Annales Univ. Sci. Budapest., Sect. Comp. 57 (2024) 47–53

ON THE EQUATION $F(p^k) = F(p^k - 1) + 1$

Imre Kátai, Bui Minh Mai Khanh and Bui Minh Phong
(Budapest, Hungary)

To the memory of Professor Pavel Varbanets
and Dr. Margit Kovács

Communicated by Gábor Farkas
(Received April 17, 2024; accepted June 28, 2024)

Abstract. We prove that if an odd positive integer k and a completely multiplicative function $F : \mathbb{N} \to \mathbb{C}$ satisfy the conditions

$$F(p^2) = F(p^2 - 1) + 1 \quad \text{and} \quad F(p^k) = F(p^k - 1) + 1 \quad \text{for every prime } p,$$

then F is the identity function. We also investigate completely functions $F : \mathbb{N} \to \mathbb{R}$ such that $F(p^2) = F(p^2 - 1) + 1$ is satisfied for every prime p.

1. Introduction

In the following let $\mathcal{P}$, $\mathbb{N}$, and $\mathbb{C}$ denote the set of primes, positive integers and complex numbers, respectively. We denote by $\mathcal{M}$ $(\mathcal{M}^*)$ the set of all complex-valued multiplicative (completely multiplicative) functions, respectively.

Let $\mathbb{I}(n) = n$ for every $n \in \mathbb{N}$ and we define the function $\mathbb{A}, \mathbb{B} \in \mathcal{M}^*$ as follows

$$\begin{cases} \mathbb{A}(2) = 0, \ \ \mathbb{A}(3) = -1, \ \ \mathbb{A}(p)^2 = 1 \quad \text{for every } \ p \in \mathcal{P}, \ p > 3 \\ \mathbb{B}(2) = -1, \ \ \mathbb{B}(3) = 0, \ \ \mathbb{B}(p)^2 = 1 \quad \text{for every } \ p \in \mathcal{P}, \ p > 3. \end{cases}$$

Key words and phrases: Completely multiplicative function, the identity function.
2010 Mathematics Subject Classification: primary 11A07, 11A25, secondary 11N25, 11N64.

In the following for a $k \in \mathbb{N}$ and $\mathbb{T} \subseteq \mathbb{C}$ we denote by $\mathcal{F}_k(\mathbb{T})$ the set of all $F \in \mathcal{M}^*$ such that $F(\mathbb{N}) \subseteq \mathbb{T}$ and

$$F(p^k) = F(p^k - 1) + 1 \quad \text{for every} \quad p \in \mathcal{P}. \tag{1.1}$$

Let $\mathcal{F}_k := \mathcal{F}_k(\mathbb{C})$. It is obvious that $\mathbb{A} \in \mathcal{F}_2$, $\mathbb{B} \in \mathcal{F}_2$ and $\mathbb{I} \in \mathcal{F}_k$ for every $k \in \mathbb{N}$.

In [1] and [2] we are given all solutions (D, G, F) of the equation

$$G(n) = F(n^2 - 1) + D \quad \text{for every} \quad n \in \mathbb{N},$$

where G, F are completely multiplicative functions and $D \in \mathbb{C}$.

In this paper we would like to give all solutions F of the equation (1.1).

The question is simple if $k = 1$.

Theorem 1. *We have*

$$\mathcal{F}_1 = \{\mathbb{I}\}.$$

It seems to be hard even in the case $k = 2$. We can solve (1.1) for $k = 2$, if additionally some other property is assumed to be hold.

Conjecture 1. *For each prime $Q > 2$ there is a prime $\pi \in \mathcal{P}$ such that*

$$Q \Big| (\pi - 1)(\pi + 1) \quad \textit{and} \quad P\Big(\frac{(\pi - 1)(\pi + 1)}{Q}\Big) < Q,$$

where $P(n)$ denotes the largest prime divisor of n.

Theorem 2. *If Conjecture 1 holds, then*

$$\mathcal{F}_2(\mathbb{R}) = \{\mathbb{A},\ \mathbb{B},\ \mathbb{I}\}.$$

Conjecture 2. *If $F \in \mathcal{M}^*$, $F(2) = 2$ and*

$$\begin{cases} F(p^2) &= F(p^2 - 1) + 1 \\ F(p^2) &= F(p^2 - 4) + 4 \end{cases} \tag{1.2}$$

for every $p \in \mathcal{P}$, then $F = \mathbb{I}$.

