Annales Univ. Sci. Budapest., Sect. Comp. 57 (2024) 321–335

EXTENDED CONVERGENCE OF A DERIVATIVE FREE FOURTH ORDER METHOD FOR EQUATIONS OR SYSTEMS

Ioannis K. Argyros (Lawton, USA)
Jayakumar Jayaraman and Jinny Ann John
(Pondicherry, India)

Communicated by László Szili
(Received April 13, 2024; accepted June 1, 2024)

Abstract. In many areas of science and engineering, solving equations or systems of equations is really important. Instead of finding exact solutions, which can be very hard or impossible in some cases, people often use iterative methods to attain the desired solutions. This article is dedicated to introducing a highly efficient derivative-free fourth order iterative technique renowned for its exceptional convergence properties. The analysis within this article deals with scrutinizing both local and semi-local convergence characteristics, taking into account the φ, ψ-continuity constraints imposed on the operators that are present in these methods. It's worth noting that the innovative methodology proposed herein isn't confined to specific techniques but has broader applicability, encompassing a wide spectrum of approaches involving the utilization of inverses of linear operators or matrices.

1. Introduction

In the realm of applied science and technology, a multitude of complex challenges can be effectively approached by framing them as non-linear equations of the following form:

$$F(x) = 0. \tag{1.1}$$

Key words and phrases: Non-linear equations, derivative-free, convergence, Banach space.
2010 Mathematics Subject Classification: 37N30, 47J25, 49M15, 65H10, 65J15.

In this context, $F : \mathcal{D} \subset \mathcal{B} \to \mathcal{B}$ represents a differentiable function according to the Fréchet sense, where $\mathcal{B}$ signifies a complete normed linear space, and $\mathcal{D}$ denotes a non-empty, open, and convex set.

The pursuit of closed-form solutions for these non-linear equations typically presents formidable difficulties. Consequently, iterative methods have emerged as the go-to approach for seeking solutions to such problems. Among these methods, Newton's method [1, 16, 21] stands out as a widely adopted choice, primarily due to its remarkable quadratic convergence rate when tackling equations like (1.1). In recent years, the fields of science and mathematics have witnessed significant advancements, leading to the discovery and application of various higher-order iterative techniques for solving non-linear equations [2–15, 17–20]. However, these advanced methods often grapple with the drawback of demanding the computation of second and higher-order derivatives, which can significantly hinder their practicality in real-world applications. The computational overhead associated with evaluating $F^{''}$ during each iteration makes classical cubic convergent schemes less suitable. Furthermore, it's worth noting that many of these methods rely on Taylor expansions, necessitating derivatives of higher order not inherently present within the method itself.

An in-depth analysis of the local and semi-local behavior of iterative methods offers invaluable insights into their convergence properties, error bounds, and the region where solutions are unique. A multitude of studies have dedicated their focus to investigating the local and semi-local convergence aspects of efficient iterative techniques, yielding substantial outcomes in the form of convergence radii, error estimations, and expanded applicability of these methods [2–4, 17]. Such findings carry significant weight, especially in guiding the selection of appropriate initial points for the iterative process.

In this article, we introduce and thoroughly scrutinize a specific fourth order derivative free iterative method. The central objective of our study revolves around establishing rigorous convergence theorems for this method, building upon the foundational work laid out in a previous research endeavor [18]. The method is formally defined for $x_0 \in \mathcal{D}$, $a \in \mathbb{R}$ and for each $n = 0, 1, 2, \ldots$ as follows:

$$
\begin{aligned}
u_n &= x_n + aF(x_n), \\
A_n &= [u_n, x_n; F], \\
y_n &= x_n - A_n^{-1} F(x_n), \\
x_{n+1} &= y_n - \left[3I - A_n^{-1}\left([y_n, x_n; F] + [y_n, u_n; F]\right)\right] A_n^{-1} F(y_n),
\end{aligned}
\tag{1.2}
$$

where $[\cdot, \cdot; F] : \mathcal{D} \times \mathcal{D} \to \mathscr{L}(\mathcal{B})$ is a divided difference of order one and $\mathscr{L}(\mathcal{B})$ stands for the space of bounded linear operators from $\mathcal{B}$ into itself.

We usually study the local as well as the semi-local analysis of convergence for an iterative method [1–4, 17]. In the local analysis, a ball is determined about the solution x^*, so if any point is selected inside it the convergence of the method is assured to x^*. The semi-local analysis uses information involving a ball centered at the starting point x_0 and provides convergence conditions based on the smallness of $\|x_1 - x_0\|$.

There are limitations in the applicability of the method (1.2) which constitute the motivation for this paper.

