Annales Univ. Sci. Budapest., Sect. Comp. 57 (2024) 297–319

NEW ZERO-FINDING METHODS BASED ON DOUBLE INTERCEPT FORM OF A STRAIGHT LINE

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Communicated by László Szili

(Received March 7, 2024; accepted June 7, 2024)

Abstract. This paper deals with quadratically convergent zero-finding methods based on double intercept form of a straight line. Hybrid zero-finding method performs better across different situations as compared to individual methods. Further, we extend hybrid approach to solve various nonlinear systems. This scheme provides advantage in the cases where the Newton's iteration process may not converge or even fails. Further, local as well as semi-local convergence analysis is done for the proposed methods. The main idea for the local convergence analysis is to estimate the bounds on convergence domain as well as the error approximations of the iterates. Depending upon the choice of initial estimate in the given domain, the semilocal convergence analysis is proved, which ensures the convergence of iterates to a unique solution in that domain.

1. Introduction

Root-finding problem is a fundamental problem encountered in various scientific and engineering disciplines. Nonlinear equations in general do not have

Key words and phrases: Iterative scheme, nonlinear equation, order of convergence, error equation.
2010 Mathematics Subject Classification: 65K10, 47H04, 65G99.

direct analytical solutions and therefore, various iterative techniques are used to approximate their solutions. One of the oldest and most popular Newton's method (NM) [10, 2, 14, 9, 4, 3] is given by

$$w_{t+1} = w_t - \frac{f(w_t)}{f'(w_t)}, \qquad t = 0, 1, 2, \ldots . \tag{1.1}$$

The Newton's method (MNM) for finding the solutions of a nonlinear system is expressed as

$$W^{(t+1)} = W^{(t)} - [F'(W^{(t)})]^{-1} F(W^{(t)}), \qquad t = 0, 1, 2, ... \tag{1.2}$$

where, $F(W) = (f_1(W), f_2(W), \ \ldots \ , f_m(W))^T$, $W = (w_1, w_2, \ \ldots \ , w_m)^T$, consisting of 'm' nonlinear equations in 'm' variables and $F'(W)$ is the Jacobian of function $F(W)$. This method has rapid convergence and converges quadratically provided that the chosen starting guess is sufficiently close to the required root or solution. There are also some other commonly used iterative methods namely fixed-point iteration methods [7, 1, 11], secant method etc. [6, 5] for solving a given problem, numerically. Each iterative scheme has its own advantages and disadvantages. Quite often the method is selected depending on some characteristic of the problem or to achieve some desired accuracy. In addition to that, it is important to consider factors such as convergence properties, namely order of convergence, computational cost, computational efficiency and robustness while selecting an iterative method for a given problem. Newton's method is often very efficient and commonly used method, but still in many situations it has been seen that the method performs poorly. Convergence to the undesired solution, convergence to a singular point and slow convergence or divergence are some problems that can arise in practical situations.

The rationale behind the work is to overcome the above mentioned faults of Newton's iteration process by using simple modification of the iterative scheme. The beauty of hybrid approach is that it works efficiently even if the derivative or Jacobian is zero. Further, local as well as semilocal convergence analysis is done for the proposed methods. The convergence of an iterative scheme is found by using the Taylor series expansions, which requires existence of higher order derivatives. This bounds the applicability of methods, whereas local and semilocal convergence involves derivatives only upto first order.

2. Development of zero-finding method

This section deals with the geometrical construction of zero-finding schemes.

(i) **First method** $(DIM1)$**.** Let

$$f(w) = 0, \tag{2.1}$$

be the equation of a nonlinear function where $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a function with properties that it is sufficiently differentiable and has simple zeros. Let $y = f(w)$ be the graph of the function $f(w)$ and $w = w^*$ be the required zero. Also, let $w_0 \neq 0$ be the starting guess and the corresponding point $(w_0, f(w_0))$ lies on the graph. A line connecting the points $(0, f(w_0))$ and $(w_0, 0)$ is drawn as shown in Figure 1. The equation of line in the intercept form can be written

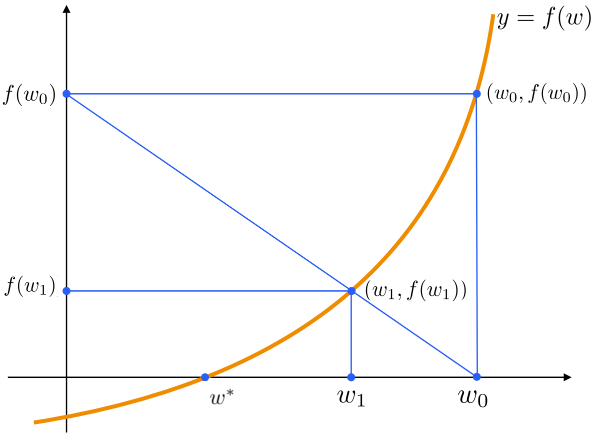

Figure 1. Graph of function $y = f(w)$

as

$$\frac{w}{w_0} + \frac{y}{f(w_0)} = 1. \tag{2.2}$$

Suppose that the point of intersection of the line and the graph $y = f(w)$ is $(w_1, f(w_1))$. Then, the point of intersection $(w_1, f(w_1))$ satisfies equation (2.2), where $w_1 = w_0 + h$. Thus, one obtains

$$\frac{w_0 + h}{w_0} + \frac{f(w_0 + h)}{f(w_0)} = 1. \tag{2.3}$$

Using Taylor's series expansion upto $O(h)$ and simplifying, we obtain

$$h = -\frac{w_0 f(w_0)}{f(w_0) + w_0 f'(w_0)}. \tag{2.4}$$

Thus, using value of h from (2.4) in w_1, gives

$$w_1 = w_0 - \frac{w_0 f(w_0)}{f(w_0) + w_0 f'(w_0)}. \tag{2.5}$$

The expression (2.5) can be generalized as

$$w_{t+1} = w_t - \frac{w_t f(w_t)}{f(w_t) + w_t f'(w_t)}, \quad t \geq 0. \tag{2.6}$$

We call this scheme as double intercept method $(DIM1)$.