Conjecture 3. *Let c be a fixed positive number. Then for every $Q \in \mathcal{P}$, $Q > c$ there exists some $\pi \in \mathcal{P}$, for which*

$$A_\pi := (\pi + 1)(\pi - 1)(\pi + 2)(\pi - 2) \equiv 0 \pmod{Q}$$

and

$$\pi + 2 < Q^2, \quad P\Big(\frac{A_\pi}{Q}\Big) < Q.$$

Theorem 3. *Assume that Conjecture 3 is true with some $c > 1$. Let $F \in \mathcal{M}^*$, for which $F(p) = p$ holds for every prime $p < c$. Assume that (1.2) is satisfied. Then $F = \mathbb{I}$.*

For $\mathcal{F}_\ell \cap \mathcal{F}_k$ we would like to prove the following

Conjecture 4. *If $\ell, k \in \mathbb{N}$ and $(\ell, k) = 1$, then*

$$\mathcal{F}_\ell \cap \mathcal{F}_k = \{\mathbb{I}\}.$$

In this note we prove Conjecture 4 for the cases $\ell = 2$ and k is odd number.

Theorem 4. *We have*

$$\mathcal{F}_2 \cap \mathcal{F}_k = \{\mathbb{I}\} \quad \textit{for every} \quad k \in \mathbb{N},\ (k, 2) = 1.$$

2. Proof of Theorem 1

Since $\mathbb{I} \in \mathcal{F}_1$, therefore we shall prove that if $F \in \mathcal{F}_1$, then $F = \mathbb{I}$.

Let $F \in \mathcal{F}_1$. Then $F \in \mathcal{M}^*$ and

$$F(p) = F(p-1) + 1 \quad \text{for every} \quad p \in \mathcal{P}, \tag{2.1}$$

which with $F(1) = 1$ implies that

$$F(2) = F(1) + 1 = 2,\ F(3) = F(2) + 1 = 3,\ F(5) = F(4) + 1 = F(2)^2 + 1 = 5.$$

Assume that $F(n) = n$ for every $n < P$, where $P > 5$. We have $F(P) = P$ if $P \notin \mathcal{P}$. Thus we can assume that $P \in \mathcal{P}$, $P \geq 7$ and $F(P-1) = P - 1$. We infer from (2.1) that

$$F(P) = F(P-1) + 1 = (P-1) + 1 = P.$$

The proof of Theorem 1 is thus completed. ∎

3. Proof of Theorem 2

First we prove the following

Lemma 1. *If $F \in \mathcal{F}_2$, then*

$$(F(2), F(3)) \in \Big\{(0, -1), (-1, 0), (2, 3)\Big\} \tag{3.1}$$

and

$$|F(n)| \leq n \quad \textit{for every} \quad n \in \mathbb{N}. \tag{3.2}$$

Proof. Let $x := F(2)$. Since $F \in \mathcal{F}_2$, we have $F(p)^2 = F(p-1)F(p+1) + 1$ for every $p \in \mathcal{P}$, therefore

$$x^2 = F(2)^2 = F(1)F(3) + 1 = F(3) + 1 \quad \text{and} \quad F(3)^2 = F(2)F(4) + 1 = x^3 + 1.$$

These imply that

$$(x^2-1)^2 = x^3 + 1 \quad \text{and so} \quad x^2(x+1)(x-2) = 0.$$

Consequently, $x \in \{0, -1, 2\}$. This with $F(3) = x^2 - 1$ proves (3.1).