Motivation: The local convergence order four is determined in [18] for $\mathcal{B} = \mathbb{R}^k$ under the Taylor series expansion technique and by assuming that the operator F is at least five time differentiable and bounded. Moreover, isolation of the solution x^* or a priori bounds on the distances $\|x_n - x^*\|$ are not developed. Notice that only the divided difference and the function appear on the method. These limitations restrict the applicability of the method. As a motivational example, consider the interval $\mathcal{D} = [-\frac{3}{2}, 2]$ and define the function $F(x) = \frac{1}{2}x^3 \log x + 4x^5 - 4x^4$ if $x \neq 0$ and $F(x) = 0$ if $x = 0$. It is clear that $F^{(3)}$ is not bounded and $x^* = 1$. Thus, the results in [18] do not guarantee the convergence of the method. But, the method converges say e.g. if $a = 1$ and $x_0 = 1.2$. The same limitations exist with other studies utilizing the Taylor series expansion approach on other method [5–15, 19, 20]. Recall that we say that a divided difference $[\cdot, \cdot; F]$ satisfies a generalized continuity condition if

$$\|[v_1, v_2; F] - [v_3, v_4; F]\| \leq \varphi(\|v_1 - v_3\|, \|v_2 - v_4\|)$$

for some non-negative, non-decreasing and continuous function (see also the functions "φ" and "ψ" that follow). These conditions generalize the usual Lipschitz or Hölder conditions, and can be used in cases these conditions do not hold. Therefore, there is a need to work on the convergence conditions by relying only on the operators on the method.

Novelty: The local convergence analysis relies on such information and the concept of generalized continuity on the divided difference $[\cdot, \cdot; F]$ [1–4,17]. This way isolation of the solution and computable a priori estimates on $\|x_n - x^*\|$ become possible. Moreover, the more challenging and interesting semi-local convergence analysis is developed based on majorizing sequences. Although, we extend the applicability of method (1.2), our technique can be used to do the same on other methods along the same lines.

The rest of the article is structured as follows: Section 2 discusses the local convergence characteristics pertaining to the method (1.2). Section 3 introduces a pivotal concept: majorizing sequences. These sequences play a crucial role in facilitating Semi-Local Convergence Analysis of (1.2). Section 4 takes the insights gained from the previous sections and puts them into practical use through numerical applications. We will leverage the convergence results derived earlier to solve real-world problems, showcasing the applicability and effectiveness of the method (1.2) in practical scenarios. Section 5 encapsulates the key takeaways and implications of our findings, providing closure to this paper's comprehensive examination of method (1.2).

2. Analysis I: Local

The notation $S(x^*, R)$ stands for the open ball with center x^* and of radius $R > 0$, whereas $S[x^*, R]$ stands for the closure of the ball $S(x^*, R)$.

The local analysis uses real functions. Let $M = [0, +\infty)$. There is a relationship between the functions "φ" listed in conditions (C_1)-(C_3) (see (C_4)-(C_6)), and the operators on the method (1.2). But first we need to list their properties.

Assume:

(C_1) There exist continuous and non-decreasing (CND) functions $\mathfrak{f} : M \to M$ and $\varphi_0 : M \times M \to M$ so that the equation $\varphi_0(\mathfrak{f}(t), t) - 1 = 0$ has a smallest positive solution (SPS). Denote this solution as ρ_0. Let $M_0 = [0, \rho_0)$.

(C_2) There exists (CND) function $\varphi : M_0 \times M_0 \to M$ so that $h_1 : M_0 \times M_0 \to M$, where

$$h_1(t) = \frac{\varphi(\mathfrak{f}(t), t)}{1 - \varphi_0(\mathfrak{f}(t), t)}$$

the equation $h_1(t) - 1 = 0$ has SPS denoted by $s_1 \in M_0$.

(C_3) There exist (CND) functions $\varphi_1 : M_0 \to M$ and $\varphi_2 : M_0 \times M_0 \times M_0 \to M$ so that for $h_2 : M_0 \to M$, where

$$h_2(t) = \left[\frac{\varphi_2(\mathfrak{f}(t), t, h_1(t)t)}{1 - \varphi_0(\mathfrak{f}(t), t)} + \frac{2\varphi_2(\mathfrak{f}(t), t, h_1(t)t)(1 + \varphi_1(h_1(t)t))}{(1 - \varphi_0(\mathfrak{f}(t), t))^2}\right] h_1(t),$$

the equation $h_2(t) - 1 = 0$ has a SPS denoted by $s_2 \in M_0$. Let

$$s^* = \min\{s_j\}, \quad j = 1, 2. \tag{2.1}$$

Set $M_1 = [0, s^*)$. It follows by the condition (C_1) and the definition of the parameter ρ_0 that for each $t \in M_1$

$$0 \le \varphi_0(\mathfrak{f}(t), t) < 1 \tag{2.2}$$

and

$$0 \le h_j(t) < 1. \tag{2.3}$$

(C_4) There exists an invertible operator $\mathscr{T}$ and $x^* \in \mathcal{D}$ with $F(x^*) = 0$ such that for each $x, y \in \mathcal{D}$

$$\|\mathscr{T}^{-1}([x, y; F] - \mathscr{T})\| \leq \varphi_0(\|x - x^*\|, \|y - x^*\|)$$

and for $u = x + aF(x)$

$$\|u - x^*\| \leq \mathfrak{f}(\|x - x^*\|) \leq \|x - x^*\|.$$

Set $\mathcal{D}_0 = \mathcal{D} \cap S(x^*, \rho_0)$. The linear operator may or may not depend on x^* and should be independent of x and y (see also Remark 2.1 (a)).

