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ON UNIVERSALITY OF COMPOSITIONS INVOLVING GRAM POINTS

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In memory of Professor Pavel Varbanets

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Abstract. Gram points t_k are solutions of the equation $\theta(t) = (k-1)\pi$, where $\theta(t)$ is the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $1/2$ and $1/2+it$. In the paper, we consider approximation of analytic functions defined in the strip $D = \{s \in \in \mathbb{C} : 1/2 < \sigma < 1\}$ by shifts $F(\zeta(s+iht_k))$, $h > 0$, and $\zeta(s)$ is the Riemann zeta-function, for some classes of operators F in the space of analytic on D functions. It is obtained that the sets of such shifts approximating a given analytic function is infinite. For proofs, a probabilistic approach is applied.

This paper is dedicated to the memory of Professor Pavel Varbanets, who passed away on May 11, 2024. Professor Pavel was not only a great mathematician and colleague but also a wonderful person, demonstrating a rare combination of modesty, intelligence, and empathy. He had a huge impact on those around him, creating an atmosphere where everyone felt special and important. His care for us when we visited Ukraine became an integral part

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of our memories of Professor Pavel – we all fondly remember his meticulous attention to organizing conferences, welcoming colleagues, and thoughtfully arranging memorable dinners. He shared warmth and attentiveness not only with those around him but especially with his family, setting an example for many young people. We will remember Professor Pavel as a mathematician and an extraordinary person who left an enduring legacy not only in science but also in our hearts.

1. Introduction

Let $s=\sigma+it$ be a complex variable. The Riemann zeta-function $\zeta(s)$, one of the most important objects of analytic number theory, is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

or by the Euler product over primes

$$\zeta(s)=\prod_{\wp}\left(1-\frac{1}{\wp^s}\right)^{-1},$$

and is analytic continued to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1. The function $\zeta(s)$ is the main tool in the investigation of the function

$$\pi(x)\stackrel{\text{def}}{=}\sum_{\wp\leqslant x}1$$

as $x\to\infty$. This property of $\zeta(s)$ was discovered by Riemann in [21], and successfully applied by Hadamard [7] and de la Vallée Poussin [4] who obtained that

$$\pi(x)=\int_2^x\frac{\mathrm{d}u}{\log u}+O\left(x\mathrm{e}^{-c\sqrt{\log x}}\right),\quad x\to\infty,$$

with $c>0$. Proofs of the latter asymptotic formula and its improvements are closely connected to zeros of the function $\zeta(s)$ in the strip $1/2\leqslant\sigma<1$. This strip is also critical for other properties of the function $\zeta(s)$ including its value denseness which ensures a possibility of approximation of a wide class of analytic functions by shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$. The last property was discovered by Voronin in [23], and is called universality.

More precisely, Voronin obtained that, for every continuous non-vanishing function $f(s)$ in the disc $|s|\leqslant r$ with $0<r<1/4$ and analytic inside of that disc, and $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s|\leqslant r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon.$$

The Voronin universality theorem was improved and extended for other zeta and L-functions by various authors, see [1, 5, 9, 12, 17, 18, 22]. For modern version of the Voronin theorem, the following notation is used. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic inside of K, and $\text{meas}A$ the Lebesgue measure of $A \subset \mathbb{R}$. Then the modern version of Voronin's theorem says that, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}>0.$$

Moreover, it was observed in [19] and [16] that "lim inf" can be replaced by "lim" for all but at most countably many $\varepsilon > 0$.

The above theorem is of continuous type because τ in shifts $\zeta(s+i\tau)$ can take arbitrary values from $[0, T]$. Also there are discrete universality theorems when τ take values from a certain discrete set, for example, from the set $\{hk : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$, $h > 0$.