(ii) **Second method** $(DIM2)$**.** Let $w = w^*$ be the required zero and $w_0 \neq 0$ the starting guess. The equation of line joining the points $(w_0, 0)$ and $(0, -f(w_0))$ is given as

$$\frac{w}{w_0} - \frac{y}{f(w_0)} = 1. \tag{2.7}$$

Then, proceeding in similar way as in first method $(DIM1)$, we get the following iterative scheme

$$w_{t+1} = w_t + \frac{w_t f(w_t)}{f(w_t) - w_t f'(w_t)}, \quad t \geq 0. \tag{2.8}$$

We name it as $(DIM2)$.

(iii) **Hybrid method** $(DIM3)$**.** As it is well-known, combining two methods involves creating a hybrid approach that leverages the strengths of each other. Therefore, hybrid form of above two iterative schemes is given as below:

$$w_{t+1} = w_t \mp \frac{w_t f(w_t)}{f(w_t) \pm w_t f'(w_t)}, \quad t \geq 0. \tag{2.9}$$

Remark 2.1. The choice of sign is dependent on the following condition: If $|f(w_0)+w_0 f'(w_0)| \geq |f(w_0)-w_0 f'(w_0)|$, we choose $w_{t+1} = w_t - \frac{w_t f(w_t)}{f(w_t)+w_t f'(w_t)}$, otherwise we will choose $w_{t+1} = w_t + \frac{w_t f(w_t)}{f(w_t)-w_t f'(w_t)}$.

Remark 2.2. If $f'(w_0) = 0$, then we choose the formula $w_{t+1} = w_t + + \frac{w_t f(w_t)}{f(w_t)-w_t f'(w_t)}$, as in this case $w_1 = 2w_0$, which is well-defined and it can generate a convergent sequence of successive approximations.

3. Multivariate double intercept method $(MDIM)$

We can usually find the solutions to system of nonlinear equations when number of unknowns matches the number of equations. Therefore, methods $(DIM1)$, $(DIM2)$ and $(DIM3)$ corresponding to nonlinear system are given as follows:

(i) **First method** $(MDIM1)$

(3.1)

$$W^{(t+1)} = W^{(t)} \; - \; \Big[D(F(W^{(t)})) \; + \; ||W^{(t)}|| \; F'(W^{(t)})\Big]^{-1} ||W^{(t)}|| \; F(W^{(t)}),$$

where, $D(F(W^{(t)}))$ is a diagonal matrix of the same order as the order of $||W^{(t)}|| \; F'(W^{(t)})$ i.e.

$$D(F(W^{(t)})) = \begin{bmatrix} f_1(W^{(t)}) & 0 & \cdots & 0 \\ 0 & f_2(W^{(t)}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_m(W^{(t)}) \end{bmatrix}.$$

Further, this can be rewritten as

$D(F(W^{(t)})) = \text{diag}\big(f_1(W^{(t)}), f_2(W^{(t)}), ..., f_m(W^{(t)})\big)$ and $||W^{(0)}|| \neq 0$.

(ii) **Second method** $(MDIM2)$

(3.2)

$$W^{(t+1)} = W^{(t)} \; + \; \Big[D(F(W^{(t)})) \; - \; ||W^{(t)}|| \; F'(W^{(t)})\Big]^{-1} ||W^{(t)}|| \; F(W^{(t)}).$$

(iii) **Modified hybrid method** $(MDIM3)$

(3.3)

$$W^{(t+1)} = W^{(t)} \; \mp \; \Big[D(F(W^{(t)})) \; \pm \; ||W^{(t)}|| \; F'(W^{(t)})\Big]^{-1} ||W^{(t)}|| \; F(W^{(t)}).$$

4. Convergence analysis

This section discusses the order of convergence of the proposed iterative schemes.

Theorem 4.1. *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function and $w = w^* \neq 0$ be a simple zero. Assume that $w_0 \neq 0$ is an initial guess for w^*. Then, the iterative scheme (2.6) converges quadratically.*

Proof. Expanding $f(w_t)$ about w^* by using Taylor's expansion, we have

$$f(w_t) = f'(w^*)(e_t + c_2 e_t^2 + c_3 e_t^3 + O(e_t^4)), \tag{4.1}$$

where $e_t = w_t - w^*$ and $c_k = \frac{f^k(w^*)}{k! f'(w^*)}$, $\quad k = 2, 3, 4....$

Furthermore, one has

$$f'(w_t) = f'(w^*)(1 + 2c_2 e_t + 3c_3 e_t^2 + O(e_t^3)), \tag{4.2}$$

and

$$\frac{f(w_t)}{f'(w_t)} = e_t - c_2 e_t^2 + 2(c_2^2 - c_3)e_t^3 + O(e_t^4). \tag{4.3}$$

Using (4.2) and (4.3) in equation (2.6), one has

$$\begin{aligned} e_{t+1} = e_t - \frac{(e_t + w^*)(e_t - c_2 e_t^2 + 2(c_2^2 - c_3)e_t^3)}{e_t - c_2 e_t^2 + 2(c_2^2 - c_3)e_t^3 + e_t + w^*} = \\ = \frac{(c_2 w^* + 1)e_t^2 - 2(c_2^2 - c_3)e_t^3 w^* + O(e_t^4)}{e_t - c_2 e_t^2 + 2(c_2^2 - c_3)e_t^3 + e_t + w^*} = \\ = \frac{(c_2 w^* + 1)e_t^2 - 2(c_2^2 - c_3)e_t^3 w^* + O(e_t^4)}{w^*(1 + w^* e_t - c_2 e_t^2 + 2(c_2^2 - c_3)e_t^3 + e_t w^*)}. \end{aligned}$$

Simplifying this, one obtains

$$e_{t+1} = \frac{(c_2 w^* + 1)e_t^2}{w^*} + O(e_t^3). \tag{4.4}$$

Hence, the iterative scheme (2.6) converges quadratically. ■

On similar lines, one can prove that the iterative scheme (2.8) converges quadratically.

Next, we discuss the convergence of multivariate iterative scheme (3.1).

Theorem 4.2. *Let $D \subset \mathbb{R}^m$ be an open convex set, $F : D \to \mathbb{R}^m$ such that $F(W^*) = 0$ and $||W^{(0)}|| \neq 0$. Let $F(W)$ be sufficiently Fréchet differentiable in some neighbourhood S of the zero. If $D(F(W)) + ||W||\ F'(W)$ is nonsingular matrix for all W in S, then iterative scheme (3.1) is quadratically convergent.*

Proof. Let $e^{(t)} = W^{(t)} - W^*$ and $F(W) = (f_1(W), f_2(W), \ldots, f_m(W))^T$.