It also follows by (C_1) that $\|\mathscr{T}^{-1}(A_0 - \mathscr{T})\| \leq \varphi_0(\|x_0 - x^*\|, \|x_0 - x^*\|) <$ < 1. Thus, the celebrated Banach Lemma for invertible operators [1, 16, 21] implies the existence of A_0^{-1} and

$$\|A_0^{-1}\mathscr{T}\| \leq \frac{1}{1 - \varphi_0(\mathfrak{f}(\|x_0 - x^*\|), \|x_0 - x^*\|)}. \tag{2.4}$$

(C_5)

$$\|\mathscr{T}^{-1}([x, x^*; F] - \mathscr{T})\| \leq \varphi_1(\|x - x^*\|)$$

and

$$\|\mathscr{T}^{-1}([u, x; F] - [y, x; F])\| \leq \varphi_2(\|u - x^*\|, \|x - x^*\|, \|y - x^*\|)$$

for each $x, y \in \mathcal{D}_0$.

(C_6) $S[x^*, s^*] \subset \mathcal{D}$, where s^* is given in (2.1).

Remark 2.1.

(a) A popular choice but not the most flexible one is $\mathscr{T} = F'(x^*)$. In this case, one finds only a simple solution of the equation $F(x) = 0$ provided that the operator F is differentiable at $x = x^*$.

(b) A possible choice for the function $\mathfrak{f}$ is motivated by the calculation

$$\begin{aligned} u - x^* &= x - x^* + aF(x) = (I + a[x, x^*; F])(x - x^*) = \\ &= (I + a\mathscr{T}\mathscr{T}^{-1}([x, x^*; F] - \mathscr{T} + \mathscr{T}))(x - x^*) = \\ &= ((I + a\mathscr{T}) + a\mathscr{T}\mathscr{T}^{-1}([x, x^*; F] - \mathscr{T}))(x - x^*) \end{aligned}$$

so that

$$\|u - x^*\| \leq [\|I + a\mathscr{T}\| + |a|\|\mathscr{T}\|\varphi_1(\|x - x^*\|)]\, \|x - x^*\|.$$

Hence, one can choose

$$\mathfrak{f}(t) = \|I + a\mathscr{T}\| + |a|\|\mathscr{T}\|\varphi_1(t). \tag{2.5}$$

The local convergence analysis can be shown under the assumptions (C_1)-(C_6).

Theorem 2.1. *Under the assumptions (C_1)-(C_6) the following assertions are valid for the method (1.2) provided that the starting iterate $x_0 \in S(x^*, s^*) - \{x^*\}$*

$$\{x_n\} \subset S(x^*, s^*), \tag{2.6}$$

$$\|y_n - x^*\| \leq h_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < s^*, \tag{2.7}$$

$$\|x_{n+1} - x^*\| \leq h_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.8}$$

and the sequence $\{x_n\}$ is convergent to x^*.

Proof. Induction method is used to show the assertions (2.6)-(2.8). Using the first sub-step of the method (1.2), (2.4), the iterate y_0 is well defined and

$$y_0 - x^* = A_0^{-1}(A_0 - [x_0, x^*; F])(x_0 - x^*). \tag{2.9}$$

It follows by (C_2), (C_4), (2.1), (2.4) and (2.9) that

$$\begin{aligned} \|y_0 - x^*\| &\leq \frac{\varphi(\mathfrak{f}(\|x_0 - x^*\|), \|x_0 - x^*\|)\|x_0 - x^*\|}{1 - \varphi_0(\mathfrak{f}(\|x_0 - x^*\|), \|x_0 - x^*\|)} \leq \\ &\leq h_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < s^*. \end{aligned} \tag{2.10}$$

Thus, the iterate $y_0 \in S(x^*, s^*)$ and the assertion (2.7) is validated if $n = 0$. Next, notice also that the iterate x_1 is well defined by (2.4) and the second sub-step of method (1.2). Consequently, one can write in turn

$$\begin{aligned} x_1 - x^* = y_0 - x^* - A_0^{-1}F(y_0) - A_0^{-1}\big[(A_0 - [y_0, x_0; F]) - \\ - (A_0 - [y_0, u_0; F])\big]A_0^{-1}F(y_0). \end{aligned} \tag{2.11}$$