In some works, more general shifts $\zeta(s + i\varphi(\tau))$, or discrete shifts $\zeta(s + i\varphi_1(k))$ with certain functions $\varphi(\tau)$ or $\varphi_1(k)$ are used. Denote by $\theta(t)$ the increment of argument of the function $\pi^{-s/2}\Gamma(s/2)$, where $\Gamma(s)$ is the Euler gamma-function, along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. It is known that the function $\theta(t)$ is increasing for $t \geqslant t^* = 6.289836\ldots$, therefore, the equation

$$\theta(t) = (k-1)\pi, \quad k \in \mathbb{N}_0,$$

has the unique solution t_k. The points t_k at first time were considered by Gram [6] in connection with imaginary parts γ_k of non-trivial zeros $1/2 + i\gamma_k$ of the Riemann zeta-function, and now they are called Gram points. He finds that $\gamma_k \in [t_{k-1}, t_k]$ for $k = 1, \ldots, 15$, and conjectured that this is not true for $k > 15$. This conjecture was confirmed by more complicated calculations, and proved analytically.

In [8], a universality theorem for $\zeta(s)$ was proved by using approximating shifts $\zeta(s + iht_k)$. More precisely, it was obtained that, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$, $h > 0$ and $\varepsilon > 0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}|\zeta(s+iht_k)-f(s)|<\varepsilon\right\}>0.$$

Moreover, the limit

$$\lim_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}|\zeta(s+iht_k)-f(s)|<\varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon>0$.

Here $\#A$ denotes the cardinality of a set A.

Universality is a useful property of analytic functions, therefore, it is important to extend the class of universal functions. One of extension ways consists of investigations of universality for compositions $F(Z(s))$, where $Z(s)$ is universal in the Voronin sense zeta-function, and F a certain operator in the space of analytic functions. This was proposed in [13] and [14]. The aim of this paper is to obtain universality theorems by using shifts $F(\zeta(s+iht_k))$, $h>0$.

2. Results

Let $H(D)$ be the space of analytic on D functions endowed with the topology of uniform convergence on compacta. The first theorem on universality of $F(\zeta(s))$ with shifts $\zeta(s+iht_k)$ uses the operator $F:H(D)\to H(D)$ of the Lipschitz type. Namely, the operator F belongs to the class $L(\alpha)$, $\alpha>0$, if it satisfies the hypotheses:

1° For every polynomial $p=p(s)$ and every $K\subset\mathcal{K}$, there exists an element $g\in F^{-1}\{p\}$ such that $g(s)\neq 0$ for $s\in K$.

2° For every $K\in\mathcal{K}$, there are a constant $c>0$ and $K_1\in\mathcal{K}$ such that

$$\sup_{s\in K}|F(g_1(s))-F(g_2(s))|\leqslant c\sup_{s\in K_1}|g_1(s)-g_2(s)|^\alpha$$

for all $g_1,g_2\in H(D)$.

Denote by $H(K)$, $K\in\mathcal{K}$, the class of continuous functions on K that are analytic in the interior of K.

Theorem 2.1. *Suppose that $F\in L(\alpha)$, $K\in\mathcal{K}$, and $f(s)\in H(K)$. Then, for every $h>0$ and $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}|F(\zeta(s+iht_k))-f(s)|<\varepsilon\right\}>0. \tag{2.1}$$

For other classes of operators F, define the set

$$S=\{g\in H(D):g(s)\neq 0 \text{ on } D, \text{ or } g(s)\equiv 0\}.$$

Theorem 2.2. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $h > 0$ and $\varepsilon > 0$, inequality (2.1) is valid. Moreover, "$\liminf$" in (2.1) can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.*

A requirement in Theorem 2.2 for F in terms of open sets can be replaced by a more convenient hypothesis in terms of polynomials. The following modification of Theorem 2.2 takes place.