Using Taylor's series expansion and considering that $f_j(W^*) = 0$ and $f_j'(W^*) \neq 0$, one can expand $f_j(W^{(t)})$ about W^* to obtain

$$f_j(W^{(t)}) = f_j'(W^*)e^{(t)} + \frac{1}{2!}f_j''(W^*)(e^{(t)})^2 + O((e^{(t)})^3), \quad j = 1, 2, \ldots, m. \tag{4.5}$$

In vector notation, one can write (4.5) as

$$F(W^{(t)}) = F'(W^*)e^{(t)} + \frac{1}{2!}F''(W^*)(e^{(t)})^2 + O((e^{(t)})^3). \tag{4.6}$$

This further implies that

$$F(W^{(t)}) = F'(W^*)[e^{(t)} + C_2(e^{(t)})^2 + O((e^{(t)})^3)], \tag{4.7}$$

where $C_i = \frac{1}{i!}[F'(W^*)]^{-1}F^{(i)}(W^*)$, $i = 2, 3, 4...$

Also,

$$F'(W^{(t)}) = F'(W^*)[I + 2C_2 e^{(t)} + O((e^{(t)})^2)]. \tag{4.8}$$

Also,

$$D\left(\frac{1}{||W^{(t)}||}F(W^{(t)})\right) = \frac{1}{||W^{(t)})||}diag\left(f_1(W^{(t)}), f_2(W^{(t)}), \dots, f_m(W^{(t)})\right). \tag{4.9}$$

From, (4.5) each $f_j(W^{(t)})$ is of order $e^{(t)}$ as

$$f_j(W^{(t)}) = \left(f_j'(W^*) + \frac{1}{2!}f_j''(W^*)e^{(t)} + O((e^{(t)})^2)\right)e^{(t)}, \quad j = 1, 2, \dots, m. \tag{4.10}$$

So, from (4.9) and (4.10)

$$D\left(\frac{1}{||W^{(t)}||}F(W^{(t)})\right) = O(e^{(t)}). \tag{4.11}$$

Further, we can write

$$\begin{aligned} &\left[D(F(W^{(t)})) \ + \ ||W^{(t)}|| \ F'(W^{(t)})\right]^{-1} = \\ &= \frac{1}{||W^{(t)}||}\left[D(\frac{1}{||W^{(t)}||}F(W^{(t)})) + F'(W^{(t)})\right]^{-1} = \\ &= \frac{1}{||W^{(t)}||} \ \left[F'(W^*)\right]^{-1}[I + O(e^t)] \, . \end{aligned} \tag{4.12}$$

Post multiplying equation (4.12) by $||W^{(t)}|| \ F(W^{(t)})$, we get

$$\begin{aligned} &\left[D(F(W^{(t)})) + ||W^{(t)}|| \ F'(W^{(t)})\right]^{-1} ||W^{(t)}|| \ F(W^{(t)}) \ = \\ &= [I + O(e^t)] \ [F'(W^*)]^{-1} \ F(W^{(t)}) = \\ &= e^{(t)} + O\left((e^{(t)})^2\right). \end{aligned} \tag{4.13}$$

Using equation (4.13) in iterative scheme (3.1), one gets

$$e^{(t+1)} = O((e^{(t)})^2). \tag{4.14}$$

Hence, the iterative scheme (3.1) converges quadratically. ■

Remark 4.1. On similar lines, one can prove that the iteration process (3.3) is quadratically convergent.

5. Alternative convergence analysis

The convergence analysis of Section 4 is based on Talyor series. According to error estimate (4.1) the function f must be atleast four times differentiable. Let us look at the motivational example:
Define the function $f : [-1.5, 1.5] \to \mathbb{R}$ by

$$f(w) = \begin{cases} c_1 w^2 \log(w) + c_2 w^5 + c_3 w^4, & \text{if } w \neq 0 \\ 0, & \text{if } w = 0, \end{cases} \tag{5.1}$$

where $c_1 \neq 0$ and $c_2 + c_3 = 0$. It follows by this definition that $f(1) = 0$, and $w^* = 1$ belongs in the domain of the function f. But the function f''' is unbounded on the interval $[-1.5, 1.5]$, since it is not continuous at $w = 0$. Thus, Theorem 4.1 cannot assure the convergence of the scheme (2.6) to the solution w^*. However the scheme (2.6) converges to w^* if for example $c_1 = c_2 = 1$, $c_3 = -1$ and $w_0 = 1.1$. Consequently, the sufficient convergence conditions of the schemes in Section 4 can be weakened. Moreover, no priori upper bounds on $|w^* - w_t|$ are available, so we do not know how many iterations are needed to arrive at a desired error tolerance $\epsilon > 0$. Furthermore, the isolation of the solution in some neighbourhood containing it is not given in Section 4. Finally, the results are local. That is the assumption of a simple solution is required to show the convergence of the scheme. Similar work can be seen in [12, 13]. That is why in this section we are motivated to address these limitations in the applicability of the scheme (2.6). In particular, we positively handle these limitations since:

- (i) The convergence is shown using conditions only on the functions on the scheme, i.e. only on f and f'.
- (ii) The number of iterations to reach the error tolerance $\epsilon > 0$ is known in advance, since computable a priori estimates become available.
- (iii) Neighbourhoods containing only one solution of the equation are provided.
- (iv) Knowledge of the solution w^* is not required.

It is worth noticing that the methodology to follow is also applicable to other schemes using inverses of functions.
It is convenient to rewrite the scheme (2.6) as

$$w_{t+1} = w_t - A_t^{-1} w_t f(w_t), \tag{5.2}$$

where $A_t = f(w_t) + f'(w_t) w_t$. Set $D = [a, b]$ for $a \geq 0$.

The convergence conditions are for our analysis.

Suppose:

(H_1) There exists a function $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$ continuous as well as nondecreasing such that the function $g_0(w)-1$ has a smallest positive zero. Denote such zero by ρ_0. Set $M_0 = [0, \rho_0)$.

(H_2) There exist positive numbers $0 < \alpha, \gamma$ with $\alpha < \gamma$, a real number δ such that for $p : M_0 \to \mathbb{R}_+$ defined by
$p(w)=\frac{1}{\alpha}\left((1+\int_0^1 g_0(\theta w)d\theta + g_0(w)+|\delta|)w + g_0(w)\gamma\right)$
the function $p(w)-1$ has smallest zero in the interval $M_0 - \{0\}$. Denote such a zero by ρ_1. Set $\rho = min\{\rho_0, \rho_1\}$, and $M = [0, \rho)$.