Then, in view of (2.1), (2.4), (2.10), (C_4), (C_5) and (2.11), one obtains

$$\begin{aligned} \|x_1 - x^*\| \leq \Bigg[\frac{\varphi_2(\mathfrak{f}(\|x_0 - x^*\|), \|x_0 - x^*\|, \|y_0 - x^*\|)}{1 - \varphi_0(\mathfrak{f}(\|x_0 - x^*\|), \|x_0 - x^*\|)} + \\ + \frac{2\varphi_2(\mathfrak{f}(\|x_0 - x^*\|), \|x_0 - x^*\|, \|y_0 - x^*\|)(1 + \varphi_1(\|y_0 - x^*\|))}{1 - \varphi_0(\mathfrak{f}(\|x_0 - x^*\|), \|x_0 - x^*\|)}\Bigg]\|y_0 - x^*\| \leq \\ \leq h_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \end{aligned} \tag{2.12}$$

Hence, the iterate $x_1 \in S(x^*, s^*)$ and the assertion (2.8) is validated if $n = 0$. These calculations can be repeated, if x_m, y_m, x_{m+1} replace x_0, y_0, x_1 in the preceding calculations. Therefore, the induction for assertions (2.6)-(2.8) is terminated. Moreover, it follows from the estimate

$$\|x_{m+1} - x^*\| \leq b\|x_m - x^*\| < s^*, \tag{2.13}$$

where $b = h_2(\|x_0 - x^*\|) \in [0, 1)$, that the iterate $x_{m+1} \in S(x^*, s^*)$ and $\lim_{m \to +\infty} x_m = x^*$. ■

The isolation of the solution x^* is developed in the next result.

Proposition 2.2. *Assume:*
There exists a solution $z \in S(x^*, s_3)$ of the equation $F(x) = 0$ for some $s_3 > 0$; The first condition in (C_4) is valid on the ball $S(x^*, s_3)$ and there exists $s_4 > s_3$ such that

$$\varphi_0(0, s_4) < 1. \tag{2.14}$$

Set $\mathcal{D}_1 = \mathcal{D} \cap S[x^*, s_4]$. Then, the equation $F(x) = 0$ is uniquely solvable by x^* in the domain $\mathcal{D}_1$.

Proof. Let $z \neq x^*$. Then, define the divided difference $T = [x^*, z; F]$. It follows by (C_4) and (2.14) that

$$\|\mathscr{T}^{-1}(T - \mathscr{T})\| \leq \varphi_0(\|x^* - x^*\|, \|z - x^*\|) \leq \varphi_0(0, s_4) < 1,$$

so T^{-1} is well defined. Therefore, one can get

$$x^* - z = T^{-1}(F(x^*) - F(z)) = T^{-1}(0).$$

Hence, it follows that $z = x^*$. ■

Remark 2.2. If $s_3 = s^*$ and all assumptions of Theorem 2.1 are valid, then take $z = x^*$.

3. Analysis II: Semi-local

The role of x^*, φ, $\mathfrak{f}$ is exchanged by x_0, ψ, g respectively in the calculations as follows:
Assume

(H_1) There exists CND functions $g : M \to \mathbb{R}$ and $\psi_0 : M \times M \to \mathbb{R}$ so that the equation $\psi_0(g(t), t) - 1 = 0$ has a SPS denoted as p. Set $M_3 = [0, p)$.

(H_2) There exists CND function $\psi : M_3 \times M_3 \times M_3 \to \mathbb{R}$.
Define the sequence $\{\alpha_n\}$ for $\alpha_0 = 0$, some $\beta_0 \geq 0$ and each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned}
q_n &= \psi(\alpha_n, \beta_n, g(\alpha_n))(\beta_n - \alpha_n),\\
\alpha_{n+1} &= \beta_n + \left[\frac{1}{1 - \psi_0(g(\alpha_n), \alpha_n)} + \frac{2\psi(\alpha_n, \beta_n, g(\alpha_n))}{(1 - \psi_0(g(\alpha_n), \alpha_n))^2}\right] q_n,\\
\gamma_{n+1} &= (1 + \psi_0(\alpha_{n+1}, \beta_n))(\alpha_{n+1} - \beta_n) + q_n, \qquad (3.1)\\
\beta_{n+1} &= \alpha_{n+1} + \frac{\gamma_{n+1}}{1 - \psi_0(g(\alpha_{n+1}), \alpha_{n+1})}.
\end{aligned}$$

A general convergence condition for this sequence is given below.

(H_3) There exists $\bar{p} \in [0, p)$ such that for each $n = 0, 1, 2, \ldots$

$$\psi_0(g(\alpha_n), \alpha_n) < 1 \quad \text{and} \quad \alpha_n \leq \bar{p}.$$

It follows that the following are valid

$$0 \leq \alpha_n \leq \beta_n \leq \alpha_{n+1} \leq \bar{p}$$

and there exists p^* such that $p^* \in [0, \bar{p}]$ and $\lim_{n\to+\infty} \alpha_n = p^*$.