Theorem 2.3. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p(s)$, the set $(F^{-1}\{p\}) \cap S \neq \varnothing$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 2.2 is valid.*

Because the non-vanishing of polynomials in a bounded region can be controlled by a constant term, it is useful to consider operators in truncated space of analytic functions $H(D_T)$, where $D_T = \{s \in \mathbb{C} : 1/2 < \sigma < 1, |t| \leqslant T\}$, $T > 0$. Let

$$S_T = \{g \in H(D_T) : g(s) \neq 0 \text{ on } D_T, \text{ or } g(s) \equiv 0\}.$$

Theorem 2.4. *Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Suppose that $T > 0$ is such that $K \subset D_T$, and $F : H(D_T) \to H(D_T)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S \neq \varnothing$. Then the same assertion as in Theorem 2.2 is valid.*

Now, we will approximate analytic functions from a certain subset of $H(D)$. For different $c_1, \dots, c_r \in \mathbb{C}$, and $F : H(D) \to H(D)$, define

$$H_{F;c_1,\dots,c_r}(D) = \{g \in H(D) : g(s) \neq c_l \text{ on } D, \, l = 1, \dots, r\} \cup \{F(0)\}.$$

Theorem 2.5. *Suppose that $F : H(D) \to H(D)$ is a continuous operator satisfying $F(S) \supset H_{F;c_1,\dots,c_r}(D)$. For $r = 1$, let $K \in \mathcal{K}$ and $f(s) - c_1 \in H_0(K)$. Then the same assertion as in Theorem 2.2 is valid. For $r \geqslant 2$, let $K \subset D$ be arbitrary compact set, and $f(s) \in H_{F;c_1,\dots,c_r}(D)$. Then the same assertion as in Theorem 2.2 is valid.*

3. Proof of Theorem 2.1

Theorem 2.1 is derived from Theorem 1.1 of [8] and Mergelyan's theorem on approximation of analytic functions by polynomials [20] which is stated as Lemma 3.1 in the frame of our aim.

Lemma 3.1. *Let $K \subset \mathcal{K}$ and $g(s) \in H(K)$. Then, for every $\varepsilon > 0$, there exists a polynomial $p_{\varepsilon,g,K}(s)$ such that*

$$\sup_{s\in K} |g(s) - p_{\varepsilon,g,K}(s)| < \varepsilon.$$

Proof of Theorem 2.1. By properties of the class $L(\alpha)$, there exists an element $g \in F^{-1}\{p\} \subset H(D)$ such that $g(s) \neq 0$ on K, where $p = p(s)$ is a polynomial satisfying, in view of Lemma 3.1, the inequality

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.1}$$

Suppose that K_1 corresponds the set K, and $k \in \mathbb{N}_0$ satisfies the inequality

$$\sup_{s\in K_1} |\zeta(s + iht_k) - g(s)| < c^{-1/\alpha} \left(\frac{\varepsilon}{2}\right)^{1/\alpha}. \tag{3.2}$$

Then, by hypothesis 2° of the class $L(\alpha)$, for such k,

$$\begin{aligned}\sup_{s\in K} |F(\zeta(s + iht_k)) - p(s)| &= \sup_{s\in K} \left|F(\zeta(s + iht_k)) - F(F^{-1}\{p\})\right| \leqslant \\ &\leqslant c \sup_{s\in K_1} |\zeta(s + iht_k) - g(s)|^{\alpha} < \\ &< c\left(c^{-1/\alpha}\left(\frac{\varepsilon}{2}\right)^{1/\alpha}\right)^{\alpha} = \frac{\varepsilon}{2}.\end{aligned}$$

By Theorem 1.1 of [8], the set of $k \in \mathbb{N}_0$ satisfying (3.2) has a positive lower density, i.e.,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s + iht_k) - g(s)| < c^{-1/\alpha}\left(\frac{\varepsilon}{2}\right)^{1/\alpha}\right\} > 0.$$

Therefore,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \sup_{s\in K} |F(\zeta(s + iht_k)) - p(s)| < \frac{\varepsilon}{2}\right\} > 0. \tag{3.3}$$

If $k \in \mathbb{N}_0$ satisfies

$$\sup_{s\in K} |F(\zeta(s + iht_k)) - p(s)| < \frac{\varepsilon}{2},$$

then, in virtue of (3.1), k satisfies inequality

$$\sup_{s\in K} |F(\zeta(s + iht_k)) - f(s)| < \varepsilon.$$

This together with (3.3) shows that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |F(\zeta(s+iht_k)) - f(s)| < \varepsilon \right\} > 0,$$

and the theorem is proved. ∎

Unfortunately, universality for operators of the class $L(\alpha)$ can't be obtained in terms of positive density.

