Annales Univ. Sci. Budapest., Sect. Comp. 56 (2024) 305–317

SPECHT MODULES AND INCIDENCE MATRICES

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Communicated by László Szili
(Received April 19, 2024; accepted May 15, 2024)

Abstract. We give representation theoretic interpretation for the terms in Wilson's rank formula (1.2) for the inclusion matrices of complete uniform families. We exhibit a dual Specht filtration of a suitable adjoint Radon map whose factors contribute the terms in the rank formula. From a representation theoretic point of view we give descriptions for the (complete) kernel and image of these maps in the case of partitions with at most two parts. The primary tools are the Specht filtration given by James in [10], and, in the dual setting, the work on polynomial functions by Friedl and Rónyai [6].

1. Introduction

Let n be a positive integer and $[n]$ stand for the set $\{1, 2, \ldots, n\}$. The family of all subsets of $[n]$ is denoted by $2^{[n]}$. For an integer $0 \leq j \leq n$ we denote by $\binom{[n]}{j}$ the family of all j element subsets of $[n]$. We write $\binom{[n]}{\leq j} := \cup_{i=0}^{j} \binom{[n]}{i}$.

Let $\mathbb{F}$ be a field. As customary, $\mathbb{F}[x_1, \ldots, x_n]$ denotes the ring of polynomials in variables $x_1, \ldots, x_n$ over $\mathbb{F}$. For a subset $F \subseteq [n]$ we write $x_F = \prod_{j \in F} x_j$. In particular, $x_\emptyset = 1$.

Let $v_F \in \{0,1\}^n$ denote the characteristic vector of a set $F \subseteq [n]$. For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, let $V(\mathcal{F}) = \{v_F : F \in \mathcal{F}\} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$. A

Key words and phrases: Inclusion matrix, tableau, dual Specht module, filtration, complete uniform family.
2010 Mathematics Subject Classification: 05E10, 05B20, 20C30.
Research supported in part by OTKA grants T42481 and T42706, and by the European Union project RRF-2.3.1-21-2022-00004 within the framework of the Artificial Intelligence National Laboratory, Hungary, and by the Program of Excellence TKP2021-NVA-02 at the Budapest University of Technology and Economics.

polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ can be considered as a function from $V(\mathcal{F})$ to $\mathbb{F}$ in the straightforward way.

Several interesting facts on finite set systems $\mathcal{F} \subseteq 2^{[n]}$ can be formulated conveniently as statements about *polynomial functions on* $V(\mathcal{F})$. For instance, certain inclusion matrices can be viewed naturally in this setting, see for example [2], [3], [6], [8], and [9] for this point of view.

For families $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ the *inclusion matrix* $I(\mathcal{F}, \mathcal{G})$ is a (0,1) matrix of size $|\mathcal{F}| \times |\mathcal{G}|$ whose rows and columns are indexed by the elements of $\mathcal{F}$ and $\mathcal{G}$, respectively. The entry at position (F, G) is 1 if $G \subseteq F$ and 0 otherwise $(F \in \mathcal{F}, G \in \mathcal{G})$. We refer to [2] for more information on them.

Two important results of Richard Wilson give the ranks of inclusion matrices of complete uniform families: if $d \leq k \leq n-d$ are natural numbers, then

$$\operatorname{rank}_{\mathbb{F}} I\left(\binom{[n]}{k}, \binom{[n]}{\leq d}\right) = \binom{n}{d}, \tag{1.1}$$

and

$$\operatorname{rank}_{\mathbb{F}} I\left(\binom{[n]}{k}, \binom{[n]}{d}\right) = \sum_i \binom{n}{i} - \binom{n}{i-1}, \tag{1.2}$$

where the summation is for those indices i, for which $\binom{k-i}{d-i}$ is not 0 in $\mathbb{F}$. For proofs see Wilson [13], Bier [4], Frankl [5], Frumkin and Yakir [7], Friedl and Rónyai [6].

In [7] A. Frumkin and A. Yakir proved (1.1) and (1.2) using the representation theory of the symmetric groups Sym_n. They worked with the linear spaces $\mathcal{M}^i$ over $\mathbb{F}$ whose basis elements are the i-subsets of $[n]$. The $\mathcal{M}^i$ can be viewed as $\mathbb{F}Sym_n$-modules. A permutation $\pi \in Sym_n$ sends a subset $H \subseteq [n]$ to $\pi(H) := \{\pi(j) : \ j \in H\}$. For $0 \leq d \leq k \leq n-d$ Frumkin and Yakir considered the Radon map $R^{d,k} : \mathcal{M}^k \to \mathcal{M}^d$ which sends a k-subset H to

$$R^{d,k}(H) := \sum_{G \subseteq H, \quad |G|=d} G, \tag{1.3}$$

and is extended linearly to $\mathcal{M}^k$.