(H_4) There exist an invertible operator $\mathscr{T}$ and $x_0 \in \mathcal{D}$ so that for each $x, y \in \mathcal{D}$ Note that the limit point p^* is the least bound (upper) of the sequence $\{\alpha_n\}$ which is unique.

$$\|\mathscr{T}^{-1}([x, y; F] - \mathscr{T})\| \leq \psi_0(\|x - x_0\|, \|y - x_0\|)$$
$$\text{and} \quad \|u - x_0\| \leq g(\|x - x_0\|) \leq \|x - x_0\|.$$

As in the local case but using (H_1) it follows that A_0^{-1} is invertible and

$$\|A_0^{-1}\mathscr{T}\| \leq \frac{1}{1 - \psi_0(g(\|x - x_0\|), \|x - x_0\|)}.$$

Choose $\|A_0^{-1}F(x_0)\| \leq \beta_0$.

(H_5) $\|\mathscr{T}^{-1}([u, x; F] - [y, x; F])\| \leq \psi(g(\|x - x_0\|), \|x - x_0\|, \|y - x_0\|)$. and

(H_6) $S[x_0, p^*] \subset \mathcal{D}$.

Remark 3.1.

(a) A possible and popular choice for $\mathscr{T}$ but not the most flexible one is $\mathscr{T} = F'(x_0)$. Therefore, the function F must be differentiable at $x = x_0$.

(b) The calculations

$$u - x_0 = x - x_0 + aF(x) = (I + a[x, x_0; F])(x - x_0) + aF(x_0) =$$
$$= [(I + a\mathscr{T}) + a\mathscr{T}\mathscr{T}^{-1}([x, x_0; F] - \mathscr{T})](x - x_0) + aF(x_0)$$

motivate the choice

$$g(t) = (\|I + a\mathscr{T}\| + |a|\|\mathscr{T}\|\psi_0(0, t))\, t + |a|\|F(x_0)\|.$$

Next, the semi-local convergence of the method (1.2) is developed using the assumptions (H_1)-(H_6).

Theorem 3.1. *Assume the conditions* (H_1)-(H_6) *are valid. Then, the following assertions hold*

$$\{x_n\} \subset S(x_0, p^*), \tag{3.2}$$

$$\|y_n - x_n\| \le \beta_n - \alpha_n, \tag{3.3}$$

$$\|x_{n+1} - y_n\| \le \alpha_{n+1} - \beta_n \tag{3.4}$$

and there exists a solution $x^* \in S[x_0, p^*]$ *of the equation* $F(x) = 0$ *so that*

$$\|x^* - x_n\| \le p^* - \alpha_n. \tag{3.5}$$

Proof. Induction is used to validate assertions (3.2)-(3.5). Clearly, (3.2) is valid if $n = 0$. Then, by the definition of β_0 in (H_4), (1.2) and (3.1) it follows

$$\|y_0 - x_0\| \le \beta_0 = \beta_0 - \alpha_0 < p^*.$$

Hence, the iterate $y_0 \in S(x_0, p^*)$ and (3.3) is valid for $n = 0$. The linear operator A_0 is invertible by (H_4), and

$$x_1 - y_0 = -A_0^{-1}F(y_0) - A_0^{-1}\left[(A_0 - [y_0, x_0; F]) - (A_0 - [y_0, u_0; F])\right] A_0^{-1}F(y_0). \tag{3.6}$$

Then, by (3.1), (H_4), (H_5) and (3.6) one gets

$$\begin{aligned}\|x_1 - y_0\| \le \Bigg[&\frac{1}{1 - \psi_0(g(\|x_0 - x_0\|, \|x_0 - x_0\|))} + \\ &+ \frac{2\psi(g(\|x_0 - x_0\|), \|x_0 - x_0\|, \|y_0 - x_0\|)}{(1 - \psi_0(g(\|x_0 - x_0\|), \|x_0 - x_0\|))^2}\Bigg] q_0 \le \alpha_1 - \beta_0\end{aligned}$$

and

$$\|x_1 - x_0\| \le \|x_1 - y_0\| + \|y_0 - x_0\| \le \alpha_1 - \beta_0 + \beta_0 - \alpha_0 = \alpha_1 < p^*,$$

where we also used

$$F(y_0) = F(y_0) - F(x_0) - A_0(y_0, x_0) = ([y_0, x_0; F] - A_0)(y_0 - x_0).$$

Thus,

$$\begin{aligned}\|\mathscr{T}^{-1}F(y_0)\| &\le \psi(g(\|x_0 - x_0\|), \|x_0 - x_0\|, \|y_0 - x_0\|) \le \\ &\le \psi(g(\alpha_0), \alpha_0, \beta_0) = q_0.\end{aligned}$$