(H_3) There exists a function $g : M \to \mathbb{R}_+$ such that for $h : M \to \mathbb{R}_+$ defined by

$$h(w) = \frac{\int_0^1 g((1-\theta)w)d\theta}{1-g_0(w)} + \frac{(1+\int_0^1 g_0(\theta w)d\theta)^2 w}{\alpha(1-p(w))(1-g_0(w))}$$

the function $h(w)-1$ has a smallest zero in the interval $M - \{0\}$. Denote such a zero by r. The parameter r is shown to be a radius of convergence for scheme (5.2) (see Theorem 5.1).

(H_4) $a \leq \alpha \leq \gamma \leq b$ and $f(\alpha)f(\gamma) < 0$. It follows by intermediate value theorem that the function f has at least one zero $w^* \in [\alpha, \gamma]$.

Next, we relate the functions g_0 and g to the functions f and f' appearing on the scheme(5.2).

(H_5) There exists a nonzero number δ such that for each $w \in D$

$$|\delta^{-1}(f'(w) - \delta)| \leq g_0(|w - w^*|).$$

(H_6) Set $D_0 = [0, \rho]$. Then

$$|\delta^{-1}(f'(u_2) - f'(u_1))| \leq g(|u_2 - u_1)|)$$

for each $u_1, u_2 \in D_0$ and

(H_7) $D_1 := [w^* - r, w^* + r] \subset D$. Set $D_2 = D_1 - \{0\}$.

The main convergence result for the scheme (5.2) follows using the developed terminology and the conditions $(H_1) - (H_7)$.

Theorem 5.1. *Suppose that the conditions $(H_1) - (H_7)$ hold. Then, the sequence $\{w_t\}$ produced by the scheme (5.2) for $w_0 \in D_2$ is well defined in the interval D_1, stays in D_1 and is convergent to w^*. Moreover, the following error estimates hold for each $t = 0, 1, 2, \ldots$.*

$$|w_{t+1} - w^*| \leq h(|w_t - w^*|)|w_t - w^*| \leq |w_t - w^*| < r. \tag{5.3}$$

Proof. Let $v \in D_2$ be arbitrary. Using the definition of the parameter r, and the conditions $(H_1)-(H_3)$ and (H_5), we have in turn that

$$|\delta^{-1}(f'(v)-\delta)| \leq g_0(|v-w^*|) \leq g_0(r) < 1.$$

So, it follows that $f'(v) \neq 0$ and

$$|f'(v)^{-1}\delta| \leq \frac{1}{1-g_0(|v-w^*|)}, \tag{5.4}$$

as a consequence of the Banach lemma on nonzero functions [2, 3].

Next, we need to show that $A(v) \neq 0$. We can write for $A(v) = f(v)+vf'(v)$ as

$$A(v) = f(v) + (f'(v)-\delta)(v-w*) + \delta(v-w*) + (f'(v)-\delta)w^* + \delta w^*.$$

Thus, using also the conditions (H_4) we get

$$\begin{aligned}|(\delta w^*)^{-1}(A(v)-\delta w^*)| \leq \frac{1}{|w^*|}\Big(1+\int_0^1 g_0(\theta|v-w^*|)d\theta\Big)\, |v-w^*|+ \\ +g_0(|v-w^*|)|v-w^*| + |v-w^*| + |w^*|g_0(|v-w^*|) \leq \\ \leq p(|v-w^*|) < 1.\end{aligned}$$

Thus, $A(v) \neq 0$ and

$$|A(v)^{-1}\delta| \leq \frac{1}{\alpha(1-g_0(|v-w^*|))}, \tag{5.5}$$

where we also used the estimate

$$f(v) = f(v) - f(w^*) = \int_0^1 f'(w^*+\theta(v-w^*))d\theta(v-w^*),$$

implying

$$\begin{aligned}|\delta^{-1}f(v)| = \left|\delta^{-1}\left(\int_0^1 f'(w^*+\theta(v-w^*)) - \delta + \delta\right) d\theta(v-w^*)\right| \leq \\ \leq \left(1+\int_0^1 g_0(\theta|v-w^*|)d\theta\right)|v-w^*|.\end{aligned}$$

By the hypothesis $w_0 \in D_2$ it follows by (5.3) and (5.4) that $f'(w_0) \neq 0$ and $A(w_0) \neq 0$. Hence, the iterate w_1 is well defined by the scheme (5.2) for $t=0$. Moreover, we can write in turn

$$w_1 - w^* = w_0 - w^* - f'(w_0)^{-1}f(w_0) + \left(f'(w_0)^{-1} - w_0A_0^{-1}\right)f(w_0) =$$

$$= w_0 - w^* - f'(w_0)^{-1}f(w_0) - A_t^{-1}(f'(w_t)w_t - A_t)f'(w_t)^{-1}f(w_t) =$$

$$= w_0 - w^* - f'(w_0)^{-1}f(w_0) + A_0^{-1}f(w_0)f'(w_0)^{-1}f(w_0). \tag{5.6}$$

The application of the condition (H_6), the definition of the function h, the parameter r, (5.4), (5.5) and (5.6) imply in turn that

$$\begin{aligned} |w_1 - w^*| &\leq \left[\frac{\int_0^1 g((1-\theta)|w_0 - w^*|)d\theta}{1 - g_0(|w_0 - w^*|)}\right]|w_0 - w^*| + \\ &+ \left[\frac{(1 + \int_0^1 g_0(\theta|w_0 - w^*|d\theta)^2|w_0 - w^*|}{\alpha(1-p_0)(1 - g_0(|w_0 - w^*|))}\right]|w_0 - w^*| \leq \end{aligned} \tag{5.7}$$

$$\leq h(|w_0 - w^*|)|w_0 - w^*| \leq |w_0 - w^*| < r.$$

Hence, the assertion (5.3) holds for $t = 0$, and the iterate $w_1 \in D_1$. Simply exchange w_0, w_1 by w_m, w_{m+1}, respectively in the preceding estimates to complete the induction for the assertion (5.3). Then, from the estimation

$$|w_{m+1} - w^*| \leq b|w_m - w^*| \leq b^{m+1}|w_0 - w^*| < r,$$

where $b = h(|w_0 - w_t|) \in [0, 1]$. We deduce that $\lim_{m\to\infty} w_m = w^*$ and that the iterate $w_{m+1} \in D_1$. ∎

Next, an interval is determined containing only one solution of the equation $f(w) = 0$.