Clearly the Radon maps are $\mathbb{F}Sym_n$-homomorphisms, and the transpose of $I\left(\binom{[n]}{k}, \binom{[n]}{d}\right)$ can be interpreted as the matrix of $R^{d,k}$. A key ingredient of their argument was that the image $\operatorname{Im}(R^{d,k})$ contains the Specht module S^d (see the next Section for the definition of Specht modules).

Here we shall consider the adjoint Radon maps $r^{k,d} : \mathcal{M}^d \to \mathcal{M}^k$ given as the linear extension of

$$r^{k,d}(G) := \sum_{H \supseteq G, \quad |H|=k} H, \tag{1.4}$$

where G is a d-subset of $[n]$. The name is justified by the following relation: let $0 \le d \le k \le n$, $v \in \mathcal{M}^d$, $w \in \mathcal{M}^k$. Then we have

$$\langle v, R^{d,k}(w)\rangle = \langle r^{k,d}(v), w\rangle. \tag{1.5}$$

Indeed, by bilinearity it suffices to check this when v is a d-subset, and w is a k-subset of $[n]$. Then we have 1 on both sides if $v \subseteq w$, and 0 if $v \not\subseteq w$.

Our aim here is to describe more precisely the image and the kernel of $r^{k,d}$. We do so by exhibiting filtrations with nice factors (dual Specht modules) in Theorems 3.1, 3.2. The factors of these filtrations give natural explanations of the terms in (1.2).

We shall use the Specht filtration of the modules M^i developed by James [10] (see also in [11]) and, in the dual setting some results of Friedl and Rónyai [6] about polynomial functions on $V\binom{[n]}{k}$.

From a combinatorial point of view, we give representation theoretic interpretation for the terms in formula (1.2). From a representation theoretic point of view we extend the work by James on Radon maps $R^{d,k}$, to adjoint Radon maps by giving descriptions for the kernel and image in the case of partitions with at most two parts.

2. Preliminaries

2.1. Partitions, Specht modules, Specht filtration

Concerning terminology related to partitions and tableaux, we follow James [11].

Throughout the paper n is a positive integer. A sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ of natural numbers is a *partition* of n, if $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$. We say that the partition λ is *proper* if $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_m > 0$. A λ-tableau t is a collection of boxes (cells) appearing in left justified rows, the ith row is with λ_i boxes for $i = 1, \ldots, m$. The boxes of t are filled with elements of $[n]$, each box contains exactly one integer, and different boxes contain different elements. There are $n!$ λ-tableaux.

For example, if $n = 7$, $\lambda = (4, 1, 2)$ then

$$\begin{array}{|c|c|c|c|}\hline 2 & 5 & 3 & 6 \\ \hline 4 \\ \cline{1-1} 7 & 1 \\ \cline{1-2}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 1 & 5 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 7 & 6 \\ \cline{1-2}\end{array} \tag{2.1}$$

are two of the $7!$ λ-tableaux.

The symmetric group Sym_n acts on the tableaux: for $\pi \in Sym_n$ and a λ-tableau t the tableau πt is also a λ-tableau and it will have $\pi(j)$ in the box where t contains j. Two tableaux t and t' associated with the same partition λ are *row equivalent* if t' can be obtained from t by permuting numbers in the same rows. The equivalence classes are called *tabloids*. The tabloid of a tableau t is denoted by $\{t\}$. As in [11], we depict the tabloid $\{t\}$ by just erasing the vertical lines from the picture of t. The tabloids corresponding to the tableaux of (2.1) may be drawn as

$$\begin{array}{cccc} \hline 2 & 5 & 3 & 6 \\ \hline 4 & & & \\ \cline{1-2} 7 & 1 & & \\ \cline{1-2} \end{array} \quad \text{and} \quad \begin{array}{cccc} \hline 1 & 2 & 4 & 5 \\ \hline 3 & & & \\ \cline{1-2} 7 & 6 & & \\ \cline{1-2} \end{array} .$$

Let $\mathbb{F}$ be a field, λ be a partition of n. We denote by M^λ the linear space over $\mathbb{F}$ whose basis elements are the λ-tabloids. M^λ carries an $\mathbb{F}Sym_n$-module structure: for $\pi \in Sym_n$ and a λ-tableau t we set $\pi\{t\} := \{\pi t\}$. There is a convenient Sym_n-invariant non-degenerate bilinear form $\langle,\rangle$ on M^λ which is defined by stating that the tabloids form an orthonormal basis.