It is not difficult to see that the operator $F(g) = g^{(r)}$ belongs to the class $L(1)$. If $p(s)$ is a polynomial, then $F^{-1}\{p\}$ again is a polynomial $p_r(s)$, and we may choose its constant term such that $p_r(s) \neq 0$ on K. Thus, hypothesis 1° is satisfied. To confirm validity of hypothesis 2° of the class $L(1)$, it suffices to use Cauchy integral formula which gives, for $s \in K$,

$$F(g_1(s)) - F(g_2(s)) = g_1^{(r)}(s) - g_2^{(r)}(s) = \frac{r!}{2\pi i} \int\limits_{\mathcal{L}} \frac{g_1(z) - g_2(z)}{(z-s)^{r+1}}\, \mathrm{d}z,$$

where $\mathcal{L}$ is a suitable closed simple contour lying in D and enclosing the set K. Thus, from the above formula, we have

$$\sup_{s\in K} |F(g_1(s)) - F(g_2(s))| \leqslant c_{K,K_1} \sup_{s\in K_1} |g_1(s) - g_2(s)|$$

with certain $c_{K,K_1} > 0$. Here $K_1 \in \mathcal{K}$ is such that $K \subset G \subset K_1$ for some open set G, and $\mathcal{L}$ is lying in $K_1 \setminus G$.

4. Weak convergence results

For the proof of Theorems 2.2 – 2.5, we will apply the probabilistic approach based on weak convergence of probability measures in the spaces of analytic functions $H(D)$ and $H(D_T)$. This method was proposed by Bagchi [1] and developed by various authors [9, 10, 11, 12, 22]. We recall basics of weak convergence.

Let $\mathcal{X}$ be a topological space, and $\mathcal{B}(\mathcal{X})$ denote the Borel σ-field of $\mathcal{X}$, and P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Then P_n converges weakly to P as $n \to \infty$ ($P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$) if, for every real continuous bounded function g on $\mathcal{X}$,

$$\lim_{n\to\infty} \int\limits_{\mathcal{X}} g\, \mathrm{d}P_n = \int\limits_{\mathcal{X}} g\, \mathrm{d}P.$$

Some equivalents of weak convergence are known. We will remind two of them. Recall that $A \in \mathcal{B}(\mathcal{X})$ is a continuity set of the probability measure P on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ if $P(\partial A) = 0$, where ∂A denotes the boundary of A.

Lemma 4.1. *The following statements are equivalent:*

1. $$P_n \xrightarrow[n\to\infty]{\mathrm{w}} P;$$
2. *For every open set* $G \subset \mathcal{X}$,
$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$
3. *For every continuity set* A *of* P,
$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 from [2], where its proof is presented.

For investigation of the composition $F(\zeta(s))$, the preservation principle of weak convergence under some mappings is very useful, therefore, we remind it. Let $\mathcal{X}_1$ be one more space. A mapping $u : \mathcal{X} \to \mathcal{X}_1$ is $(\mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}_1))$-measurable if
$$u^{-1}\mathcal{B}(\mathcal{X}_1) \subset \mathcal{B}(\mathcal{X}).$$
In this case, every probability measure P on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ defines the unique probability measure Pu^{-1} on $(\mathcal{X}_1, \mathcal{B}(\mathcal{X}_1))$ by $Pu^{-1}(A) = P(u^{-1}A)$, $A \in \mathcal{B}(\mathcal{X}_1)$. It is well known that every continuous mapping is $(\mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}_1))$-measurable [2]. Moreover, the following lemma holds.

Lemma 4.2. *Suppose that a mapping* $u : \mathcal{X} \to \mathcal{X}_1$ *is continuous, and* $P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$ *in the space* $\mathcal{X}$. *Then* $P_n u^{-1} \xrightarrow[n\to\infty]{\mathrm{w}} Pu^{-1}$ *in the space* $\mathcal{X}_1$.