Proposition 5.2. *Suppose: $G : \mathbb{R} \to \mathbb{R}$ is a differentiable function. The function G has a zero v^*. The condition (H_5) holds (for $w^* = v^*$) on the interval $(w^* - r_1, w^* + r_1) = D_3$ for some $r_1 > 0$, and*

$$\int_0^1 g_0(\theta r_1)d\theta < 1. \tag{5.8}$$

Then, the only solution of the only zero of the function f in the interval D_3 is v^*.

Proof. Suppose that there exists $u \in D_3$ such that $f(u) = 0$ and $u \neq v^*$. Define the function

$$G = \int_0^1 f'(v^* + \theta(u - v^*))d\theta.$$

Using the conditions (H_4) and (5.8), we get in turn that

$$|\delta^{-1}(G-\delta)| \leq \int_0^1 g_0(\theta|u-v^*|)d\theta \leq \int_0^1 g_0(\theta r_1)d\theta < 1,$$

so $G \neq 0$. Then, by the identity $u - v^* = G^{-1}(f(u) - f(v^*)) = G^{-1}(0) = 0$, we conclude that $u = v^*$. ∎

Remark 5.1.

(i) A possible choice for $\delta = f'(w^*)$. In this case the zero w^* is simple. However, such a condition is not assumed in the Theorem 5.1. Therefore, the results of the Theorem 5.1 can be used to approximate non simple zeros of the function f using scheme (5.2). Another choice for $\delta = 1$ or any other choice as long as the conditions (H_5) and (H_6) hold.

(ii) Under all the hypothesis of the Theorem 5.1 we can set $v^* = w^*$ and $r_1 = r$ in the Propositions 5.2.

(iii) The second condition in (H_4) can be dropped if the estimate $\alpha \leq |w^*| \leq \gamma$ can be established some way other than the intermediate value theorem.

Next, we develop the semilocal analysis of convergence by relying on majorizing sequences, and in a way analogous to the local case. But, the role of w^* is exchanged by w_0. However, the computations are similar. It is convenient for the analysis to rewrite the scheme (5.2) as

$$\begin{aligned} u_t &= w_t - f'(w_t)^{-1} f(w_t), \\ w_{t+1} &= w_t - A_t^{-1} w_t \; f(w_t). \end{aligned} \tag{5.9}$$

Suppose:

(B_1) There exists a function $g_1 : \mathbb{R}_+ \to \mathbb{R}_+$ continuous as well as nondecreasing such that the function $g_1(w) - 1$ has a smallest positive zero. Denote such zero by ρ_2. Set $M_1 = [0, \rho_2]$.

(B_2) There exists a function $g_2 : M_1 \to \mathbb{R}_+$. Define the sequences $\{\lambda_t\}, \{\mu_t\}$ for some d, d_1 satisfying $0 < d < d_1$, $\lambda_0 = 0, \mu_0 \geq 0$, and each $t = 0, 1, 2, \ldots$ by

$$q_t = \frac{1}{d}\left[(1 + g_1(\lambda_t))(u_t - \lambda_t) + g_1(\lambda_t)\mu_t + \lambda_t + g_1(\lambda_t)d_1\right],$$

$$\lambda_{t+1} = \mu_t + \frac{(1 + g_1(\lambda_t))(\mu_t - \lambda_t)^2}{d(1 - q_t)},$$

$$\beta_{t+1} = \int_0^1 g_2((1-\theta)(\lambda_{t+1}-\lambda_t))d\theta(\lambda_{t+1}-\lambda_t)+ \\ +(1+g_1(\lambda_t))(\lambda_{t+1}-\mu_t). \tag{5.10}$$

These sequences $\{\lambda_t\}$, $\{\mu_t\}$ are shown to be majorizing for scheme (5.9) in Theorem 5.3. A convergence condition is required for these sequences.

(B_3) There exists $\rho_3 \in [0, \rho_2)$ such that for each $t = 0, 1, 2, \dots$ $g_1(\lambda_t) < < 1$, $q_t < 1$ and $\lambda_t \leq \rho_3$. It follows by (5.10) and this condition that $0 \leq \lambda_t \leq \mu_t \leq \lambda_{t+1} < \rho_3$ and there exists $\lambda^* \in [0, \rho_3]$ such that $\lim_{n\to\infty} \lambda_t = \lim_{t\to\infty} \mu_t = \lambda^*$. This limit is the unique least upper bound of the sequence $\{\lambda_t\}$.

(B_4) There exists a nonzero number Δ such that for each $\Delta \in D$ and some $w_0 \in D$

$$|\Delta^{-1}(f'(w) - \Delta)| \leq g_1(|w - w_0|).$$

Then, the end points of the interval a, b do not have to satisfy the condition $0 < a \leq b$. It follows by this condition and definition of ρ_2 that

$$\Delta^{-1}(f'(w_0) - \Delta)| \leq g_1(0) < 1,$$

so $f'(w_0) \neq 0$ and we can set $|f'(w_0)^{-1} f(w_0)| \leq \mu_0$.

(B_5) $d \leq |w_0| \leq d_1$.

(B_6) $|\Delta^{-1}(f'(v_2) - f'(v_1))| \leq g_2(|v_2 - v_1|)$ for each $v_1, v_2 \in D_3 = D \cap (w_0 - -\rho_2, w_0 + \rho_2)$.

(B_7) $D_4 := [w_0 - \lambda^*, w_0 + \lambda^*] \subset D$.