In the paper we shall consider partitions of shape $\lambda = (n-j, j)$, where n is a positive integer, and $0 \le j \le n$. We write M^j instead of $M^{(n-j,j)}$.

For $0 \le i \le n$ let $P^i \subseteq \mathbb{F}[x_1, \dots, x_n]$ denote the $\mathbb{F}$-subspace spanned by the square-free monomials of degree i. We write $P^{\le i} := P^0 \oplus P^1 \oplus \dots \oplus P^i$. P^i and $P^{\le i}$ are $\mathbb{F}Sym_n$-modules by the rule $\pi(x_{l_1} x_{l_2} \cdots x_{l_j}) := x_{\pi(l_1)} x_{\pi(l_2)} \cdots x_{\pi(l_j)}$.

For $0 \le \ell \le n$ M^ℓ and P^ℓ are isomorphic $\mathbb{F}Sym_n$-modules. The map ϕ_ℓ which sends the tabloid

$$\{t\} = \begin{array}{ccccccc} \hline j_1 & j_2 & \dots & j_k & \dots & j_{n-\ell} \\ \hline i_1 & i_2 & \dots & i_\ell & & \\ \cline{1-4} \end{array} \tag{2.2}$$

to $\phi_\ell(\{t\}) := x_{i_1} x_{i_2} \cdots x_{i_\ell}$ is an $\mathbb{F}Sym_n$-isomorphism. Actually, ϕ_ℓ can be turned into an isometry as well, by defining $\langle x_H, x_K \rangle := \delta_{H,K}$, for ℓ-subsets $H, K \subseteq [n]$. We have also $\mathcal{M}^\ell \cong P^\ell$, as $\mathbb{F}Sym_n$-modules with the map $H \longleftrightarrow$ $\longleftrightarrow x_H$. Thus, M^ℓ can be viewed as the linear space spanned by the ℓ-subsets of $[n]$, and we have for $d \le k$ the Radon maps $R^{d,k} : M^k \to M^d$ defined by (1.3). We shall also consider the adjoint Radon maps $r^{k,d} : M^d \to M^k$ given by (1.4).

In the rest of the paper we shall always use the one from the isomorphic $\mathbb{F}Sym_n$-modules M^ℓ, $\mathcal{M}^\ell$ and P^ℓ which allows the most convenient exposition.

Let $\lambda = (n-\ell, \ell)$ be a partition of n and t be a λ-tableau and put $m = \min\{n-\ell, \ell\}$. For an integer $1 \le i \le m$ we denote by $e_{t,i}$ the sum of tabloids

$$e_{t,i} := \sum \mathrm{sign}\pi \cdot \pi\{t\},$$

where the summation is for those permutations $\pi \in Sym_n$ which stabilize the first i columns of t (as subsets of $[n]$), and fix all other boxes of t.

Example. Let $n = 7$, $\lambda = (4,3)$, $i = 2$ and

$$t = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 3 & 6 \\ \hline 7 & 4 & 1 \\ \cline{1-3}\end{array}.$$

Then

$$(2.3)\quad e_{t,i} = \begin{array}{cccc}\hline 2 & 5 & 3 & 6 \\ \hline 7 & 4 & 1 \\ \cline{1-3}\end{array} - \begin{array}{cccc}\hline 7 & 5 & 3 & 6 \\ \hline 2 & 4 & 1 \\ \cline{1-3}\end{array} - \begin{array}{cccc}\hline 2 & 4 & 3 & 6 \\ \hline 7 & 5 & 1 \\ \cline{1-3}\end{array} + \begin{array}{cccc}\hline 7 & 4 & 3 & 6 \\ \hline 2 & 5 & 1 \\ \cline{1-3}\end{array}$$

Let $S^{\ell,i}$ denote the linear subspace of M^ℓ spanned by the tabloid sums $e_{t,i}$ where t runs through the $(n-\ell,\ell)$ tableaux. The modules $S^m := S^{\ell,m} \cong S^{n-\ell,m}$ are called *Specht modules* and play a central role in the representation theory of the symmetric group Sym_n. For notational convenience we put $S^{\ell,m+1} := (0)$.

Next we summarize some of the basic properties of the modules $S^{\ell,i}$. These were obtained by Gordon James as a part of his characteristic-free treatment of the representation theory of symmetric groups, see [10], [11].

Please note that for simplicity we work here with the notation $S^{\ell,i}$ and M^ℓ instead of the more precise $S^{(n-\ell,i),(n-\ell,\ell)}$ and $M^{(n-\ell,\ell)}$ used by James. This will cause no confusion as we always consider partitions of $[n]$ and into at most two parts.