The lemma is a particular case of Theorem 5.1 of [2].

For statement of a limit theorem for $F(\zeta(s))$, we need one $H(D)$-valued random element. Let $\mathbb{P}$ denote the set of all prime numbers and
$$\mathbb{T} = \prod_{\wp\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\}.$$
Then the torus $\mathbb{T}$ is a compact topological Abelian group, therefore, on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ the probability Haar measure m_H exists. This gives the probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m_H)$. Let $\omega = (\omega(\wp) : \wp \in \mathbb{P})$ be elements of $\mathbb{T}$. On the probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m_H)$, define the $H(D)$-valued random element
$$\zeta(s,\omega) = \prod_{\wp\in\mathbb{P}} \left(1 - \frac{\omega(\wp)}{\wp^s}\right)^{-1}.$$
Note that the latter product, for almost all $\omega \in \mathbb{T}$, converges uniformly on compact sets of the strip D [12].

Denote by P_ζ the distribution of the random element, thus,

$$P_\zeta(A) = m_H \{\omega \in \mathbb{T} : \zeta(s,\omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

For $A \in \mathcal{B}(H(D))$ and $h > 0$, define

$$P_N(A) = \frac{1}{N+1} \# \{0 \leqslant k \leqslant N : \zeta(s + iht_k) \in A\}.$$

Then in [8], the following result was obtained.

Lemma 4.3. *The relation*

$$P_N \xrightarrow[N\to\infty]{\text{w}} P_\zeta$$

is true.

Now, define

$$P_{N,F}(A) = \frac{1}{N+1} \# \{0 \leqslant k \leqslant N : F(\zeta(s + iht_k)) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Then Lemmas 4.2 and 4.3 imply the following statement.

Lemma 4.4. *Suppose that $F : H(D) \to H(D)$ is a continuous operator. Then $P_{N,F}$ converges weakly to the measure $P_\zeta F^{-1}$ as $N \to \infty$.*

Proof. By the definitions of P_N and $P_{N,F}$, we have

$$\begin{aligned} P_{N,F}(A) &= \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \zeta(s + iht_k) \in F^{-1}A\right\} \\ &= P_N(F^{-1}A) = P_N F^{-1}(A) \end{aligned}$$

for all $A \in \mathcal{B}(H(D))$. Thus, the equality $P_{N,F} = P_N F^{-1}$ holds. Therefore, in view of Lemmas 4.2 and 4.3,

$$P_{N,F} \xrightarrow[N\to\infty]{\text{w}} P_\zeta F^{-1}. \qquad \blacksquare$$

Remind that $D_T = \{s \in \mathbb{C} : 1/2 < \sigma < 1, |t| \leqslant T\}$ for $T > 0$. For $A \in \mathcal{B}(H(D_T))$, define

$$P_{N,F,T}(A) = \frac{1}{N+1} \# \{0 \leqslant k \leqslant N : F(\zeta(s + iht_k)) \in A\},$$

and

$$P_{\zeta,T}(A) = m_H\{\omega \in \mathbb{T} : \zeta(s,\omega) \in A\}.$$

Lemma 4.5. *Suppose that $F : H(D_T) \to H(D_T)$ is a continuous operator. Then $P_{N,F,T}$ converges weakly to the distribution of the random element $F(P_{\zeta,T})$ as $N \to \infty$.*

Proof. Obviously, the mapping $u_T : H(D) \to H(D_T)$ given by $u_T(g(s)) = = g(s)\big|_{s\in D_T}$ is continuous. Therefore, Lemmas 4.2 and 4.3 imply that

$$P_{N,T}(A) = \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \zeta(s+iht_k) \in A\right\}, \quad A \in \mathcal{B}(H(D_T)),$$

as $N \to \infty$, converges weakly to $P_{\zeta,T}$. This together with Lemma 4.2 again gives that

$$P_{N,F,T} \xrightarrow[N\to\infty]{\text{w}} P_{\zeta,T}F^{-1},$$

and the lemma follows. ∎

5. Supports

Proofs of universality theorems by probabilistic method require to know explicit form of supports of limit measures in limit theorems for probability measures in the space of analytic functions. Let $H(G)$ be the space of analytic functions a certain region $G \subset \mathbb{C}$ endowed with the topology of uniform convergence on compacts. The space $H(G)$ is separable. Let P be a probability measure on $(H(G), \mathcal{B}(H(G)))$. We remind that the support of P is a minimal closed set $S_P \subset H(G)$ such that $P(S_P) = 1$. The set S_P consists of all elements g whose every open neighbourhood has a positive P-measure. The support of the distribution of a $H(G)$-valued random element is called a support of that element.

