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SPECTRAL SYNTHESIS ON CONTINUOUS IMAGES

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Abstract. Recently we introduced the concept of localizability of ideals in the Fourier algebra of locally compact Abelian groups. It turns out that localizability can be used to characterize synthesizability of varieties. Based on this we show that spectral synthesis holds on continuous images of varieties which have spectral synthesis.

1. Introduction

Let G be a locally compact Abelian group. Spectral synthesis deals with uniformly closed translation invariant linear spaces of continuous complex valued functions on G. Such a space is called a *variety*. We say that *spectral analysis* holds for a variety, if every nonzero subvariety contains a one dimensional subvariety. We say that a variety is *synthesizable*, if its finite dimensional subvarieties span a dense subspace in the variety. Finally, we say that *spectral synthesis* holds for a variety, if every subvariety of it is synthesizable. On commutative topological groups, finite dimensional varieties of continuous functions are completely characterized: they are spanned by exponential monomials. *Exponential polynomials* on a topological Abelien group are defined as the elements of the complex algebra of continuous complex valued functions generated by all continuous homomorphisms into the multiplicative group of nonzero complex numbers (*exponentials*), and all continuous homomorphisms

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into the additive group of all complex numbers (*additive functions*). An *exponential monomial* is a function of the form

$$x \mapsto P\big(a_1(x), a_2(x), \dots, a_n(x)\big)m(x),$$

where P is a complex polynomial in n variables, the a_i's are additive functions, and m is an exponential. If $m = 1$, then we call it a *polynomial*. Every exponential polynomial is a linear combination of exponential monomials. One dimensional varieties are exactly those spanned by an exponential, and finite dimensional varieties are exactly those spanned by exponential monomials (see [5]). The *variety of the continuous function* f, denoted by $\tau(f)$, is the intersection of all varieties including f. For more about spectral analysis and synthesis on groups see [4, 5].

In [3], the authors characterized those discrete Abelian groups having spectral synthesis: spectral synthesis holds for every variety on the discrete Abelian group G, if and only if G has finite torsion free rank. In particular, from this result it follows, that if spectral synthesis holds on G and H, then it holds on $G \oplus H$. Unfortunately, such a result does not hold in the non-discrete case. Namely, by the fundamental result of L. Schwartz [1], spectral synthesis holds on $\mathbb{R}$, but D. I. Gurevich showed in [2] that spectral synthesis fails to hold on $\mathbb{R} \times \mathbb{R}$. A complete description of those locally compact Abelian groups where spectral synthesis holds for the space of all continuous functions was obtained in [7], where the present author proved the following two theorems:

Theorem 1.1. *Spectral synthesis holds on the compactly generated locally compact Abelian group G if and only if it is topologically isomorphic to $\mathbb{R}^a \times Z^b \times C$, where C is compact, and a, b are nonnegative integers with $a \leqslant 1$.*

Theorem 1.2. *Spectral synthesis holds on the locally compact Abelian group G if and only if G/B is topologically isomorphic to $\mathbb{R}^a \times Z^b \times F$, where B is the subgroup of all compact elements in G, F is a discrete Abelian group of finite rank, and a, b are nonnegative integers with $a \leqslant 1$.*

These characterization theorems describe those groups where all varieties are synthesizable – another question is if spectral synthesis holds for a given particular variety, even if it does not hold on the whole group. In [8], we introduced the concept of localization, which is an effective tool to prove spectral synthesis on varieties. In this paper we apply this method to show that any continuous image of a synthesizable variety is synthesizable as well. It will follow that spectral synthesis holds for every continuous image of a variety, for which spectral synthesis holds.

2. Preliminaries

Given a locally compact Abelian group, the space of all continuous complex valued functions on G is denoted by $\mathcal{C}(G)$. It is a locally convex topological vector space, when equipped with the topology of uniform convergence on compact sets. It is known that the dual space of $\mathcal{C}(G)$ can be identified with the space $\mathcal{M}_c(G)$ of all compactly supported complex Borel measures on G. This space is called the *measure algebra* of G – it is a topological algebra with the linear operations, with the convolution of measures and with the weak*-topology arising from $\mathcal{C}(G)$. On the other hand, the space $\mathcal{C}(G)$ is a *topological vector module* over the measure algebra under the action realized by the convolution of measures and functions. The annihilators of subsets in $\mathcal{C}(G)$ and the annihilators of subsets in $\mathcal{M}_c(G)$ play an important role in our investigation. For each closed ideal I in $\mathcal{M}_c(G)$ and for every variety V in $\mathcal{C}(G)$, $\operatorname{Ann} I$ is a variety in $\mathcal{C}(G)$ and $\operatorname{Ann} V$ is a closed ideal in $\mathcal{M}_c(G)$. Further, we have

$$\operatorname{Ann}\operatorname{Ann} I = I \ \text{ and } \ \operatorname{Ann}\operatorname{Ann} V = V$$

(see [5, Section 11.6], [6, Section 1]).