The $\mathbb{F}Sym_n$-modules $S^{\ell,i}$ form a *Specht filtration* of M^ℓ. To be more specific, assume that $0 \le \ell \le n$, $m = \min\{n-\ell, \ell\}$ and $0 \le i \le m$. We have then

$$(2.4)\qquad M^\ell = S^{\ell,0} \ge S^{\ell,1} \ge \ldots \ge S^{\ell,m} \ge S^{\ell,m+1} = (0),$$

with

$$(2.5)\qquad S^{\ell,i}/S^{\ell,i+1} \cong S^i,$$

for $0 \le i \le m$, where the isomorphism is induced by the Radon map $R^{i,\ell}$, restricted to $S^{\ell,i}$. Moreover,

$$(2.6)\qquad \dim_{\mathbb{F}} S^{\ell,i} = \binom{n}{\ell} - \binom{n}{i-1}.$$

We record separately an important special case of (2.6). For $0 \leq i \leq n/2$ we have

$$\dim_{\mathbb{F}} S^i = \binom{n}{i} - \binom{n}{i-1}. \tag{2.7}$$

Thus, $S^{\ell,i}$, $i = 0, 1, \ldots, m+1$ is a descending series of $\mathbb{F}Sym_n$-submodules of M^ℓ whose factors are Specht modules of the form S^i. As for proofs, (2.4) follows directly from the definitions. The isomorphism (2.5) was proved by a beautiful combinatorial argument by James (Corollary 9.2, [10]) in a much more general setting, see also Theorem 17.13 in [11]. Formula (2.7) is a special case of the Frame-Robinson-Thrall Theorem (Theorem 20.1 in [11]). Finally, (2.6) is a simple consequence of (2.4), (2.5) and (2.7).

2.2. Modules of functions, dual Specht filtration

Here we introduce yet another chain of submodules in M^ℓ. It is based on spaces of $\mathbb{F}$-valued functions and gives a setting dual to the Specht filtration introduced earlier.

Let $0 \leq \ell \leq n$, and let $F_{\ell,i} := F_{\ell,i}(\mathbb{F})$ stand for the $\mathbb{F}$-space of functions from $V\binom{[n]}{\ell}$ to $\mathbb{F}$ which are polynomials of degree at most i. Put $F_\ell := F_{\ell,\ell}$. Clearly $F_{\ell,i}$ is an $\mathbb{F}Sym_n$-module: a permutation $\pi \in Sym_n$ acts on $f \in F_{\ell,i}$ as

$$(\pi f)(v_1, \ldots, v_n) := f(v_{\pi(1)}, \ldots, v_{\pi(n)}).$$

We have a chain of submodules

$$F_\ell = F_{\ell,\ell} \geq F_{\ell,\ell-1} \geq \ldots \geq F_{\ell,0}. \tag{2.8}$$

As before, we write $m = \min\{\ell, n-\ell\}$. The rank formula (1.1) implies that $\dim_{\mathbb{F}} F_{\ell,i} = \binom{n}{i}$ for $i = 0, \ldots, m$, therefore $F_{\ell,m} = \cdots = F_{\ell,\ell} = F_\ell$. It is known also that the evaluation map ev which sends a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ to the corresponding polynomial function on $V\binom{[n]}{\ell}$ provides an isomorphism of the $\mathbb{F}$-spaces P^ℓ and F_ℓ (cf. Proposition 8 in [6] for a simple proof). It is easy to check that this map is actually an $\mathbb{F}Sym_n$-isomorphism as well. Let us define $P_{\ell,i} := ev^{-1}(F_{\ell,i}) \cap P^\ell$. Now (2.8) gives the following chain of $\mathbb{F}Sym_n$-submodules

$$P^\ell = P_{\ell,\ell} = P_{\ell,m} \geq P_{\ell,m-1} \geq \ldots \geq P_{\ell,0} \geq P_{\ell,-1} := (0), \tag{2.9}$$

where for $i = 0, 1, \ldots, m$ we have

$$\dim_{\mathbb{F}} P_{\ell,i} = \binom{n}{i}. \tag{2.10}$$

Definition. Let $H \subseteq [n]$, $|H| \leq \ell$. We put

$$y_{H,\ell} := \sum_{H \subseteq G,\ \ |G|=\ell} x_G.$$

Note that $y_{H,\ell} = r^{\ell,j}(x_H)$, where $j = |H|$.

Recall, that a tableau t is *standard* if it belongs to a proper partition λ and the numbers in the boxes of t are increasing from left to right and from top to bottom. We call a j-subset $H \subset [n]$ *standard*, if $x_H = \phi_j(\{t\})$ holds for a standard tableau t.

