2}, \end{aligned} \tag{3.7}$$

where $\gamma_0 = \min\{\theta_j/\log q\}$. The Corollary follows from (3.7) and by setting $q \to \infty$. ■

In the following corollary, we determine an upper bound of $\|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})}$ for any $t \geq 0$.

Corollary 3.2. *Assume that $\xi_K(1/2) \neq 0$ and $4\min_j(\theta_j/\log q)^2 > 1$. For $t \geq 0$, we have*

$$0 \leq \|\mathfrak{G}_{t,K}\|^2_{L^2(\mathbb{R})} \leq \frac{4\log^2 q}{q}\left[\frac{L''}{L}(q^{-1/2}) - \left(\frac{L'}{L}(q^{-1/2})\right)^2 + q^{1/2}\frac{L'}{L}(q^{-1/2})\right].$$

Proof. By (3.5), (3.6) and using that for $x \in \mathbb{R}$ we have $-1 \leq \cos(x) \leq 1$. Therefore, we obtain the result. ■

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