Hence, the iterate $x_1 \in S(x_0, p^*)$ and (3.4) is valid for $n = 0$. Then, we can write

$$F(x_1) = F(x_1) - F(y_0) + F(y_0)$$

leading to

$$\begin{aligned}\|\mathscr{T}^{-1}F(x_1)\| &\leq \|\mathscr{T}^{-1}[x_1,y_0;F]\|\|x_1-y_0\|+q_0 \leq \\ &\leq \|\mathscr{T}^{-1}([x_1,y_0;F]-\mathscr{T}+\mathscr{T})\|\|x_1-y_0\|+q_0 \leq \\ &\leq (1+\psi_0(\|x_1-x_0\|,\|y_0-x_0\|))\|x_1-y_0\|+q_0 \leq \\ &\leq (1+\psi_0(\alpha_1-\beta_0))(\alpha_1-\beta_0)+q_0=\gamma_1, \\ \|y_1-x_1\| &\leq \|A_1^{-1}\mathscr{T}\|\|\mathscr{T}^{-1}F(x_1)\| \leq \\ &\leq \frac{\gamma_1}{1-\psi_0(g(\|x_1-x_0\|),\|x_1-x_0\|)} \leq \frac{\gamma_1}{1-\psi_0(g(\alpha_1),\alpha_1)} \\ &= \beta_1-\alpha_1\end{aligned}$$

and

$$\|y_1-x_0\| \leq \|y_1-x_1\|+\|x_1-x_0\| \leq \beta_1-\alpha_1+\alpha_1-\alpha_0=\beta_1<p^*.$$

So, the iterate $y_1 \in S(x_0,p^*)$ and (3.3) is valid for $n=1$. The preceding calculations can be repeated with x_0,y_0,x_1 replaced by x_m,y_m,x_{m+1} to terminate the induction. It follows by these assertions that the sequence $\{x_n\}$ is complete in Banach space $\mathcal{B}$ and as such it is convergent to some $x^* \in S[x_0,p^*]$. Moreover, by letting $m \to +\infty$ in the estimate $\|\mathscr{T}^{-1}F(x_m)\| \leq \gamma_m$ and using the continuity of F, we deduce $F(x^*)=0$. Furthermore, by letting $i \to +\infty$ in the estimate

$$\|x_{m+i}-x_m\| \leq \alpha_{m+i}-\alpha_m$$

we obtain the assertion (3.5). ∎

The isolation of the solution follows, as shown below:

Proposition 3.2. *Assume:*
There exists a solution $v \in S(x_0,r_1)$ of the equation $F(x)=0$ for some $r_1>0$; the first condition in (H_4) is valid in $S(x_0,r_1)$ and there exists $r_2 \geq r_1$ so that

$$\psi_0(r_1,r_2)<1.$$

Set $\mathcal{D}_3=\mathcal{D}\cap S[x_0,r_2]$.
Then, the equation $F(x)=0$ is uniquely solvable by v in the domain $\mathcal{D}_3$.

Proof. Let $v_0 \in \mathcal{D}_3$ with $F(v_0)=0$ and $v_0 \neq v$. Define the divided difference $T_0=[v,v_0;F]$. It follows that

$$\|\mathscr{T}^{-1}(T_0-\mathscr{T})\| \leq \psi(\|v-x_0\|,\|v_0-x_0\|) \leq \psi_0(r_1,r_2)<1.$$

Thus, it follows that $v_0=v$. ∎

Remark 3.2.

(a) Under all the assumptions of Theorem 3.1, set $x^*=v$ and $p^*=r_1$ in Proposition 3.2.

(b) The limit point p^* can be replaced by p in assumption (H_6).

4. Numericals

Let $\mathscr{T} = F'(x^*)$ and define the divided difference

$$[x, y; F] = \int_0^1 F'(x + \theta(y - x))d\theta.$$

Consider $a = 1$.

Example 4.1. Consider the following simultaneous differential equations which dictates the movement of an object defined by

$$\begin{aligned} F_1'(w_1) &= e^{w_1}, \\ F_2'(w_2) &= (e-1)w_2 + 1, \\ F_3'(w_3) &= 1. \end{aligned}$$

These equations are subject to initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. We define a vector function $F(w) = (e^{w_1} - 1, \frac{e-1}{2}w_2^2 + w_2, w_3)^T$ on the interval $\mathcal{D} = [0, 1]$, where $w = (w_1, w_2, w_3)^T$.