Example. Let $n = 7$, $\lambda = (4, 3)$, and

$$t = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 7 & \\ \cline{1-3} \end{array}.$$

Then t is a standard tableau and the corresponding standard set is $H = \{3, 4, 7\}$.

The following was proved in Theorem 13 of [6].

Proposition 2.1. *Suppose that* $0 \leq \ell \leq n$ *and* $0 \leq i \leq m = \min\{\ell, n - \ell\}$. *Then a basis of* $P_{\ell,i}$ *over* $\mathbb{F}$ *is*

$$\{y_{H,\ell} :\ |H| \leq i,\ H \textit{ is a standard subset of } [n]\}.$$

The simple lemma below will be useful in calculations with the Radon images of tabloid sums $e_{t,i}$. Recall, that the map ϕ_ℓ is defined at (2.2).

Lemma 2.1. *Let* $0 \leq \ell \leq n$, $m = \min\{\ell, n - \ell\}$. *Suppose that* i *is an integer,* $0 \leq i \leq m$, *and* t *is an* $(n - \ell, \ell)$*-tableau*

$$t = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & a_2 & \ldots & a_\ell & \ldots & a_{n-\ell} \\ \hline b_1 & b_2 & \ldots & b_\ell & & \\ \cline{1-4} \end{array}. \tag{2.11}$$

Let H *be a subset of* $[n]$, *and* $1 \leq j \leq i$ *be such that* $H \cap \{a_j, b_j\} = \emptyset$. *Then the polynomial* $\phi_\ell(e_{t,i})$ *can be written as*

$$\phi_\ell(e_{t,i}) := w + \sum_G \epsilon_G (x_{a_j} - x_{b_j}) x_G, \tag{2.12}$$

where G *runs through the* $\ell - 1$*-subsets of* $[n] \setminus \{a_j, b_j\}$ *which contain* H, $\epsilon_G \in \{-1, 0, 1\}$, *and* w *is a linear combination of monomials not divisible by* x_H.

Proof. Write $\phi_\ell(e_{t,i})$ as a linear combination of monomials

$$\phi_\ell(e_{t,i}) = \sum_{F\subseteq[n],\ |F|=\ell} \alpha_F x_F. \tag{2.13}$$

We may put $w = \sum_{H\not\subseteq F} \alpha_F x_F$. We focus now on the monomials x_F from (2.13), where $H \subset F$ and $\alpha_F \neq 0$. These come in pairs. With $G = F \cap \cap([n] \setminus \{a_j, b_j\})$ there are exactly two ℓ-subsets $F_1, F_2 \subset [n]$ which intersect $[n] \setminus \{a_j, b_j\}$ in the $\ell-1$-subset G, namely $F_1 = G \cup \{a_j\}$, and $F_2 = G \cup \{b_j\}$. Clearly the monomials x_{F_1} and x_{F_2} occur in $\phi_\ell(e_{t,i})$ with opposite signs, hence their contribution to the sum is $\epsilon_G(x_{F_1} - x_{F_2}) = \epsilon_G(x_{a_j} - x_{b_j})x_G$. The proof is complete. ∎

Corollary 2.1. *Let ℓ, i, t be as above. If $H \subseteq [n]$ with $|H| < i$ then there exists a j such that $1 \leq j \leq i$ and $H \cap \{a_j, b_j\} = \emptyset$, therefore the conclusion of the Lemma holds: $\phi_\ell(e_{t,i})$ has an expression of the form (2.12).* ∎

To simplify notation, we shall denote also by $S^{\ell,i}$ the (isomorphic) image $\phi_\ell(S^{\ell,i}) \leq P^\ell$ of the module $S^{\ell,i} \leq M^\ell$. The next theorem states that $P_{\ell,i}$ is the orthogonal complement in $P^\ell \cong M^\ell$ of a member of the chain (2.4).

Theorem 2.2. *Suppose that $0 \leq \ell \leq n$, $m = \min\{\ell, n-\ell\}$, and $-1 \leq i \leq m$. Then we have $P_{\ell,i} = (S^{\ell,i+1})^\perp$.*