Theorem 5.3. *Suppose that the conditions (B_1) – (B_7) hold. Then, the sequences $\{w_t\}$,$\{u_t\}$ produced by the scheme (5.9) are well defined in the interval D_4, remain in D_4 for each $t = 0, 1, 2, \dots$, and are convergent to a zero $w^* \in D_4$ of the function f. Moreover, the following assertions hold for each $t = 0, 1, 2, \dots$*

$$|u_t - w_t| \leq \mu_t - \lambda_t, \tag{5.11}$$

$$|w_{t+1} - u_t| \leq \lambda_{t+1} - \mu_t, \tag{5.12}$$

and

$$|\lambda^* - w_t| \leq \lambda^* - \lambda_t. \tag{5.13}$$

Proof. The assertion (5.11) holds for $t=0$ by the definition of μ_0, since

$$|u_0 - w_0| = |f'(w_0)^{-1} f(w_0)| \le \mu_0 - \lambda_0 = \mu_0 < \lambda^*,$$

and the iterate $u_0 \in D_4$. As in local case but using $w_0, (B_4)$ instead of $w^*, (H_4)$, we get

$$|f'(v)^{-1}\Delta| \le \frac{1}{1 - g_1(|w - w_0|)}, \tag{5.14}$$

and

$$\begin{aligned} |A(v)^{-1}\Delta| &\le \frac{1}{|w_0|(1 - q(|v - w_0|))} \le \\ &\le \frac{1}{d(1 - q(|v - w_0|))}. \end{aligned} \tag{5.15}$$

By subtracting the first from the second substep of the scheme (5.9) we have

$$\begin{aligned} |w_{m+1} - u_m| &= (f'(w_m)^{-1} - w_m A_m^{-1}) f(w_m) = \\ &= -(w_m A_m^{-1} - f'(w_m)^{-1}) f(w_m) = \\ &= -A_m^{-1}(w_m f'(w_m) - A_m) f'(w_m)^{-1} f(w_m) = \\ &= -A_m f(w_m)(u_m - w_m) = \\ &= A_m f'(w_m)(u_m - w_m)^2. \end{aligned} \tag{5.16}$$

Thus, we get by (5.10), (5.15) and (5.16) that

$$\begin{aligned} |w_{m+1} - u_m| &\le \frac{1 + g_1(|w_m - x_0|)|u_m - w_m|^2}{d(1 - q(|w_m - w_0|))} \le \\ &\le \frac{(1 + g_1(\lambda_m))(u_m - \lambda_m)^2}{d(1 - q(\lambda_m))} = \\ &= \lambda_{m+1} - \mu_m, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} |w_{m+1} - w_0| &\le |w_m - u_m| + |u_m - w_0| \le \\ &\le \lambda_{m+1} - \mu_m + \mu_m - \lambda_0 = \lambda_{m+1} \le \lambda^*. \end{aligned} \tag{5.18}$$

Hence, the assertion (5.12) holds for $t = m$, and the iterate $w_{m+1} \in D_4$. Next, from the first substep of the scheme (5.9) we can write in turn that

$$\begin{aligned} f(w_{m+1}) &= f(w_{m+1}) - f(w_m) - f'(w_m)(u_m - w_m) = \\ &= f(w_{m+1}) - f(w_m) - f'(w_m)(w_{m+1} - w_m) + \\ &\quad + f'(w_m)(w_{m+1} - u_m), \end{aligned} \tag{5.19}$$

leading by (B_6) and the induction hypothesis that

$$
\begin{aligned}
|\Delta^{-1} f(w_{m+1})| &\leq \int_0^1 g_2((1-\theta)(w_{m+1}-w_m))d\theta \; |(w_{m+1}-w_m)| + \\
&\quad +(1+g_1(|w_m-w_0|))|w_{m+1}-u_m| \leq \\
&\leq \int_0^1 g_2((1-\theta)(\lambda_{m+1}-\lambda_m))d\theta \; (\lambda_{m+1}-\lambda_m) + \\
&\quad +(1+g_1(\lambda_m))(\lambda_m-\mu_m) = \beta_{m+1}. \qquad (5.20)
\end{aligned}
$$

Therefore, by (5.9), (5.14) and the induction hypotheses we obtain

$$
\begin{aligned}
|u_{m+1}-w_{m+1}| &\leq |f'(w_{m+1})^{-1}\Delta| \; |\Delta^{-1} f(w_{m+1})| \leq \\
&\leq \frac{\beta_{m+1}}{1-g_1(|w_{m+1}-w_0|)} \leq \frac{\beta_{m+1}}{1-g_1(\lambda_{m+1})} = \\
&= \mu_{m+1}-\lambda_{m+1}, \qquad (5.21)
\end{aligned}
$$

and

$$
\begin{aligned}
|u_{m+1}-w_0| &\leq |u_{m+1}-w_{m+1}| + |w_{m+1}-w_0| \leq \\
&\leq \mu_{m+1}-\lambda_{m+1}+\lambda_{m+1}-\lambda_0 = \mu_{m+1} \leq \lambda^*. \qquad (5.22)
\end{aligned}
$$

Thus, the assertion (5.11) holds and the iterate $u_{m+1} \in D_4$. By the triangle inequality, (5.11) and (5.12) we get

$$
|w_{m+1}-w_m| \leq \lambda_{m+1}-\lambda_m. \qquad (5.23)
$$

Consequently the sequence $\{w_m\}$ is complete, since $\{\lambda_m\}$ is convergent to λ^*. Therefore, there exists $w^* \in D_4$ such that $\lim_{m\to+\infty} w_m = w^*$. By letting $m \to +\infty$ in (5.20) and using the continuity of f, we deduce that $f(w^*) = 0$. Moreover, the estimate (5.23) for i a natural number gives

$$
|w_{m+i}-w_m| \leq \lambda_{m+i}-\lambda_m. \qquad (5.24)
$$

By letting $\lim_{i\to+\infty}$ in (5.24) we show that the assertion (5.13). ■

Proposition 5.4. *Suppose: there exists a solution u of equation $f(w) = 0$. The condition (B_4) holds in the interval $D_5 = [w-\rho_5, w+\rho_5]$ for some $\rho_5 > 0$, and there exists $\rho_6 \geq \rho_5$ such that*

$$
\int_0^1 g_1(\theta\rho_5 + (1-\theta)\rho_6)d\theta < 1. \qquad (5.25)
$$

Set $D_6 = D \cap D_5$.
Then, the only zero of the function f in the set D_6 is u.

Proof. Suppose that there exists $u_1 \in D_6$ such that $f(u_1) = 0$. Define the function $G_1 = f'(u + \theta(u_1 - u))d\theta$. Then, it follows by (B_4) and (5.25) that

$$\begin{aligned} |\Delta^{-1}(G_1 - \Delta)| &\leq \int_0^1 g_1(\theta|u - x_0| + (1-\theta)|u_1 - x_0|)d\theta \leq \\ &\leq \int_0^1 g_1(\theta\rho_5 + (1-\theta)\rho_6)d\theta < 1, \end{aligned}$$

so, $G_1 \neq 0$. Finally, from the identity

$$u_1 - u = G_1^{-1}(f(u_1) - f(u)) = G_1^{-1}(0) = 0$$

we conclude that $u_1 = u$. ∎

Remark 5.2.