We will use the following lemma, see, for example, [12].

Lemma 5.1. *The support of P_ζ is the set*

$$S = \{g \in H(D) : g(s) \neq 0 \text{ on } D, \text{ or } g(s) \equiv 0\}.$$

Lemma 5.2. *Let the operator F satisfy the hypotheses of Theorem 2.2. Then the support of the measure $P_\zeta F^{-1}$ is the whole of $H(D)$.*

Proof. We take an arbitrary element $g \in H(D)$ and its open neighbourhood G. Since F is continuous, the set $F^{-1}G$ is open as well. By the hypothesis $(F^{-1}G) \cap S \neq \varnothing$, there is an element $\widehat{g} \in F^{-1}G$ which, by Lemma 5.1, is an element of the support S of $\zeta(s,\omega)$. Therefore, by the support property,

$$P_\zeta(F^{-1}G)) = P_\zeta F^{-1}(G) > 0.$$

This together with an obvious equality $P_\zeta F^{-1}(H(D)) = 1$ proves the lemma. ∎

Lemma 5.3. *Let the operator F satisfy the hypotheses of Theorem 2.3. Then the support of the measure $P_\zeta F^{-1}$ is the whole of $H(D)$.*

Proof. It suffices to show that the hypothesis $(F^{-1}\{p\}) \cap S \neq \varnothing$ for every polynomial $p = p(s)$ implies the hypothesis $(F^{-1}G) \cap S \neq \varnothing$ for every open set $G \subset H(D)$.

The set $H(D)$ is metric [12, 22]. Let $\{K_m : m \in \mathbb{N}\} \subset D$ be a sequence of embedded compact sets such that

$$D = \bigcup_{m=1}^{\infty} K_m$$

and every compact set K lies in some K_m. The existence of such a sequence is proved in [3]. Then

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|}$$

is a metric in $H(D)$ inducing its topology of uniform convergence on compacta. We notice that, in the case of the strip D, the set K_m can be chosen with connected complements, for example, we can take closed rectangles.

The form of the metric ρ shows that the function g_1 is approximated with a given accuracy by a function g_2 if g_1 is approximated by g_2 with suitable accuracy uniformly on the st K_m with sufficiently large m.

Let $G \subset H(D)$ be arbitrary open non-empty set, and $g \in G$. Fix ε. Then, in view of Lemma 3.1, there exists a polynomial $p = p(s)$ such that

$$\sup_{s \in K_m} |g(s) - p(s)| < \varepsilon.$$

Thus, if ε is small enough, $p \in G$ as well. Since $(F^{-1}\{p\}) \cap S \neq \varnothing$ we obtain that $(F^{-1}G) \cap S \neq \varnothing$ as well, and this proves the lemma. ■

Lemma 5.4. *Suppose that the operator F satisfies the hypotheses of Theorem 2.4. Then the support of the measure $P_{\zeta,T}F^{-1}$ is the whole $H(D_T)$.*

Proof. Clearly, the support of the measure $P_{\zeta,T}$ is the set S_T. First, suppose that a continuous operator $F : H(D_T) \to H(D_T)$ satisfies the hypothesis $(F^{-1}G) \cap S_T \neq \varnothing$ for all open sets G of D_T.

Then repeating the arguments of the proof of Lemma 5.2 shows that the support of $P_{\zeta,T}F^{-1}$ is the whole of $H(D_T)$. Hence, as in the proof of Lemma 5.3, we obtain the assertion of the lemma.

