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ON THE ITERATES OF SOME MULTIPLICATIVE FUNCTIONS

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Abstract. We summarize some known results concerning the iterates of multiplicative functions and prove two new results.

1. Introduction

We shall use the following standard notation: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$. Let $\mathcal{P}$ the set of primes. Let

$$\omega(n)=\sum_{\substack{p|n\\ p\in\mathcal{P}}}1 \quad \text{and} \quad \Omega(n)=\sum_{\substack{p^k|n\\ p\in\mathcal{P}}}1.$$

Let $\mathcal{M}$ be the set of multiplicative, $\mathcal{M}^*$ be the set of completely multiplicative functions.

Let

$$\overline{\mathcal{M}}=\{f\in\mathcal{M} \;\mid\; f(n)\in\mathbb{N} \quad \text{for every} \quad n\in\mathbb{N}\}, \quad \overline{\mathcal{M}^*}=\overline{\mathcal{M}}\cap\mathcal{M}^*.$$

One can easily show that

$$\text{if} \quad f_1\in\overline{\mathcal{M}},\ f_2\in\mathcal{M}^*,\ h(n)=f_2(f_1(n))\in\mathcal{M}.$$

Let $\overline{\mathcal{M}_C}$ be the set of those $f\in\overline{\mathcal{M}}$, for which

$$f(p)=\text{constant} \quad \text{for every} \quad p\in\mathcal{P}.$$

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Let
$$\tau^{(k)}(n) = \sharp\{n = u_1 \cdots u_k \mid u_j \in \mathbb{N},\ j = 1, \dots, k\}.$$
It is clear that $\tau^{(k)}(p) = k$ if $p \in \mathcal{P}$, and so $\tau^{(k)} \in \overline{\mathcal{M}_C}$.

It is known furthermore that
$$\sum_{n\in\mathbb{N}} \frac{\tau^{(\ell)}(n)}{n^s} = \zeta(s)^\ell \quad (\mathrm{Re}\ s > 1).$$
Furthermore, we have
$$\tau^{(\ell)}(p^r) = \binom{r+\ell-1}{\ell-1} \quad \text{if} \quad p \in \mathcal{P}.$$
Let $\tau(n) = \tau^{(2)}(n)$. Then the above relation implies that
$$\tau(p^r) = r + 1.$$

2. On the iteration of $\tau(n)$

Let $\tau_1(n) = \tau(n),\ \tau_{r+1}(n) = \tau(\tau_r(n))$ for every $r, n \in \mathbb{N}$. Let
$$D_r(x) = \sum_{n\le x} \tau_r(n).$$
Bellman and Shapin [2] formulated the conjecture that
$$D_r(x) = (1 + o_x(1))x \log_r x.$$
This conjecture is proved up to $r \le 4$. We refer to the works of P. Erdős [3] for case $r = 2$, of I. Kátai [6], [7] for $r = 2, 3$, of P. Erdős and I. Kátai [4] for case $r = 4$.

The basic observation to get these results was the following:
$$\text{If} \quad n = Km,\ \mu(m) \ne 0,\ (K, m) = 1, \text{then} \quad \tau(n) = \tau(K)2^{\omega(m)}.$$

3. The Selberg–Delange method. The method of K. Ramachandra

Let $K \in \mathbb{N}$ be fixed,
$$N_K(x) = \sum_{\substack{m\le x\\(m,K)=1}} |\mu(m)|.$$

Let $z \in \mathbb{C}$, $|z| < 2$, $s \in \mathbb{C}$, Re s > 1. Let

$$F_K(s,z) = \sum_{\substack{m=1 \\ (m,K)=1}}^{\infty} \frac{z^{\omega(m)}|\mu(m)|}{m^s}.$$

Then

$$F_K(s,z) = a_K(s,z)b(s,z)\zeta(s)^z,$$

where

$$a_K(s,z) = \prod_{p|K} \frac{1}{1+\frac{z}{p^s}},$$

$$b(s,z) = \prod_{p\in\mathcal{P}} \Big(1-\frac{1}{p^s}\Big)^z \Big(1+\frac{z}{p^s}\Big).$$

Hence, by using the standard method, we can deduce that

$$N_K(x) = \frac{6}{\pi^2}\Big(\prod_{p|K} \frac{1}{1+\frac{1}{p}}\Big)x + O(\sqrt{x}).$$

Let

$$N_K(x,k) = \sharp\{m \le \ x: \ (m,K)=1, \ \omega(m)=k\}.$$

Let

$$\Theta_k(x) = \frac{1}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}.$$

Repeating the argument of A. Selberg [11] we can deduce that

$$\frac{N_K(x,k)}{N_K(x)} = \Theta_k(x)\Big(1+O(\frac{1}{\log\log x})\Big)$$

uniformly as

$$k \le R_x := \log\log x + \rho_x\sqrt{\log\log x}. \tag{3.1}$$

Here $\rho_x \to \infty$, appropriately slowly. Especially, we obtain that

$$\lim_{x\to\infty} \frac{1}{N_K(x)}\sharp\Big\{m \le x \,:\, (m,K)=1,\, \mu(m)\ne 0,\, \frac{\omega(m)-\log\log x}{\sqrt{\log\log x}} < y\Big\} = \Phi(y),$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{u^2}{2}}\,du$$

is the Gaussian law.