Proof. The claim is immediate for $i = -1$ and for $i = m$. We assume now that $0 \leq i < m$. By (2.6) and (2.10) both sides have dimension $\binom{n}{i}$ over $\mathbb{F}$. It is, therefore, enough to show that $P_{\ell,i} \subseteq (S^{\ell,i+1})^\perp$. To this end, we verify that $\langle \phi_\ell(e_{t,i+1}), y_{H,\ell}\rangle = 0$ whenever t is an $(n-\ell, \ell)$-tableau, $e_{t,i+1}$ is the corresponding sum of tabloids, and $H \subset [n]$, $|H| \leq i$. This suffices because the elements $e_{t,i+1}$ generate $S^{\ell,i+1}$ by definition and the $y_{H,\ell}$ span $P_{\ell,i}$ by Proposition 2.1. We apply Corollary 2.1 (with $i+1$ in the place of i). We have

$$\langle \phi_\ell(e_{t,i+1}), y_{H,\ell}\rangle = \langle w + \sum_G \epsilon_G(x_{a_j} - x_{b_j})x_G, y_{H,\ell}\rangle =$$

$$= \sum_G \epsilon_G \langle (x_{a_j} - x_{b_j})x_G, y_{H,\ell}\rangle = \sum_G \epsilon_G \langle (x_{a_j} - x_{b_j})x_G, (x_{a_j} + x_{b_j})x_G\rangle = 0.$$

Here we used that the monomials in $y_{H,\ell}$ are all divisible by x_H, hence we have $\langle w, y_{H,\ell}\rangle = 0$. The proof is complete. ∎

Definition 2.1. Let $0 \leq \ell \leq n$, $m = \min\{\ell, n-\ell\}$. The $\mathbb{F}Sym_n$-module $S_m := P^\ell/(S^m)^\perp \cong P^{n-\ell}/(S^m)^\perp$ is called a *dual Specht module.*

Clearly we have $\dim_{\mathbb{F}} S_m = \dim_{\mathbb{F}} S^m = \binom{n}{m} - \binom{n}{m-1}$. An important consequence of Theorem 2.2 is that (2.9) is a dual Specht filtration of P^ℓ.

Corollary 2.2. *Suppose that* $0 \le \ell \le n$, $m = \min\{\ell, n-\ell\}$, *and* $0 \le i \le m$. *Then*

$$P_{\ell,i}/P_{\ell,i-1} \cong S_i.$$

Proof. The statement is immediate for $i = 0$ because $P^0 = S^0$. Henceforth we assume that $0 < i \le m$. Clearly we have $P_{\ell,i}/P_{\ell,i-1} \cong F_{\ell,i}/F_{\ell,i-1}$, hence it is enough to prove the claim for $F_{\ell,i}/F_{\ell,i-1}$. Obviously, $F_{\ell,i}$ is an epimorphic image of $P^{\le i}$ with respect to the evaluation map $ev : P^{\le i} \to F_{\ell,i}$, which sends a polynomial to the corresponding polynomial function on $V\binom{[n]}{\ell}$. Let $M = ev^{-1}(F_{\ell,i-1})$ be the inverse image of the submodule $F_{\ell,i-1} \le F_{\ell,i}$. We have $ev(P^{\le i-1}) \le F_{\ell,i-1}$, hence

$$M = P^{\le i-1} \oplus N$$

for some submodule $N \le P^i$.

By the second isomorphism theorem we have

$$F_{\ell,i}/F_{\ell,i-1} \cong P^{\le i}/M \cong (P^{\le i}/P^{\le i-1})/(P^{\le i-1} \oplus N/P^{\le i-1}) \cong P^i/N.$$

We show next that $P_{i,i-1} = N$. This will follow from $P_{i,i-1} \le N$, because both N and $P_{i,i-1}$ have codimension $\binom{n}{i} - \binom{n}{i-1}$ in P^i (see (2.10) and the discussion after (2.8)).

Let $H \subset [n]$, $|H| \le i-1$. The polynomial $y_{H,i}$ is the same function on $V\binom{[n]}{\ell}$ as $\binom{\ell-|H|}{i-|H|} x_H$, implying that $y_{H,i} \in N$. On the other hand, the polynomials $y_{H,i}$ span $P_{i,i-1}$ by Proposition 2.1, implying that $P_{i,i-1} \le N$.

Now Theorem 2.2 gives that

$$P_{\ell,i}/P_{\ell,i-1} \cong F_{\ell,i}/F_{\ell,i-1} \cong P^i/P_{i,i-1} = P^i/(S^i)^{\perp} = S_i. \qquad \blacksquare$$

3. Filtrations for the adjoint Radon maps

Throughout this section we assume that $d, k \in \mathbb{Z}$ such that $0 \le d \le k \le$ $\le n-d$. We write $m = \min\{k, n-k\}$. Note that $d \le m \le n/2$. Our aim is to describe dual Specht filtrations for the image and kernel of the adjoint Radon map $r^{k,d} : M^d \to M^k$. We put

$$X = \{i :\ 0 \le i \le d \text{ and } \tbinom{k-i}{d-i} \text{ is not 0 in } \mathbb{F}\},$$

and $Y := \{0, 1, \dots, d\} \setminus X$. It will be convenient to view $r^{k,d}$ as a map from P^d to P^k. We have then

$$r^{k,d}(x_H) := \sum_{H \subseteq G,\ |G|=k} x_G,$$

where $H \subseteq [n]$, $|H| = d$.