(i) A possible choice for $\Delta = f'(w_0)$ or $\Delta = I$. Other choices are possible as long as the conditions (B_4) and (B_5) hold.

(ii) The limit point λ^* can be replaced by ρ_2 in the condition (B_7).

(iii) Under all the conditions $(B_1) - (B_7)$, we can take $u = w^*$ and $\rho_5 = \lambda^*$ in the Proposition 5.4.

6. Numerical testing

To check the effectiveness of the proposed iteration processes, the schemes are tested on several scalar and system of nonlinear equations. We compare the proposed iterative schemes $(DIM1)$, $(DIM2)$ and $(DIM3)$ with Newton's method (NM) in case of scalar nonlinear equations and compare $(MDIM1)$, $(MDIM2)$ and $(MDIM3)$ with Newton's method (MNM) for nonlinear system.

1. **Scalar nonlinear equations**

   For this comparison, we consider different scalar nonlinear equations given in Table 1, where functions, their zeros and starting guesses are mentioned. For better comparison, we compute absolute residual error and error between two consecutive approximations at the end of seventh iteration (i.e. $t = 7$) for each test problem mentioned in Table 1. The calculations have been performed using *Mathematica* 12.0. One can easily observe from Table 1 that hybrid scheme $(DIM3)$ performs better than Newton's method.

Problem	Root	Initial guess	NM	**DIM1**	**DIM2**	**DIM3**
			$\|w_{t+1}-w_t\|$ $\|f(w_t)\|$	$\|w_{t+1}-w_t\|$ $\|f(w_t)\|$	$\|w_{t+1}-w_t\|$ $\|f(w_t)\|$	$\|w_{t+1}-w_t\|$ $\|f(w_t)\|$
w^3+4w^2-10	1.3652300134	0.5	3.3(-29)	NC	5.7(-51)	**5.7(-51)**
			5.4(-28)		9.4(-50)	**9.4(-50)**
$w-3\log(w)$	1.8571838602	2	3.4(-125)	1.4(-200)	8.2(-93)	**1.4(-200)**
			2.1(-125)	8.6(-201)	5.0(-93)	**8.6(-201)**
$(\sin(w))^2-w^2+1$	-1.4044916482	-1	4.2(-51)	1.4(-9)	9.3(-199)	**9.3(-199)**
			1.0(-50)	3.6(-9)	2.3(-198)	**2.3(-198)**
$(w-2)(w+2)^2$	2.0000000000	1.4	5.2(-54)	2.2(-7)	2.4(-1328)	**2.4(-1328)**
			8.3(-53)	3.5(-6)	3.8(-1327)	**3.8(-1327)**
$10we^{-w^2}-1$	1.6796306371	2	2.7(-55)	9.6(-105)	1.0(-27)	**9.6(-105)**
			7.5(-55)	2.7(-104)	2.9(-27)	**2.7(-104)**
$\sin(w)e^w+\log(w^2+1)$	-0.6032319715	-0.8	1.5(-106)	2.8(-58)	1.7(-115)	**1.7(-115)**
			1.1(-106)	2.1(-58)	1.2(-115)	**1.2(-115)**
$sin(w)-\frac{w}{2}$	1.8954942226	1.5	1.4(-66)	9.4(-27)	1.2(-190)	**1.2(-190)**
			1.1(-66)	7.7(-27)	9.5(-191)	**9.5(-191)**

Table 1. Comparison results for scalar nonlinear equations

2. **Pathological problems**
Now, we consider the test problems where the hybrid method converges, but Newton's method does not converge. These test problems with their roots and initial guesses are mentioned in Table 2.

- For instance, consider $f(w) = -w^4 + 3w^2 + 2$ with starting guess $w_0 = \sqrt{\frac{3}{2}}$ such that $f'(w_0) = 0$. In this case, Newton's method does not converge whereas the proposed iterative scheme converges to the desired root.
- In case of $y = \log(w)$, Newton's method starting with $w_0 = 3$ fails in the second iteration as the approximated value of w_t is a negative number whose logarithm is not defined.
- In case of $y = \tan^{-1}(w)$, Newton's method always diverges to increasingly large number for any starting value of w_0, while the new method converges.
- Similarly, in case of $y = sin(w)$, Newton's method with a starting guess $w_0 = 1.51$, fails to converge the desired root, while the new method converges to the desired root $w^* = 0$.

Problem	Root	Initial guess	NM	**DIM1**	**DIM2**	**DIM3**
$-w^4 + 3w^2 + 2$		0.5	Oscillate	NC	7	**7**
	1.88	1	Oscillate	NC	7	**7**
		$\sqrt{\frac{3}{2}}$	CUR	NC	7	**7**
	- 1.88	$-\sqrt{\frac{3}{2}}$	CUR	NC	7	**7**
$\log(w)$	1	3	Fails	6	Diverges	**6**
$\tan^{-1}(w)$	0	3	Diverges	51	Diverges	**51**
		-3	Diverges	51	Diverges	**51**
$\sin(w)$	0	1.51	CUR	49	CUR	**4**
	$-\pi$	$-\frac{\pi}{2}$	Fails	Fails	1	**1**
	π	$\frac{\pi}{2}$	Fails	Fails	1	**1**
$w^{1/3}$	0	1	Diverges	75	Diverges	**75**
		-1	Diverges	75	Diverges	**75**
$e^{-w} - \sin(w)$	6.28	5	CUR	CUR	6	**6**

*CUR means converges to undesired root and NC means not convergent.

Table 2. Comparison results for pathological problems

We find the number of iterations for each problem in Table 2, needed to meet the stopping criteria mentioned as follows:

$$|w_{t+1} - w_t| + |f(w_t)| < 10^{-15}. \tag{6.1}$$

It can be observed that in all the problems mentioned in Table 2, Newton's iteration method fails to converge or converges to an undesired root whereas the hybrid method ($DIM3$) converges to the desired root in all the problems.

3. **System of nonlinear equations**
   There are different ways to calculate vector norm and each norm has its own properties. Here, we use constant $L1$ norm of initial guesses for the computational purpose. For Newton's method, we have used supremum norm because accuracy is slightly higher in this case. The results obtained at the end of seventh iteration (i.e. $t = 7$) of examples (6.1-6.4) are given in Table 3.