By using the method of K. Ramachandra [10] and the observation of I. Kátai [8] we can prove the following

Theorem A. *Let $\epsilon > 0, K \in \mathbb{N}$ be fixed. Then*

$$\max_{x^{7/12+\epsilon} \le h \le x^{2/3}} \left| \frac{N_K(x+h,k) - N_K(x,k)}{h} - \frac{N_K(x,k)}{x} \right| \le c \frac{\Theta_k(x)}{\log\log x}$$

uniformly as k satisfies (3.1), $\rho_x \to \infty$ slowly.

4. On the function $R(n)$

Let $R(n) \ge 0$ for every $n \in \mathbb{N}$, $R(p) = A > 0$ for every $p \in \mathcal{P}$, and assume that

$$R(up) = R(u)R(p) \quad \text{if} \quad (u,p) = 1.$$

It implies that if $n = Km$, $(K,m) = 1, \mu(m) \ne 0$, then $R(n) = R(K)A^{\omega(m)}$. Let

$$F_K(s,A) = \sum_{\substack{m=1 \\ (m,K)=1}}^{\infty} \frac{A^{\omega(m)}|\mu(m)|}{m^s}.$$

From Theorem 1 of A. Selberg [11], we have

$$S_K(x) = \sum_{\substack{m \le x \\ (m,K)=1}} A^{\omega(m)}|\mu(m)| =$$

(4.1)
$$= \prod_{p|K} \frac{1}{1+\frac{A}{p}} \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^A \left(1 + \frac{A}{p}\right) - \frac{x(\log x)^{A-1}}{\Gamma(A)} + \\ + O\Big(x(\log x)^{A-2}\Big).$$

The error term is true up to $K \le \log x$, say.

Let $\mathcal{K}$ be the set of squarefull integers. Let

$$E(x) = \sum_{n \le x} R(n).$$

From (4.1) we obtain:

Theorem B. *Assuming that $R(K) = O(K^{1/4})$ $(K \in \mathcal{K})$ we have*

$$E(x) = dx\Big(\log x\Big)^{A-1} + O\Big(x\Big(\log x\Big)^{A-2}\Big),$$

where

$$d = \frac{1}{\Gamma(A)} \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^A \left(1 + \frac{A}{p}\right) \sum_{K \in \mathcal{K}} \frac{R(K)}{K} \prod_{p|K} \frac{1}{1+\frac{A}{p}}.$$

5. On the distribution of $\tau^{(\ell)}(\tau^{(k)}(n))$

Let

$$r(n) = \tau^{(\ell)}(\tau^{(k)}(n)) \quad \ell, k \geq 2.$$

Assume that $n = Km, (K, m) = 1, \mu(m) \neq 0, K \in \mathcal{K}$.

Let $k = \pi_1^{a_1} \cdots \pi_r^{a_r}$, where $\pi_1, \cdots, \pi_r$ are distinct primes. Let $\tau^{(k)}(K) = K_1 \Delta_{K,k}$, where $(K_1, k) = 1$ and $\Delta_{K,k} = \pi_1^{b_1} \cdots \pi_r^{b_r}, b_j \geq 0$.

Then

$$\tau^{(k)}(n) = K_1 \prod_{j=1}^{r} \pi_j^{a_j \omega(m) + b_j}.$$

For given K let

$$Pol_K(u) = \prod_{j=1}^{r} \binom{a_j u + b_j + \ell - 1}{\ell - 1}.$$

Then we have

$$r(m) = \tau^{(\ell)}(K_1) Pol_K(\omega(n)).$$

Let $K \in \mathcal{K}$ be fixed, $N_K(x)$ and $N_K(x,k)$ as in Section 3. Then we have

Theorem C. *We have*

$$\max_{x^{7/12+\epsilon} \leq h \leq x^{2/3}} \left| \frac{N_K(x+h,t) - N_K(x,t)}{h} - \frac{N_K(x,t)}{x} \right| \leq c \frac{\Theta_t(x)}{\log\log x}$$

up to $t \leq \log\log x + \rho_x \sqrt{\log\log x}$.

We define the interval $J_{K,x}(u,v)$ as follows:

$$J_{K,x}(u,v) = \left[Pol_K(\log\log x + u\sqrt{\log\log x}, \; Pol_K(\log\log x + v\sqrt{\log\log x} \right].$$

From the previous results we have

Theorem D. *Let* $K \in \mathcal{K}, (u,v) \in (-\infty, \infty)$. *Then*

$$\frac{1}{N_K(x)} \sharp \Big\{ m \leq x : (m,K) = 1, \; \mu(m) \neq 0, \\ \frac{r_K(Km)}{\tau^{(\ell)}(K_1)} = Pol_K(\omega(m)) \subset J_{K,x}(u,v) \Big\} \to \Phi(v) - \Phi(u).$$

The same is true if m runs in the short interval $[x, x+y]$ when $x^{7/12+\epsilon} \leq y \leq x^{2/3}$.