Lemma 5.5. *Suppose that the operator F satisfies the hypotheses of Theorem 2.5. Then the support of the measure $P_\zeta F^{-1}$ contains the closure of the set $H_{F;c_1,\dots,c_r}(D)$.*

Proof. The hypothesis

$$H_{F;c_1,\dots,c_r}(D) \subset F(S)$$

implies that, for every element $g \in H_{F;c_1,\dots,c_r}(D)$, there is an element $\widehat{g} \in S$ such that $F(\widehat{g}) = g$. Let G be arbitrary open neighbourhood of g. Therefore, in view of Lemma 5.1,

$$P_\zeta(F^{-1}G) = P_\zeta F^{-1}(G) > 0.$$

This shows that g is an element of the support of the measure $P_\zeta F^{-1}$. Therefore, the set $H_{F;c_1,\dots,c_r}(D)$ lies in the support of $P_\zeta F^{-1}$. Since the support is a closed set, this proves the lemma. ■

6. Proofs of Theorems 2.2 – 2.5

Proof of Theorem 2.2. Consider the set

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\},$$

where $p(s)$ is a polynomial satisfying inequality (3.1) The set $\mathcal{G}_\varepsilon$ is open in the space $H(D)$. Moreover, by Lemma 5.2, the polynomial $p(s)$ is an element of the support of the measure $P_\zeta F^{-1}$. Hence,

$$P_\zeta F^{-1}(\mathcal{G}_\varepsilon) > 0. \tag{6.1}$$

Define one more set

$$\widehat{\mathcal{G}}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Suppose that $g \in \mathcal{G}_\varepsilon$. Then, in view of (3.1),

$$\sup_{s \in K} |g(s) - f(s)| < \varepsilon.$$

Thus, $\mathcal{G}_\varepsilon \subset \widehat{\mathcal{G}}_\varepsilon$. This and (6.1) show that $P_\zeta F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0$. From this and Lemmas 4.1 and 4.4, we have

$$\liminf_{N \to \infty} P_{N,F}(\widehat{\mathcal{G}}_\varepsilon) \geqslant P_\zeta F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0,$$

and definitions of $\widehat{\mathcal{G}}_\varepsilon$ and $P_{N,F}$ prove the first assertion of the theorem.

To prove the second assertion of the theorem, observe that the boundary $\partial\widehat{\mathcal{G}}_\varepsilon$ lies in the set

$$\left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Hence, we obtain that $\partial\widehat{\mathcal{G}}_{\varepsilon_1} \cap \partial\widehat{\mathcal{G}}_{\varepsilon_2} = \varnothing$ for different positive ε_1 and ε_2. This remark shows that the set $\widehat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure $P_\zeta F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemmas 4.1 and 4.4,

$$\lim_{N\to\infty} P_{N,F}(\widehat{\mathcal{G}}_\varepsilon) = P_\zeta F^{-1}(\widehat{\mathcal{G}}_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$, and the positivity of $P_\zeta F^{-1}(\widehat{\mathcal{G}}_\varepsilon)$ prove the second assertion of the theorem. ■

Proof of Theorem 2.3. We repeat the arguments of the proof of Theorem 2.2 with using Lemma 5.3 in place of Lemma 5.2. ■

Theorems 2.2 and 2.3, in certain sense, are theoretical because it is difficult to check their hypotheses.

Proof of Theorem 2.4. We use Lemmas 4.5 and 5.4, and follow with obvious changes the lines of the proof of Theorem 2.3. ■

It is not difficult to give an example of operators satisfying hypotheses of Theorem 2.4. For $g \in D_T$, set

$$F(g) = \sum_{j=1}^{n} c_j g^{(j)} \quad \text{with } c_j \in \mathbb{C} \setminus \{0\}.$$

The Cauchy integral formula implies the continuity of the operator F.