The inclusion matrix $I = I_{\mathbb{F}}\left(\binom{[n]}{k}, \binom{[n]}{d}\right)$ can be viewed as the matrix of the map $r^{k,d}$, hence a description of $\mathrm{Im}(r^{k,d})$ gives an *interpretation of the terms* in the rank formula (1.2) for I, as dimensions of dual Specht modules.

Theorem 3.1. *The module $Q := \mathrm{Im}(r^{k,d})$ admits a filtration with $\mathbb{F}Sym_n$-modules*

$$Q =: Q_d \geq Q_{d-1} \geq \ldots \geq Q_0 \geq Q_{-1} = (0), \tag{3.1}$$

where $Q_i = Q_{i-1}$ if $i \in Y$, and $Q_i/Q_{i-1} \cong S_i$, when $i \in X$ Thus, after deleting multiple terms, (3.1) gives rise to a dual Specht filtration of Q. In particular, we have

$$\dim_{\mathbb{F}} \mathrm{Im}(r^{k,d}) = \sum_{i \in X} \binom{n}{i} - \binom{n}{i-1}.$$

Remark 3.1. *The (dimensions of the) dual Specht factors S_i in the Theorem can be seen to contribute the terms in Wilson's rank formula (1.2).*

Proof. For the sake of simplicity we write simply r instead of $r^{k,d}$. We set $Q_i := r(P_{d,i})$ for $i = -1, 0, \ldots, d$. Clearly we have

$$P_{k,d} \geq Q_d \geq Q_{d-1} \geq \ldots \geq Q_0 \geq Q_{-1} = (0),$$

and

$$\dim_{\mathbb{F}} Q_i/Q_{i-1} \leq \dim_{\mathbb{F}} P_{d,i}/P_{d,i-1} = \binom{n}{i} - \binom{n}{i-1},$$

for $0 \leq i \leq d$, see also (2.10). Simple counting shows that

$$r(y_{H,d}) = \binom{k-|H|}{d-|H|} y_{H,k}, \tag{3.2}$$

whenever $H \subseteq [n]$, $|H| \leq d$. Now Proposition 2.1 and (3.2) imply that

$$Q_i = r(P_{d,i}) \leq P_{k,i}. \tag{3.3}$$

Let i be an integer such that $0 \leq i \leq d$ and $\binom{k-i}{d-i}$ is not 0 in $\mathbb{F}$. Then $Q_i + P_{k,i-1} = P_{k,i}$, because by (3.2) we have $y_{H,k} \in Q_i$ for every standard i-set H, and these polynomials form a basis of $P_{k,i}$ modulo $P_{k,i-1}$ (see Proposition 2.1).

By Corollary 2.2

$$S_i \cong P_{k,i}/P_{k,i-1} = (Q_i + P_{k,i-1})/P_{k,i-1} \cong Q_i/(Q_i \cap P_{k,i-1}). \tag{3.4}$$

We know also that $Q_{i-1} \leq Q_i \cap P_{k,i-1}$, hence

$$\binom{n}{i} - \binom{n}{i-1} = \dim_{\mathbb{F}} Q_i/(Q_i \cap P_{k,i-1}) \leq \dim_{\mathbb{F}} Q_i/Q_{i-1} \leq \binom{n}{i} - \binom{n}{i-1}.$$

We must have equalities above, implying that

$$Q_{i-1} = Q_i \cap P_{k,i-1}, \tag{3.5}$$

and $Q_i/Q_{i-1} \cong S_i$.

Finally, let $i \in Y$. Then $r(y_{H,d}) = 0$ by (3.2), when H is an i-set, hence $Q_i = Q_{i-1}$. This completes the proof. ∎

Corollary 3.1. *For $-1 \le i \le d$ we have*

$$Q_i = Q_d \cap P_{k,i}.$$

Proof. This is immediate for $i = d$, and we may proceed by induction for $i = d-1, d-2, \ldots, 0$. Using (3.5) and the induction hypothesis, we obtain

$$Q_i = Q_{i+1} \cap P_{k,i} = Q_d \cap P_{k,i+1} \cap P_{k,i} = Q_d \cap P_{k,i}. \qquad \blacksquare$$