6. New results

Theorem 1. *Let $H \in \mathcal{M}$, $|H(n)| = 1$ for every $n \in \mathbb{N}$, furthermore $H(p) = \kappa$ for every $p \in \mathcal{P}$, $\kappa \neq 1$. Assume that $Y = Y(x) \in [x^{7/12+\epsilon}, x^{2/3}]$. Then*

$$\lim_{x\to\infty} \frac{1}{Y} \sum_{x\leq n\leq x+y} H(\tau_r(n)) \to 0.$$

Proof. The first observation is the following.

(1) If $\kappa^\ell = 1$, then $\sum_{k=0}^{\ell-1} \kappa^j = 0$

(2) If $\kappa = e^{2\pi i\alpha}, \alpha$ is an irrational number, then

$$\lim_{L\to\infty} \inf \frac{1}{L} \Big| \sum_{h=0}^{L-1} \kappa^h \Big| = 0,$$

thus for every $\delta > 0$ there exists an integer $L = L_\delta$ for which

$$\frac{1}{L_\delta} \Big| \sum_{h=0}^{L_\delta - 1} \kappa^h \Big| < \delta.$$

We have

$$\sum_{x\leq n\leq x+y} H(\tau_r(n)) = \sum_{K\in\mathcal{K}} H(K) \sum_{\substack{x\leq Km\leq x+y \\ (K,m)=1}} \kappa^{\omega(m)} |\mu(m)| = \sum_{K\in\mathcal{K}} H(K)\Sigma_K.$$

Let T be a positive number. We have

$$\sum_{\substack{K\in\mathcal{K} \\ K>T}} |H(K)\Sigma_K| \leq \sum_{T<K\leq y} \frac{y}{K} + \sum_{\substack{K\in\mathcal{K} \\ y<K<x+y}} 1. \tag{6.1}$$

Since

$$\sum_{\substack{K\in\mathcal{K} \\ K>u}} 1 \leq c\sqrt{u},$$

the right hand side of (6.1) tends to 0 as $T \to \infty$.

Let $\delta > 0$ be a given small number, $T = T_\delta$ be such a number for which the right hand side of (6.1) is less than δ. We shall estimate

$$\Sigma_K = \sum_{\substack{\frac{x}{K}\leq m\leq \frac{x}{K}+\frac{y}{K} \\ (K,m)=1}} \kappa^{\omega(m)} |\mu(m)|$$

for $K \le T_\delta$. Let $x_K = \frac{x}{K}, y_K = \frac{y}{K}$. Then

$$x_K^{7/2+\epsilon/2} \le y_K \le x_K^{2/3}$$

holds if x is large enough.

We shall prove that

$$\lim_{x\to\infty} \max_{K \le T_\delta} \left| \frac{1}{y_K} \sum_{\substack{(m,K)=1 \\ x_K \le m \le x_K + y_K}} \kappa^{\omega(m)} |\mu(m)| \right| = 0.$$

Let

$$U_x = \Big[(1-\sigma_x)x_2, x_2 + \rho_x \sqrt{x_2}\Big],$$

where $x_2 = \log\log x$, $\rho_x \to \infty$ and $\sigma_x \to 0$ slowly.

From Theorem A we obtain that

$$\max_{K \le T_\delta} \frac{1}{y_K} \sum_{\substack{(m,K)=1 \\ \omega(m) \notin U_x}} |\mu(m)| \to 0 \quad \text{as} \quad x \to \infty.$$

By using Theorem A we obtain that

$$\Sigma_K = y_K \sum_{k \in U_x} \kappa^h \rho_h(x) + o(y_K).$$

Since

$$\max_{\ell \le L} \left| \frac{\rho_{h+1}(x)}{\rho_h(x)} - 1 \right| \to 0 \quad \text{as} \quad x \to \infty,$$

therefore

$$\rho_h(x) = \frac{1}{L} \sum_{\ell=1}^{L} \rho_{h+\ell}(x) + o_x(1)$$

and so

$$\Big| \sum_{k \in U_x} \kappa^h \rho_h(x_K) \Big| \le \frac{1}{L} \sum_{t \in U_x} \rho_t(x_K) | \sum_{j=1}^{t} \kappa^{t-j} | \le \delta + o_x(1).$$

Thus

$$\frac{1}{y_K} \Big| \Sigma_K \Big| \to 0 \quad \text{for every} \quad K \in \mathcal{K}, K \le T_\delta.$$

The proof of Theorem 1 is therefore complete. ■

The following theorem is a corollary of Theorem 1:

Theorem 2. *Let λ be the Liouville function, $\lambda(r) = -1$. Then*

$$\frac{1}{y}\sum_{x\le n\le x+y}\lambda(\tau_r(n)) \to 0 \quad as \quad x\to\infty,$$

when $x^{7/12+\epsilon} \le y \le x^{2/3}$.

Proof. We have $\lambda(\tau_r(p)) = \lambda(r) = -1$. Thus, Theorem 2 is a special case of Theorem 1. ■

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