The Fourier–Laplace transformation (shortly: Fourier transformation) on the measure algebra is defined as follows: for every exponential m on G and for each measure μ in $\mathcal{M}_c(G)$ its *Fourier transform* is

$$\widehat{\mu}(m) = \int \check{m}\, d\mu,$$

where $\check{m}(x) = m(-x)$ for each x in G. The Fourier transform $\widehat{\mu}$ is a complex valued function defined on the set of all exponentials on G. As the mapping $\mu \mapsto \widehat{\mu}$ is linear and $(\mu * \nu)\widehat{\ } = \widehat{\mu} \cdot \widehat{\nu}$, all Fourier transforms form a function algebra. By the injectivity of the Fourier transform, this algebra is isomorphic to $\mathcal{M}_c(G)$. If we equip the algebra of Fourier transforms by the topology arising from the topology of $\mathcal{M}_c(G)$, then we get the *Fourier algebra* of G, denoted by $\mathcal{A}(G)$. In fact, $\mathcal{A}(G)$ can be identified with $\mathcal{M}_c(G)$. We utilize this identification: for instance, every ideal in $\mathcal{A}(G)$ is of the form $\widehat{I}$, where I is an ideal in $\mathcal{M}_c(G)$. Based on this fact, we say that *spectral synthesis holds for the ideal* $\widehat{I}$ *in* $\mathcal{A}(G)$, if spectral synthesis holds for $\operatorname{Ann} I$ in $\mathcal{C}(G)$.

We shall use the polynomial derivations on the Fourier algebra. A *derivation* on $\mathcal{A}(G)$ is a linear operator $D : \mathcal{A}(G) \to \mathcal{A}(G)$ such that

$$D(\widehat{\mu} \cdot \widehat{\nu}) = D(\widehat{\mu}) \cdot \widehat{\nu} + \widehat{\mu} \cdot D(\widehat{\nu})$$

holds for each $\widehat{\mu}, \widehat{\nu}$. We say that D is a *first order derivation*. Higher order derivations are defined inductively: for a positive integer n we say that linear operator D on $\mathcal{A}(G)$ is a *derivation of order* $n+1$, if the two variable operator

$$(\widehat{\mu}, \widehat{\nu}) \mapsto D(\widehat{\mu} \cdot \widehat{\nu}) - D(\widehat{\mu}) \cdot \widehat{\nu} - \widehat{\mu} \cdot D(\widehat{\nu})$$

is a derivation of order n in both variables. The identity operator id is considered a derivation of order 0. All derivations form an algebra of operators, and the derivations in subalgebra generated by all first order derivations are called *polynomial derivations.* They have the form $P(D_1, D_2, \ldots, D_k)$, where $D_1, D_2, \ldots, D_k$ are first order derivations, and P is a complex polynomial in k variables. In [8], we proved the following result:

Theorem 2.1. *The linear operator D on $\mathcal{A}(G)$ is a polynomial derivation if and only if there exists a unique polynomial f_D such that*

$$D\widehat{\mu}(m) = \int f_D(x) m(-x)\, d\mu(x)$$

holds for each $\widehat{\mu}$ in $\mathcal{A}(G)$ and for every exponential m on G.

Iin [8], we introduced the following concepts. Given an ideal $\widehat{I}$ in $\mathcal{A}(G)$ and an exponential m, we denote by $\mathcal{P}_{\widehat{I},m}$ the family of all polynomial derivations $P(D_1, D_2, \ldots, D_k)$ which annihilate $\widehat{I}$ at m. This means that

$$\partial^\alpha P(D_1, D_2, \ldots, D_k)\widehat{\mu}(m) = 0$$

for each multi-index α in $\mathbb{N}^k$, for every exponential m, and for every $\widehat{\mu}$ in $\widehat{I}$. The dual concept is the following: given a family $\mathcal{P}$ of polynomial derivations and an exponential m we denote by $\widehat{I}_{\mathcal{P},m}$ the set of all functions $\widehat{\mu}$ which are annihilated by every derivation in the family $\mathcal{P}$ at m. Then $\widehat{I}_{\mathcal{P},m}$ is a closed ideal. Obviously,

$$\widehat{I} \subseteq \bigcap_m \widehat{I}_{\mathcal{P}_{\widehat{I},m},m}$$

holds for every ideal $\widehat{I}$. We call $\widehat{I}$ *localizable*, if we have equality in this inclusion. In other words, the ideal $\widehat{I}$ in $\mathcal{A}(G)$ is localizable if and only if it has the following property: if $\widehat{\mu}$ is annihilated by all polynomial derivations, which annihilate $\widehat{I}$ at each m, then $\widehat{\mu}$ is in $\widehat{I}$. The main result in [8] is the following:

Theorem 2.2. *Let G be a locally compact Abelian group. The ideal $\widehat{I}$ in the Fourier algebra is localizable if and only if* $\operatorname{Ann} I$ *is synthesizable.*

3. Main result

Let G be a locally compact Abelian group. Given a variety V in $\mathcal{C}(G)$ a *continuous image of V* is a variety W on a locally compact Abelian group H

such that there exists a continuous surjective homomorphism $\Phi : G \to H$ such that the function φ is in W if and only if the function $\varphi \circ \Phi$ is in V.

Theorem 3.1. *Let G be a locally compact Abelian group and V a variety on G. If V is synthesizable, then every continuous image of V is synthesizable.*

Proof. In the light of Theorem 2.2, it is enough to show that if V is synthesizable, then, for every continuous image W of V, the ideal $(\operatorname{Ann} W)\hat{}$ is localizable.

Assume that W is a variety on the locally compact Abelian group H, and $\Phi : G \to H$ is a continuous surjective homomorphism such that the function φ is in W if and only if the function $\varphi \circ \Phi$ is in V. We denote $\operatorname{Ann} V$, resp. $\operatorname{Ann} W$ by I, resp. J.

First we observe that for every exponential m on H, the function $m \circ \Phi$ is an exponential on G. Similarly, for every additive function a on H, the function $a \circ \Phi$ is an additive function on G. From this we conclude that for every polynomial p, resp. m-exponential monomial φ on H, the function $p \circ \Phi$, resp. the function $\varphi \circ \Phi$ is a polynomial, resp. an $m \circ \Phi$-exponential monomial on G.

The mapping Φ induces a continuous algebra homomorphism Φ_H of the measure algebra $\mathcal{M}_c(G)$ into the measure algebra $\mathcal{M}_c(H)$ in the following manner: for each measure μ on G we let

$$\langle \Phi_H(\mu), \varphi \rangle = \langle \mu, \varphi \circ \Phi \rangle$$

whenever φ is in $\mathcal{C}(H)$. It is easy to se that Φ_H is a continuous linear functional on $\mathcal{C}(H)$, hence it is in $\mathcal{M}_c(H)$. We can check easily that Φ_H is a continuous algebra homomorphism.

In fact, Φ_H is surjective. Indeed, for each u in H there is an x in G such that $u = \Phi(x)$. It follows, for each φ in $\mathcal{C}(H)$,

$$\langle \Phi_H(\delta_x), \varphi \rangle = \langle \delta_x, \varphi \circ \Phi \rangle = \varphi(\Phi(x)) = \varphi(u) = \langle \delta_u, \varphi \rangle,$$

hence $\Phi_H(\delta_x) = \delta_{\Phi(x)}$. As each measure in $\mathcal{M}_c(H)$ is a weak*-limit of finitely supported measures, and all finitely supported measures are in the image of Φ_H, we conclude that Φ_H is surjective.

The adjoint mapping of Φ_H is a linear mapping from $\mathcal{M}_c(H)^*$ onto $\mathcal{M}_c(G)^*$. As these spaces are identified by $\mathcal{C}(H)$, resp. $\mathcal{C}(G)$, we realize the adjoint of Φ_H as the mapping

$$\Phi_H^*(\varphi) = \varphi \circ \Phi$$

for each φ in $\mathcal{C}(H)$. As Φ_H is surjective, so is Φ_H^*, and we infer that every function in $\mathcal{C}(G)$ is of the form $\varphi \circ \Phi$ with some φ in $\mathcal{C}(H)$. We note that, in particular, every exponential on G is of the form $m \circ \Phi$, where m is an exponential on H.