Similarly to $\mathrm{Im}(r^{k,d})$, we have a dual Specht filtration for the kernel K of $r := r^{k,d}$. We set for $-1 \le i \le d$

$$K_i := \mathrm{Ker}(r) \cap P_{d,i}.$$

This provides a chain of submodules

$$K = K_d \ge K_{d-1} \ge \ldots \ge K_0 \ge K_{-1} = (0). \tag{3.6}$$

Theorem 3.2. *Suppose that $0 \le i \le d$. Then $K_i = K_{i-1}$ if $i \in X$, and $K_i/K_{i-1} \cong S_i$, when $i \in Y$. Consequently we have*

$$\dim_{\mathbb{F}} \mathrm{Ker}(r^{k,d}) = \sum_{i \in Y} \binom{n}{i} - \binom{n}{i-1}.$$

Proof. We note first that

$$K_i \cap P_{d,i-1} = \mathrm{Ker}(r) \cap P_{d,i} \cap P_{d,i-1} = \mathrm{Ker}(r) \cap P_{d,i-1} = K_{i-1},$$

hence

$$K_i/K_{i-1} \cong K_i/(K_i \cap P_{d,i-1}) \cong (K_i + P_{d,i-1})/P_{d,i-1}.$$

Now if $i \in Y$, then $r(y_{H,d}) = 0$ holds by (3.2) for every i-subset $H \subset [n]$, giving by Proposition 2.1 that $K_i + P_{d,i-1} = P_{d,i}$, and then $K_i/K_{i-1} \cong S_i$ by Corollary 2.2.

Let us settle now the case $i \in X$. Consider an arbitrary element $z \in K_i$. Then

$$z = \sum \alpha_G y_{G,d}$$

where G runs through the standard subsets of $[n]$ with $|G| \le i$ and $\alpha_G \in \mathbb{F}$. Using (3.2) we infer

$$0 = r(z) = \sum \binom{k-|G|}{d-|G|} \alpha_G y_{G,k}.$$

The polynomials $y_{G,k}$ are independent by Proposition 2.1, hence $\binom{k-|G|}{d-|G|}\alpha_G = 0$ for every G. By assumption, $\binom{k-i}{d-i} \neq 0$, therefore $\alpha_G = 0$, if G is an i-subset. This implies that $z \in K_{i-1}$, giving that $K_i = K_{i-1}$ in this case. ■

The results of this subsection bear some resemblance to a theorem of Lusztig and Stanley [12, Proposition 4.11], which gives the multiplicities of the Specht factors in the homogeneous components of $\mathbb{C}[x_1, \dots, x_n]/(\sigma_1, \dots, \sigma_n)$, where σ_i is the i-th elementary symmetric polynomial in $x_1, \dots, x_n$. In both cases the object of study is the Sym_n-module structure of a combinatorially interesting module of polynomial functions.

Similar filtrations for $R^{d,k}$ (with Specht module factors) will be considered in a forthcoming paper.

References

[1] **Anstee, R.P., L. Rónyai and A. Sali,** Shattering news, *Graphs and Combinatorics* **18** (2002), 59–73.

[2] **Babai, L. and P. Frankl,** *Linear algebra methods in combinatorics,* September 1992.

[3] **Bernasconi, A. and L. Egidi,** Hilbert function and complexity lower bounds for symmetric Boolean functions, *Information and Computation* **153** (1999), 1–25.

[4] **Bier, T.,** Remarks on recent formulas of Wilson and Frankl, *Europ. J. Combin.* **14** (1993), 1–8.

[5] **Frankl, P.,** Intersection theorems and mod p rank of inclusion matrices, *J. Combin. Theory A* **54** (1990), 85–94.

[6] **Friedl, K. and L. Rónyai,** Order-shattering and Wilson's theorem, *Discrete Math..* **270** (2003), 127–136.

[7] **Frumkin, A. and A. Yakir,** Rank of inclusion matrices and modular representation theory, *Israel journal of Math.* **71** (1990), 309–320.

[8] **Hegedűs, G. and L. Rónyai,** Gröbner bases for complete uniform families, *J. Algebraic Combin.* **17** (2003), 171–180.

[9] **Hegedűs, G. and L. Rónyai,** Standard monomials for q-uniform families and a conjecture of Babai and Frankl, *Cent. Eur. J. Math.* **1** (2003), 198–207.

[10] **James, G.D.,** A Characteristic-Free Approach to the Representation Theory of S_n, *Journal of Algebra,* **46** (1977), 430–450.

[11] **James, G.D.,** *The Representation Theory of the Symmetric Groups,* Springer Lecture Notes in Mathematics 682, Springer-Verlag, 1978.

[12] **Stanley, R.P.,** Invariants of finite groups and their applications to combinatorics, *Bulletin of the AMS,* **1** (1979), 475–511.

[13] **Wilson, R.M.,** A diagonal form for the incidence matrices of t-subsets vs. k-subsets, *European Journal of Combinatorics,* **11** (1990), 609–615.

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