Example 6.1. Consider

$$\begin{aligned} 3w_1 + w_2^2 &= 0, \\ w_1 - w_2(1 + w_2) &= 0, \end{aligned} \tag{6.2}$$

with $W^{(0)} = (1, 2)^T$, $W^{(0)} = (0.2, -0.375)^T$ as initial guesses and $W^* = (0, 0)^T$, $W^* = (-0.1875000000, -0.7500000000)^T$ as the solutions, respectively.

Example 6.2. Consider

$$\begin{aligned} e^{w_1} - w_2 - 2 &= 0, \\ w_2 - w_1 + cos(w_1) - 1 &= 0, \end{aligned} \tag{6.3}$$

with $W^{(0)} = (0.5, 0.5)^T$ and $W^{(0)} = (0, 0.5)^T$ as the starting guess. The corresponding solution correct upto 10 digits is
$W^* = (1.47848895998, 2.38631249609)^T$.

Example 6.3. Consider the following system of equations [8]

$$\sum_{s=1,\ s\neq r}^{30} w_s - e^{-w_r} = 0,\ 1 \leq r \leq 30, \tag{6.4}$$

with initial guess as $W^{(0)} = (\frac{3}{2}, \frac{3}{2}, \ldots, \frac{3}{2})^T$. The required root is

$$W^* = (0.0333516678\ldots, 0.0333516678\ldots, \ldots, 0.0333516678\ldots)^T.$$

Example 6.4. Here, a nonlinear boundary value problem is considered [8].

$$\begin{aligned} y''(w) &= y(w)^3 + \sin(y'(w)^2), \quad w \in [0,1], \\ y(0) &= 0, \quad y(1) = 1. \end{aligned} \tag{6.5}$$

Let the interval $[0,1]$ be partitioned as $b_0 = 0 < b_1 < b_2 < \cdots <$ $< b_k = 1,\ b_{i+1} = b_i + h,\ h = \frac{1}{k}$. Further, we define $y_0 = y(b_0) = 0,\ y_1 =$ $= y(b_1), \dots, y_{k-1} = y(b_{k-1}),\ y_k = y(b_k) = 1$. Discretization of the problem (6.5) with help of first and second order derivatives formula results in the following system of equations:

$$y_{r-1} - 2y_r + y_{r+1} - h^2 y_r^3 - h^2 \sin\left(\left(\frac{y_{r-1} - y_{r+1}}{2h}\right)^2\right) = 0, \tag{6.6}$$

where $r = 1, 2, 3, \dots, k-1$. Consider $k = 51$ to obtain a 50×50 system of nonlinear equations. Choose the starting guesses as $Y^{(0)} = (\frac{1}{15}, \frac{1}{15}, \dots, \frac{1}{15})^T$ and $Y^{(0)} = (\frac{5}{4}, \frac{5}{4}, \dots, \frac{5}{4})^T$. The solution of the problem in both cases is given as follows:

$$\begin{aligned} W^* = (&0.0012825138\dots, 0.0258116762\dots, \dots, \\ &, \dots, 0.9396112972\dots, 0.9694966179\dots)^T. \end{aligned}$$

Next, we consider an example which is an application of Section 5.

Example 6.5. Consider, $f(w) = e^w - 1$, $D = [0,1]$. Then, for $w^* = 0$, $f(w^*) = 0$. The conditions (H_5) and (H_6) hold if $g_0(w) = (e-1)w$ and $g(w) = e^{\frac{1}{e-1}} w$, respectively. Choose $\alpha = 0.5, \delta = 0.2$ and $\gamma = 1$. The radius of convergence for the method (5.2) is obtained by solving $h(w) - 1 = 0$ is $r = 0.103857$.

7. Conclusion

This paper presents geometrically constructed root-finding methods for approximating roots of scalar and system of nonlinear equations having convergence of second order. We tested the proposed iterative schemes numerically and the numerical experiments demonstrates that new iteration procedures are comparable with Newton's method. Further, hybrid method also works for the test problems where Newton's iteration process fails to converge. Thus, hybrid scheme can be considered as an alternative of Newton's iteration process. We also explored the local convergence as well semilocal convergence of proposed methods.

Problem	Initial guess	NM	**DIM1**	**DIM2**	**DIM3**
		$\|\|W^{(t+1)}-W^{(t)}\|\|_\infty$	$\|\|W^{(t+1)}-W^{(t)}\|\|_1$	$\|\|W^{(t+1)}-W^{(t)}\|\|_1$	$\|\|W^{(t+1)}-W^{(t)}\|\|_1$
		$\|\|F(W^{(t)})\|\|_\infty$	$\|\|F(W^{(t)})\|\|_1$	$\|\|F(W^{(t)})\|\|_1$	$\|\|F(W^{(t)})\|\|_1$
(1)	$(1,2)^T$	1.0(-9)	4.7(-11)	1.3(-27)	**1.3(-27)**
		7.9(−10)	2.2(-10)	1.5(-27)	**1.5(-27)**
	$(0.2,-0.375)^T$	Fails	8.8(-23)	Diverges	**8.8(-23)**
			1.8(-22)		**1.8(-22)**
(2)	$(0.5,0.5)^T$	8.5(-1)	CUR	1.7(-10)	**1.7(-10)**
		2.0		1.0(-10)	**1.0(-10)**
	$(0,0.5)^T$	Fails	CUR	3.4(-1)	**3.4(-1)**
				3.3(-1)	**3.3(-1)**
(3)	$(\frac{3}{2},\frac{3}{2},\ \dots,\frac{3}{2})^T.$	2.7(-115)	5.3(-115)	4.2(-81)	**5.3(-115)**
		8.2(-114)	1.6(-113)	1.3(-79)	**1.6(-113)**
(4)	$(\frac{1}{15},\frac{1}{15},\ \dots,\frac{1}{15})^T$	5.2(-29)	1.6(-39)	1.0(-21)	**1.6(-39)**
		5.4(-29)	7.9(-41)	4.7(-23)	**7.9(-41)**
	$(\frac{5}{4},\frac{5}{4},\ \dots,\frac{5}{4})^T$	4.8(-11)	2.1(-14)	3.9(-19)	**2.1(-14)**
		1.1(-11)	6.9(-16)	8.9(-21)	**6.9(-16)**

*CUR means converges to undesired root and NC means not convergent.

Table 3. Results of comparison of system of nonlinear equations

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