The derivative of the function $F(w)$ with respect to w is given by:

$$F'(w) = \begin{bmatrix} e^{w_1} & 0 & 0 \\ 0 & (e-1)w_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the definition of F, we find that $F'(x^*) = \mathbb{I}$, where $x^* = (0, 0, 0)^T$. Now, we aim to validate conditions (C_1)-(C_6). These conditions are valid for

$$\begin{aligned} \varphi_0(t_1, t_2) &= \frac{1}{2}(e-1)\left(t_1 + t_2\right), \\ \varphi_1(t) &= \frac{1}{2}(e-1)t, \\ \varphi_2(t_1, t_2, t_3) &= \frac{1}{2}(e-1)\left(\mathfrak{f}(t_1) + t_2 + t_3\right), \\ \varphi(t_1, t_2) &= \frac{1}{2}(e-1)\left(\mathfrak{f}(t_1) + t_2\right), \\ \mathfrak{f}(t) &= 2 + \frac{1}{2}(e-1)t. \end{aligned}$$

Upon solving, we find $\rho_0 = 0.352417$, which implies $M_0 = [0, \rho_0)$. The radii are further determined as $s_1 = 0.184268$, and $s_2 = 0.118707$. Utilizing equation (2.1), we obtain the radius of convergence as $s^* = 0.118707$.

Example 4.2. Consider the set $\mathcal{D} = \mathcal{B} = \mathbb{R}$, and let's define a function F on this set as follows: $F(x) = \sin(x)$. Consequently, the derivative of F, denoted as $F'(x)$, is $F'(x) = \cos(x)$. We find that $x^* = 0$ is a solution of this function. Now, let's examine the conditions (C_1) to (C_6), and we observe that they are satisfied when:

$$\begin{aligned} \varphi_0(t_1, t_2) &= t_1 + t_2, \\ \varphi_1(t) &= t, \\ \varphi_2(t_1, t_2, t_3) &= \mathfrak{f}(t_1) + t_2 + t_3, \\ \varphi(t_1, t_2) &= \mathfrak{f}(t_1) + t_2, \\ \mathfrak{f}(t) &= 2 + t. \end{aligned}$$

After solving, we obtain $\rho_0 = 0.302776$. This result indicates that $M_0 = [0, \rho_0)$. The radii are subsequently calculated as $s_1 = 0.158312$ and $s_2 = 0.101986$. By applying equation (2.1), we determine the radius of convergence to be $s^* = = 0.101986$.

All computations are performed in Mathematica programming package version 11.3.0.0 with 600 digits. These computations were carried out on an Intel(R) Core(TM) i5 - 8250U CPU @ 1.60 GHz 1.80 GHz with 8 GB of RAM, running on Windows 11 Home version 22H2. To stop the iterative process, we have used the criterion: $error = |x_N - x_{N-1}| < \epsilon$, where $\epsilon = 10^{-50}$ and N represents the number of iterations required for convergence.

Example 4.3. Take into account the pair of equations:

$$(x-1)^4 + e^{-y} - y^2 + 3y + 1 = 0$$
$$4\sin(x-1) - \log(x^2 - x + 1) - y^2 = 0.$$

These equations are governed by the initial conditions $x_0 = \{2, -2\}^T$, and the solution $x^* = \{2.07044337667988070\ldots, -1.53017120230005783\ldots\}^T$.

Detailed error estimates for the solution when $a = 0.01$ can be found in Table 1. Following a comprehensive examination of the equation system, it becomes evident that convergence towards the solution represented as x^* occurs within a maximum of three iterations.

$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$
$3.65 * 10^{-2}$	$2.38 * 10^{-5}$	$1.12 * 10^{-18}$

Table 1. Error estimates for Example 4.3

Example 4.4. Now, let's delve into a system of five equations,

$$\sum_{j=1, j\neq i}^{5} x_j - e^{-x_i} = 0, \quad 1 \leq i \leq 5.$$

We start with an initial value $x_0 = \{1, 1, 1, 1, 1\}^T$. The solution to this particular problem is represented as:

$$x^* = \{0.20388835470224016, \cdots, 0.20388835470224016\}^T.$$

Table 2 provides error estimates for the solution in the case where $a = 0.01$. After conducting a comprehensive analysis of the equation system, it becomes clear that convergence towards x^* is achieved in a maximum of three iterations.

$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$
$2.73 * 10^{-5}$	$4.77 * 10^{-19}$	$6.44 * 10^{-83}$

Table 2. Error estimates for Example 4.4

5. Conclusion

We have introduced a new methodology that thoroughly examines how well high-order methods converge, both locally and semi-locally. What sets our approach apart is that it doesn't rely on extra derivatives beyond what the method itself already uses. Unlike previous methods that assumed the existence of high-order derivatives, ours doesn't require that, opening up new possibilities for convergence, error estimation, and determining uniqueness. Importantly, our method is highly adaptable, working well with a wide range of high-order methods, including both single-step and multi-step ones. This versatility expands the potential applications across various scientific and engineering fields. These encompass not only single-step but also multi-step methods, as exemplified by references [2–4, 17]. This approach is also applicable for the extension of methods found in [5–15, 19, 20], where our future research plans will uncover them.