Obviously, Φ_H induces a continuous algebra homomorphism from the Fourier algebra $\mathcal{A}(G)$ onto the Fourier algebra $\mathcal{A}(H)$, which we denote by $\widehat{\Phi}_H$, satisfying

$$\widehat{\Phi}_H(\widehat{\mu}) = \Phi_H(\mu)\widehat{\ }$$

for each μ in $\mathcal{M}_c(G)$. We claim that the ideal $\widehat{I}$ is mapped onto $\widehat{J}$ by $\widehat{\Phi}_H$. Let $\widehat{\mu}$ be in $\widehat{I}$, then $\mu * f = 0$ for each f in V. We need to show that $\widehat{\Phi}_H(\widehat{\mu})$ is in $\widehat{J}$, that is, $\Phi_H(\mu)$ annihilates W. If φ is in W, then $f = \varphi \circ \Phi$ is in V, hence

$$\Phi_H(\mu) * \varphi(u) = \int_H \varphi(u - v)\, d\Phi_H(\mu)(v) = \int_G \varphi(\Phi(x) - \Phi(y)\, d\mu(y) =$$

$$\int_G \varphi(\Phi(x - y))\, d\mu(y) = \int_G (\varphi \circ \Phi)(x - y)\, d\mu(y) =$$

$$\int_G f(x - y)\, d\mu(y) = \mu * f(x) = 0.$$

On the other hand, if $\widehat{\nu}$ is in $\widehat{J}$, then ν is in $J = \operatorname{Ann} W$, further $\nu = \Phi_H(\mu)$ for some μ in $\mathcal{M}_c(G)$. We want to show that μ is in $\operatorname{Ann} V = \operatorname{Ann} \operatorname{Ann} I$. Assuming the contrary, there exists an f in V such that $\mu * f \neq 0$. We have $f = \varphi \circ \Phi$ for some φ in W, and this implies

$$\nu * \varphi = \Phi_H(\mu) * \varphi = \mu * (\varphi \circ \Phi) = \mu * f \neq 0,$$

a contradiction, as ν is in $\operatorname{Ann} W$ and φ is in W. We conclude that μ is in $\operatorname{Ann} V$, hence the mapping $\widehat{\Phi}_H : \widehat{I} \to \widehat{J}$ is onto.

Now we are ready to show that $\widehat{J}$ is localizable, if V is synthesizable, i.e. $\widehat{I}$ is localizable. Let $\widehat{\nu}$ be in $\widehat{J}_{\mathcal{P}_{m,\widehat{J}},m}$ – we need to show that $\widehat{\nu}$ is in $\widehat{J}$. Here m is an arbitrary exponential on H, hence $m \circ \Phi$ is an exponential on G. Let $\widehat{\nu} = \widehat{\Phi}_H(\widehat{\mu})$, where $\widehat{\mu}$ is in $\mathcal{A}(G)$. It is enough to show that $\widehat{\mu}$ is in $\widehat{I}$. We note that Using the localizability of $\widehat{I}$, it is enough to show that every derivation in $\widehat{I}_{\mathcal{P}_{m\circ\Phi,\widehat{I}},m\circ\Phi}$ annihilates $\widehat{\mu}$ at $m \circ \Phi$. Let D be a polynomial derivation in $\widehat{I}_{\mathcal{P}_{m\circ\Phi,\widehat{I}},m\circ\Phi}$. It has the form

$$D\widehat{\mu}(m \circ \Phi) = \int_G f_D(x)(m \circ \Phi)(-x)\, d\mu(x)$$

for each exponential m on H, where $f_D : G \to \mathbb{C}$ is a polynomial. We have seen above that f_D can be written as $f_D = p_D \circ \Phi$ with some polynomial $p_D : H \to \mathbb{C}$. Hence we have

$$D\widehat{\mu}(m \circ \Phi) = \int_G f_D(x)(m \circ \Phi)(-x)\, d\mu(x) = \int_G (p_D \circ \Phi)(x)(m \circ \Phi)(-x)\, d\mu(x),$$

or

$$D\widehat{\nu}(m) = D\widehat{\Phi}_H(\widehat{\mu})(m) = D\Phi_H(\mu)\widehat{\ }(m) = \int_G p_D(u)m(-u)\,d\Phi_H(\mu)(u).$$

This means that D induces a polynomial derivation on $\mathcal{A}(H)$, which is in $\widehat{J}_{\mathcal{P}_{m,\widehat{J}},m}$. By assumption, this derivation annihilates $\widehat{\nu}$ at m, which implies that D annihilates $\widehat{\mu}$ at $m \circ \Phi$. As this holds for each D in $\widehat{I}_{\mathcal{P}_{m\circ\Phi,\widehat{I}},m\circ\Phi}$, by the localizability of $\widehat{I}$, we conclude that $\widehat{\mu}$ is in $\widehat{I}$, thus $\widehat{\nu}$ is in $\widehat{J}$, and our theorem is proved. ■

Corollary 3.1. *If spectral synthesis holds for a variety on a locally compact Abelian group, then it holds on every continuous image of it.*

Proof. Spectral synthesis means that every subvariety is synthesizable. If spectral synthesis holds for V on G, then every subvariety of V is synthesizable, which implies that every subvariety of every continuous image of V is synthesizable and our statement follows. ■

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