Now we prove (3.2). It is clear from (3.1) that (3.2) holds for $n \in \{1,2,3,4\}$. Assume that (3.2) holds for every $n < N$, where $N \geq 5$. Thus,

$$|F(N)| \leq N \quad \text{if} \quad N \notin \mathcal{P}.$$

If $N \in \mathcal{P}$, then we infer from $F \in \mathcal{F}_2$ that

$$\begin{aligned} |F(N)^2| &= |F(N-1)F(N+1) + 1| \leq \\ &\leq |F(N-1)||F(N+1)| + 1 \leq (N-1)(N+1) + 1 = N^2. \end{aligned}$$

This proves (3.2), therefore the proof of Lemma 1 is finished. ∎

Now we prove Theorem 2. It is obvious that $\{\mathbb{A}, \ \mathbb{B}, \ \mathbb{I}\} \subseteq \mathcal{F}_2(\mathbb{R})$. Now we prove that $\mathcal{F}_2(\mathbb{R}) \subseteq \{\mathbb{A}, \ \mathbb{B}, \ \mathbb{I}\}$.

Let $F \in \mathcal{F}_2(\mathbb{R})$. According (3.1), we need to consider three cases:

Case 1. $(F(2), F(3)) = (0, -1)$. In this case, $p^2 - 1 \equiv 0 \pmod 2$ for every $p \in \mathcal{P}$, $p > 2$, and so $F(p^2-1) = 0$ for every $p \geq 3$. Consequently

$$F(p)^2 = F(p^2) = F(p^2-1) + 1 = 1 \quad \text{for every} \quad p \in \mathcal{P}, p \geq 3,$$

which proves that $F = \mathbb{A}$.

Case 2. $(F(2), F(3)) = (-1, 0)$. It is easily seen that

$$p^2 - 1 \equiv 0 \pmod 3 \quad \text{and so} \quad F(p^2-1) = 0 \quad \text{for every} \quad p \in \mathcal{P}, p \neq 3.$$

This implies that

$$F(p)^2 = F(p^2) = F(p^2-1) + 1 = 1 \quad \text{for every} \quad p \in \mathcal{P}, p \geq 3,$$

which proves that $F = \mathbb{B}$.

Case 3. $(F(2), F(3)) = (2, 3)$.

Now we prove $F = \mathbb{I}$. It is clear that $F(n) = n$ holds for $n \in \{1,2,3,4\}$. Assume that $F(n) = n$ holds for every $n < Q$, where $Q \geq 5$. Assuming, by contradiction, that $F(Q) \neq Q$. Then $Q \in \mathcal{P}$ and

$$\begin{aligned} F(Q)^2 &= F(Q^2) = F(Q^2-1) + 1 = F(2)^2 F\Big(\frac{Q-1}{2}\Big) F\Big(\frac{Q+1}{2}\Big) + 1 = \\ &= 4\frac{Q-1}{2}\frac{Q+1}{2} + 1 = Q^2, \end{aligned}$$

consequently $F(Q) = -Q$.

On the other hand, it follows from Conjecture 1 that there exists $\pi \in \mathcal{P}$ such that

$$Q \Big| (\pi - 1)(\pi + 1) \quad \text{and} \quad P\Big(\frac{(\pi - 1)(\pi + 1)}{Q}\Big) < Q.$$

Consequently

$$F\Big(\frac{\pi^2 - 1}{Q}\Big) = \frac{\pi^2 - 1}{Q}$$

and

$$F(\pi)^2 = F(\pi^2) = F(\pi^2 - 1) + 1 = F(Q)F\Big(\frac{\pi^2 - 1}{Q}\Big) + 1 = -Q\frac{\pi^2 - 1}{Q} + 1 =$$
$$= -\pi^2 + 2 < 0.$$

This is impossible by our assumption $F \in \mathcal{F}_2(\mathbb{R})$.

Theorem 2 is proved. ∎

4. Proof of Theorem 3

Assume that $F(n) = n$ holds for every $n < N$. If $N \leq c$ or N is not a prime, then $F(N) = N$ is true as well. Assume that $N = Q \in \mathcal{P}$ and $Q > c$. Since $P((Q - 1)(Q + 1)) < Q$, then (1.1) implies that $F(Q) = Q^2$.

Let π be according to Conjecture 3. Thus $\pi < Q^2, Q|A_\pi, P(\frac{A_\pi}{Q}) < Q$. We have two cases:

Case a): $Q|(\pi - 1)(\pi + 1)$,

Case b): $Q|(\pi - 2)(\pi + 2)$.