Now, let $p = p(s)$ be arbitrary polynomial. Since a derivative of a polynomial is again a polynomial $p_1 = p_1(s)$ such that $p_1 = F^{-1}\{p\}$. Moreover, because the region D_T is bounded, we can found a constant term of the polynomial $p_1(s)$ such that $p_1(s) \neq 0$ on D_T. This shows that $p_1 \in S_T$, and we have the hypothesis $(F^{-1}\{p\}) \cap S_T \neq \varnothing$ of the theorem.

Proof of Theorem 2.5. Consider separately two cases $r = 1$ and $r \geqslant 2$.

Case $r = 1$. By Lemma 3.1, there is a polynomial $p(s)$ satisfying

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{6.2}$$

By hypothesis of the theorem, the function $f(s) - c_1$ is non-vanishing on K. Thus, in view of (6.2), $p(s) - c_1 \neq 0$ on K if $\varepsilon > 0$ is small sufficiently. This and Lemma 3.1 show that there is a polynomial $q(s)$ such that

$$\sup_{s\in K} \left| p(s) - c_1 - \mathrm{e}^{q(s)} \right| < \frac{\varepsilon}{4}. \tag{6.3}$$

Clearly, the function $\widehat{g}(s) = \mathrm{e}^{q(s)} + c_1 \neq c_1$, thus, $\widehat{g}(s) \in H_{F;c_1}(D)$. Therefore, by Lemma 5.5, the function $\widehat{g}(s)$ is an element of the support of the measure $P_\zeta F^{-1}$. Let

$$\mathcal{G}_{1,\varepsilon} = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - \widehat{g}(s)| < \frac{\varepsilon}{2} \right\}.$$

Then, by a property of the support, the inequality

$$P_\zeta F^{-1}(\mathcal{G}_{1,\varepsilon}) > 0 \tag{6.4}$$

is valid. Let $\widehat{\mathcal{G}}_\varepsilon$ be the same as above. Then (6.2) and (6.3) show that $\mathcal{G}_{1,\varepsilon} \subset \widehat{\mathcal{G}}_\varepsilon$. Therefore, by (6.4),

$$P_\zeta F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0. \tag{6.5}$$

This, and Lemmas 4.1 and 4.4 imply that

$$\liminf_{N\to\infty} P_{N,F}(\widehat{\mathcal{G}}_\varepsilon) \geqslant P_\zeta F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0,$$

and we have the first assertion of the theorem in the case $r = 1$.

The second assertion of this case is obtained by using a remark that the set $\widehat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure $P_\zeta F^{-1}$ for all but at most countably many $\varepsilon > 0$.

Case $r \geqslant 2$. This case is simpler than that $r = 1$. Since $f(s) \in H_{F;c_1,\dots,c_r}(D)$, $f(s)$, in virtue of Lemma 5.5, is an element of the support of the measure $P_\zeta F^{-1}$. Hence, (6.5) follows. Thus, the proof is completed in the same way as in the case $r = 1$. ∎

We give an example of application of Theorem 2.5. Let

$$F(g) = \cosh(g) = \frac{\mathrm{e}^g + \mathrm{e}^{-g}}{2}.$$

In this case, $F(0) = 1$. Obviously, F is continuous. Suppose that $c_1 = 1$ and $c_2 = -1$. We will prove the inclusion $H_{F;1,-1}(D) \subset F(S)$. Let $v \in H_{F;1,-1}(D)$. We solve the equation

$$\mathrm{e}^g + \mathrm{e}^{-g} = 2v.$$

We find that

$$\mathrm{e}^g = v \pm \sqrt{v^2 - 1}.$$

Thus, we can take a solution

$$g = \log(v - \sqrt{v^2 - 1}).$$

Since $v \neq 1$ and $v \neq -1$, $\sqrt{v^2 - 1}$ is well defined. Moreover, $v - \sqrt{v^2 - 1} \neq 0$ and $v - \sqrt{v^2 - 1} \neq 1$. Therefore, $g \in S$, and the application of Theorem 2.5 shows that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\cosh(\zeta(s + iht_k)) - f(s)| < \varepsilon \right\} > 0$$

for every $K \in \mathcal{K}$, $f(s) \in H(K)$ and $\varepsilon > 0$.

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