References

[1] **Argyros, I.K.,** *The Theory and Applications of Iteration Methods*, 2nd edition, CRC Press/Taylor and Francis Publishing Group Inc., Boca Raton, Florida, USA,2022

[2] **Argyros, I.K., C.I. Argyros, J.A. John and J. Jayaraman,** On two competing methods with optimal eighth order convergence, *Int. J. Appl. Comput. Math.*, **9** (2023), 72.

[3] **Argyros, I.K., S. Regmi, J.A. John and J. Jayaraman,** Efficient derivative-free class of seventh order method for non-differentiable equations, *European Journal of Mathematical Analysis*, **3** (2023), 19–19.

[4] **Argyros, I.K., S. Regmi, J.A. John and J. Jayaraman,** Extended convergence for two sixth order methods under the same weak conditions, *Foundations*, **3** (2023), 127–139.

[5] **Behl, R., A. Cordero, S.S. Motsa and J.R. Torregrosa,** Stable high-order iterative methods for solving nonlinear models, *Appl. Math. Comput.*, **303** (2017), 70–88.

[6] **Chun, C. and B. Neta,** Developing high order methods for the solution of systems of nonlinear equations, *Appl. Math. Comput.*, **342** (2019), 178–190.

[7] **Cordero, A., J.G. Maimó, J.R. Torregrosa and M.P. Vassileva,** Iterative methods with memory for solving systems of nonlinear equations using a second order approximation, *Mathematics*, **7** (2019), 1069.

[8] **Ezquerro, J.A., M.A. Hernández, N. Romero and A.I. Velasco,** On Steffensen's method on Banach spaces, *J. Comput. Appl. Math.*, **349** (2013), 9–23.

[9] **Hueso, J.L., E. Martínez and C. Teruel,** Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems, *J. Comput. Appl. Math.*, **275** (2015), 412–420.

[10] **Jangveladze, T., Z. Kiguradze and B. Neta,** Large time behavior of solutions and finite difference scheme to a nonlinear integro-differential equation, *Comput.Math.Appl.*, **57** (2009), 799–811.

[11] **Kumar, Abhimanyua., D.K. Gupta, E. Martínez and J.L. Hueso,** Convergence and dynamics of improved Chebyshev-secant-type methods for non differentiable operators, *Numer. Algorithms*, **86** (2021), 1051–1070.

[12] **Liu, Z., Q. Zheng and P. Zhao,** A variant of Steffensen's method of fourth-order convergence and its applications, *Appl. Math. Comput.*, **216** (2010), 1978–1983.

[13] **Montazeri, H., F. Soleymani, S. Shateyi, S.S. Motsa, et al.,** On a new method for computing the numerical solution of systems of nonlinear equations, *J. Appl. Math.*, **2012** (2012).

[14] **Narang, M., S. Bhatia, A.S. Alshomrani and V. Kanwar,** General efficient class of Steffensen type methods with memory for solving systems of nonlinear equations, *J. Comput. Appl. Math.*, **352** (2019), 23–39.

[15] **Narang, M., S. Bhatia and V. Kanwar,** New two-parameter Chebyshev–Halley-like family of fourth and sixth-order methods for systems of nonlinear equations, *Appl. Math. Comput.*, **275** (2016), 394–403.

[16] **Petković, M.S., B. Neta, L.D. Petković and J. Džunić,** *Multipoint Methods for Solving Nonlinear Equations*, Elsevier, Amsterdam, 2013.

[17] **Regmi, S., I.K. Argyros, J.A. John and J. Jayaraman,** Extended convergence of two multi-step iterative methods, *Foundations*, **3** (2023), 140–153.

[18] **Sharma, J.R. and H. Arora,** An efficient derivative free iterative method for solving systems of nonlinear equations, *Appl. Anal. Discrete Math.*, (2013), 390–403.

[19] **Sharma, J.R. and H. Arora,** Efficient Jarratt-like methods for solving systems of nonlinear equations, *Calcolo*, **51** (2014), 193–210.

[20] **Sharma, J.R., R. Sharma and N. Kalra,** A novel family of composite Newton–Traub methods for solving systems of nonlinear equations, *Appl. Math. Comput.*, **269** (2015), 520–535.

[21] **Traub, J.F.,** *Iterative Methods for the Solution of Equations*, vol. 312. American Mathematical Soc., Providence, 1982.

I.K. Argyros
Department of Computing and Mathematical Sciences
Cameron University
Lawton, OK 73505
USA
iargyros@cameron.edu

J. Jayaraman and J.A. John
Department of Mathematics
Puducherry Technological University
Pondicherry, 605014
India
jjayakumar@ptuniv.edu.in
jinny3@pec.edu