In case a) we have $F(\pi^2) = F(\pi^2 - 4) + 4 = (\pi^2 - 4) + 4 = \pi^2$, since $P(\pi^2 - 4) < Q$. Then we infer from the following relation

$$\pi^2 = F(\pi^2) = F(\pi^2 - 1) + 1 = F\Big(\frac{\pi^2 - 1}{Q}\Big)F(Q) + 1 = \frac{\pi^2 - 1}{Q}F(Q) + 1$$

that $F(Q) = Q$.

The proof in the case b) is similarly. Since $Q|(\pi - 2)(\pi + 2)$ and $P(\pi^2 - 1) < < Q$, thus $F(\pi^2) = F(\pi^2 - 1) + 1 = (\pi^2 - 1) + 1 = \pi^2$ and

$$\pi^2 = F(\pi^2) = F(\pi^2 - 4) + 4 = F\Big(\frac{\pi^2 - 4}{Q}\Big)F(Q) + 4 = \frac{\pi^2 - 4}{Q}F(Q) + 4,$$

which implies that $F(Q) = Q$.

By using infinite induction we proved that $F(Q) = Q$ for every $Q \in \mathcal{P}$. The proof of Theorem 3 is completed. ∎

5. Proof of Theorem 4

Lemma 2. *If*

$$F \in \mathcal{F}_2 \cap \mathcal{F}_k \quad \textit{for some} \quad k \in \mathbb{N}, (k,2)=1, \tag{5.1}$$

then

$$F(n) = n \quad \textit{for every} \quad n \in \{1,2,3,4\}.$$

Proof. We shall prove that if F satisfies (5.1) then $(F(2),F(3)) = (2,3)$. According (3.1), we need to show that

$$(F(2),F(3)) \notin \Big\{(0,-1),(-1,0)\Big\}.$$

Assume first that $(F(2),F(3)) = (0,-1)$. Since $(k,2)=1$, $F(2)=0$ and $F \in \mathcal{F}_k$, we have

$$-1 = (-1)^k = F(3)^k = F(3^k-1)+1 = 1,$$

which is a contradiction.

Now assume that $(F(2),F(3)) = (-1,0)$. Then

$$p^2-1 \equiv 0 \pmod 3 \quad \text{and} \quad F(p^2-1)=0 \quad \text{for every} \quad p \in \mathcal{P}, p \neq 3,$$

consequently

$$F(p)^2 = F(p-1)F(p+1)+1 = 1 \quad \text{for every} \quad p \in \mathcal{P}, p \neq 3. \tag{5.2}$$

On the other hand, we have

$$-1 = F(2)^k = F(2^k) = F(2^k-1)+1 \quad \text{and so} \quad F(2^k-1) = -2. \tag{5.3}$$

Since $(2^k-1,3)=1$, the relation (5.2) implies that $F(2^k-1) = \pm 1$, which contradicts to (5.3). The proof of Lemma 2 is completed. ■

Now we prove Theorem 4.

By using Lemma 2, we assume that $F(n) = n$ for every $n < P$, where $P > 4$. We shall prove that $F(P) = P$.

Assume by contradiction that $F(P) \neq P$. Then $P \in \mathcal{P}$ and $P \geq 5$. It is clear from $F \in \mathcal{F}_2$ and our assumptions that

$$F(P)^2 = F(P-1)F(P+1)+1 = F(2)(P-1)F\Big(\frac{P+1}{2}\Big)+1 =$$
$$= 2(P-1)\Big(\frac{P+1}{2}\Big)+1 = P^2.$$

Thus we infer from our assumption $F(P) \neq P$ that $F(P) = -P$. Then from $(k, 2) = 1$ and $F \in \mathcal{F}_k$, we have

$$-P^k = F(P)^k = F(P^k) = F(P^k - 1) + 1$$

consequently

$$|F(P^k - 1)| = |-P^k - 1| = P^k + 1,$$

which contradicts to (3.2).

Theorem 4 is proved. ■

References

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I. Kátai, B.M.M. Khan and B.M. Phong
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest
Pázmány Péter sétány 1/C
Hungary
katai@inf.elte.hu
mbuiminh@yahoo.com
bui@inf.elte.hu